Stabilization of optical solitons in chirped $\mathscr{PT}$-symmetric lattices*

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We investigate the stability properties of optical solitons in a chirped $\mathscr{PT}$-symmetric lattice whose frequency changes in the transverse direction. Linear-stability analysis together with the direct propagation simulations demonstrates that the chirped lattice can improve the stability of optical solitons dramatically. The instability of fundamental solitons can be completely suppressed if the chirp rate exceeds a critical value. A broad stability area of dipole solitons appears if the lattice is appropriately chirped. Thus, we propose an effective way to suppress the instability of solitons in $\mathscr{PT}$-symmetric potentials.

Keywords: stability, chirp rate, $\mathscr{PT}$-symmetric lattices

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1. Introduction

Solitons play a crucial role in many branches of nonlinear science, including solid-state physics, Bose–Einstein condensates, and nonlinear optics.[1] During the past decades, lattice solitons have been intensively studied because optical lattices can be used to guide and control the dynamics of solitons.[2–4] Various types of lattice solitons have been theoretically predicted and experimentally observed.[5,6]

Very recently, dynamics of solitons in complex lattices with a $\mathscr{PT}$ symmetry have been reported in a variety of nonlinear systems.[7–18] Diverse optical potentials with gain or loss regions were suggested to support stable spatial solitons. These works include, for example, solitons in linear lattices,[7,9–11] solitons in purely nonlinear lattices,[12,13] solitons in mixed linear-nonlinear lattices,[14] multi-stable solitons in cubic-quintic media,[15] and two-dimensional (2D) bright solitons in defocusing Kerr media.[16,17]

Kartashov *et al.*[19–21] predicted that the amplitude and frequency modulation of transversally chirped photonic lattices offer new opportunities for the controllable soliton steering and routing. Thresholdless surface waves can be supported by a semi-infinite lattice with a linear spatial frequency chirp.[20] One can control the position and localization of a discrete surface soliton by manipulating the lattice chirp.[22] Surface defect gap solitons were found in chirped optical lattices.[23]

Comparing with that of solitons in purely real lattices, the stability domain of solitons in $\mathscr{PT}$- symmetric lattices usually shrinks, due to the presence of imaginary parts of lattices.[16] On the other hand, the stability domain of scalar and two-color solitons expands when the purely real lattices are appropriately chirped.[24,25] Thus, a question arises: can the instability of solitons be suppressed by the chirped $\mathscr{PT}$ lattices? If they can, how do they influence the evolution of solitons?

To answer these questions, we consider the dynamics of fundamental and higher-order solitons in a $\mathscr{PT}$-symmetric lattice whose frequency changes in the transverse direction. To illustrate the stabilization effect of the chirped lattice, we select a special type of $\mathscr{PT}$ lattices in which even fundamental solitons are unstable in their whole existence domain. The existence domain of solitons can be altered by the chirped lattice. In comparison with the entirely unstable solitons in unchirped periodic lattices, fundamental solitons can be completely stable provided that the chirp rate exceeds a critical value. Meanwhile, the instability of dipole solitons can also be remarkably suppressed if the chirp rate is large.

2. Theoretical model

We consider the propagation of optical radiation along the z axis in a focusing Kerr medium with modulation of linear refractive index along the transverse x axis, described by the nonlinear Schrödinger equation for the dimensionless complex field amplitude ψ

$$\mathrm{i}\frac{\partial\psi}{\partial z}+\frac{\partial^2\psi}{\partial x^2}+[V(x)+\mathrm{i}W(x)]\psi+|\psi|^2\psi=0. \qquad (1)$$

Here, the longitudinal z and transverse x coordinates are scaled to the diffraction length and input beam width, respectively. The functions $V(x)=V_0\cos[\Omega_x(1+\alpha|x|)x]$ and $W(x)=W_0\sin[\Omega_x(1+\alpha|x|)x]$ describe the transverse real and imaginary profiles of the complex refractive index, where V_0 and W_0 are proportional to the depths of real and imaginary

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refractive index modulations, Ω_x is the modulation frequency, and α is the chirp rate. When $\alpha = 0$, the lattice diverges to a periodic one.

We search for soliton solutions in the form: $\psi(x,z) = \phi(x)\exp(-\mathrm{i}bz)$, where $\phi(x)$ is a complex function describing the stationary solutions of solitons and b is a propagation constant. We assume that the depth of the refractive index modulation is small compared with the unperturbed index and is of the order of the nonlinear correction to the refractive index due to the Kerr effect, so that the lattices addressed here are relatively shallow. In the present paper, we will fix $\Omega_x = 2$, $V_0 = 2$, $W_0 = 1$ and vary b, α. Profiles of solitons can be found numerically by using the relaxation method. The power or energy flow of light field can be defined as: $U = \int_{-\infty}^{\infty} |q|^2 \mathrm{d}x$.

To analyze the dynamic stability of the obtained soliton families, we search for the perturbed solutions of Eq. (1) in the form: $\psi(x,z) = \{\phi(x) + [v(x) - w(x)]\exp(\lambda z) + [v(x) + w(x)]^* \exp(\lambda^* z)\}\exp(-\mathrm{i}bz)$, where perturbation components v, w can grow upon propagation with a complex growth rate λ. Linearization of Eq. (1) around a stationary solution ϕ yields a linear eigenvalue problem

$$-\mathrm{i}\begin{bmatrix} -\mathrm{i}W - \mathrm{i}\,\mathrm{Im}(\phi^2) & \widehat{L} - \mathrm{Re}(\phi^2) \\ \widehat{L} + \mathrm{Re}(\phi^2) & -\mathrm{i}W + \mathrm{i}\,\mathrm{Im}(\phi^2) \end{bmatrix}\begin{bmatrix} v \\ w \end{bmatrix} = \lambda \begin{bmatrix} v \\ w \end{bmatrix}, \tag{2}$$

where $\widehat{L} = \mathrm{d}^2/\mathrm{d}x^2 + b + V + 2|\phi|^2$. The coupled equations can be solved numerically by using a finite-difference method. Linear stability of a soliton is determined by the spectrum of the above linearization operator, and the existence of any eigenvalue with a positive real part implies the linear instability of the soliton. Stationary solutions can be completely stable only when all real parts of eigenvalues equal zero.

3. Numerical results and discussion

Before we discuss the properties of solitons in a chirped lattice with a $\mathscr{P}\mathscr{T}$ symmetry, it is instructive to consider solitons in a periodic $\mathscr{P}\mathscr{T}$ lattice. Typical profiles of fundamental solitons in a lattice with $\alpha = 0$ are shown in Fig. 1(a). The real parts of soliton profiles are even symmetric while the imaginary parts exhibit an odd symmetry. The symmetry of fundamental solitons is the same as that of the lattice. As the propagation constant grows, the peak of the solitons decreases. When the frequency of lattice is modulated, the lattice features as a nonperiodic distribution which changes the localization and existence domain of solitons. For example, solitons in the lattice with chirp rate $\alpha = 0.15$ are more localized than those in the periodic lattice [see Fig. 1(b)]. The reason is that the central lattice site is wider than the lattice sites around it, and thus it can capture more light energy. Meanwhile, due to the redistribution of the lattice, the peak value of solitons at fixed propagation constant increases slightly with the growth of chirp rate α (see Figs. 1(b) and 1(c)].

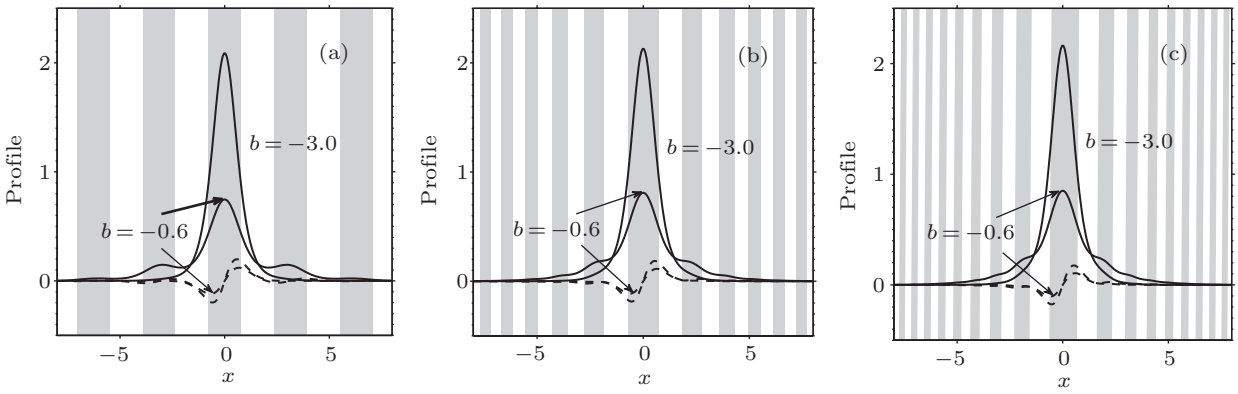


Fig. 1. Profiles of fundamental solitons in lattices with (a) $\alpha = 0$, (b) $\alpha = 0.15$, and (c) $\alpha = 0.3$. Solid and dashed curves represent the real and imaginary parts of solitons. Gray regions correspond to $V(x) \geq 0$.

Figure 2(a) displays an example of band-gap structure of a periodic $\mathscr{P}\mathscr{T}$-symmetric lattice with $V_0 = 2$, $W_0 = 1$, and $\Omega_x = 2$. Solitons in periodic lattices can only exist in the semi-infinite and finite gaps separated by the edges of bands. Due to the restriction of the upper edge of the semi-infinite gap ($b = -0.349$), there exists an upper cutoff of propagation constant for each family of solitons in focusing media. For example, the upper cutoff of propagation constant of fundamental solitons in the periodic lattice is $b_{\mathrm{co}} = -0.4$ [see Fig. 2(b)]. When the frequency of the lattice is chirped, the periodicity of the lattice is destroyed. The restriction of the semi-infinite gap on the existence domain of solitons becomes weaker. To illustrate this, we plot the upper cutoff of propagation constant b_{co} versus the chirp rate α in Fig. 2(b). The propagation constant cutoff is a monotonously increasing function of chirp rate. The existence domain of solitons expands slowly with the growth

of chirp rate. The propagation constant cutoff penetrates into the first band of the periodic lattice.

The dependence of power of solitons on the propagation constant b at different chirp rates are shown in Fig. 2(c). The power decreases with the increase of propagation constant. Near the upper propagation constant cutoff, the power increases with the growth of chirp rate. When the propagation constant is far away from the upper edge of the semi-infinite gap, the powers of solitons at fixed b approach to the same value for different chirp rates. This is because solitons become more localized at the small propagation constant, and the energy is captured mainly in the central lattice channel. At this time, the influence of frequency modulation of the lattice on the distribution of solitons becomes weaker and can be ignored.

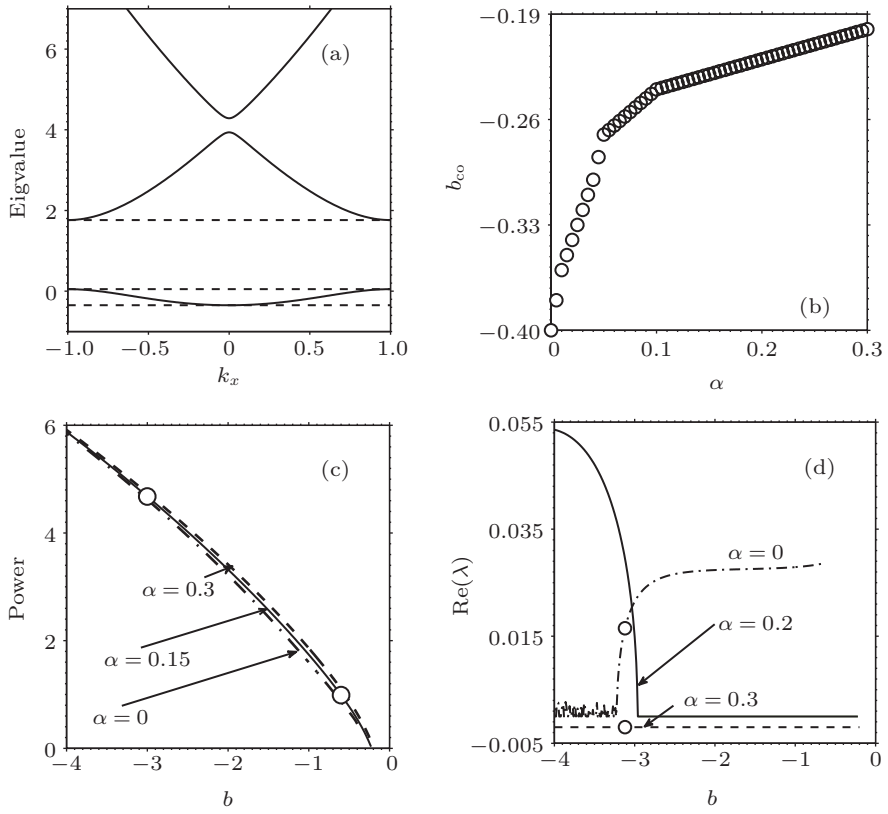


Fig. 2. (a) Band gap structure of periodic $\mathscr{P}\mathscr{T}$ lattice with $V_0 = 2$, $W_0 = 1$. (b) Upper cutoff of propagation constant versus the chirp rate α. (c) Power diagrams versus the propagation constant for solitons in lattices with different chirp rates. (d) Instability growth rate versus propagation constant. To distinguish it from other curves, the curve with $\alpha = 0.3$ is lowered by 0.002.

In the studies of optical solitons, the stability is always a crucial issue. To elucidate the stability properties of fundamental solitons in $\mathscr{P}\mathscr{T}$ lattices, based on Eq. (2), we numerically solve the instability growth rates of solitons in lattices with different chirp rates. Some representative results are shown in Fig. 2(d). In the periodic lattice, though the perturbation growth rates of fundamental solitons at lower propagation constants are very small, the solitons are rigorously unstable. They can survive for a long propagation distance due to the small instability growth rates. Note that the $\mathscr{P}\mathscr{T}$ lattice in our model is different from that in Ref. [6], where fundamental solitons are always stable. When the chirp rate increases, the dependence of instability growth rate on the propagation constant alters. That is to say, the moderate chirp rate $\alpha = 0.2$ leads to the instability of solitons at lower propagation constant but the stability near the upper cutoff of propagation constant. With the further increase of chirp rate (e.g., $\alpha = 0.3$), solitons will be completely stable in their entire existent domain. Thus, we conclude the central idea of this paper, i.e., the stability of solitons in $\mathscr{P}\mathscr{T}$ lattice can be dramatically improved by appropriate modulation of lattice frequency.

To illustrate the detail of the stability of fundamental solitons, we show two typical examples of spectra of the linearization operator for solitons at $b = -3.12$ in lattices with $\alpha = 0.3$ and 0 in Figs. 3(a) and 3(b), respectively. The corresponding propagation simulation results, shown in Figs. 3(c) and 3(d), are in good agreement with the prediction of linear-stability analysis. The unstable soliton survives a very long distance (hundreds of diffraction lengths), due to the small perturbation growth rate $[\mathrm{Re}(\lambda)_{\max} \approx 0.021]$.

Besides the fundamental solitons, we also find a class of higher-order solitons in chirped $\mathscr{P}\mathscr{T}$ lattices as shown in Fig. 4. Since the amplitude distribution of higher-order solitons is similar to that of dipole solitons in

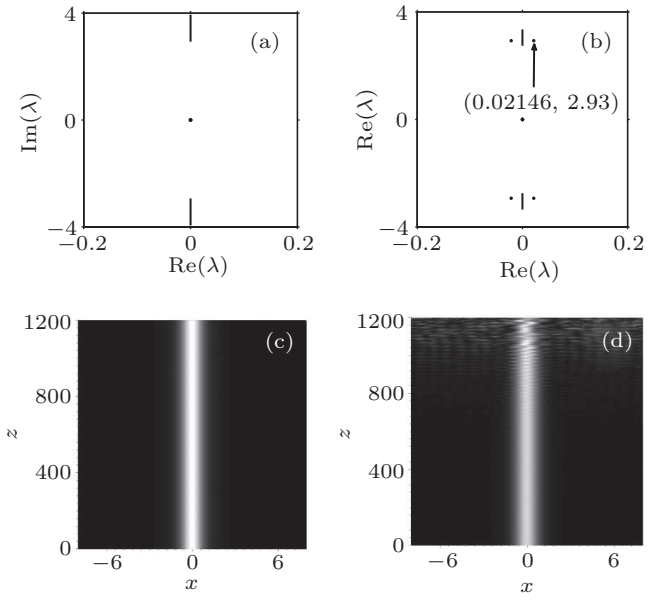


Fig. 3. Spectra of linearization operator of Eq. (2) for solitons at $b = -3.12$ in lattices with (a) $\alpha = 0.3$ and (b) $\alpha = 0$, respectively. (c) and (d) show the stable and unstable propagations of solitons with eigenvalues shown in (a) and (b), respectively.

conventional real lattices or bulk media, we can treat them as nonlinear dipole modes. The real parts of such solitons exhibit an odd symmetry while the imaginary parts are of even symmetry. The symmetries of dipole solitons are exactly opposite to those of lattices. A dipole soliton can also be seen as a nonlinear superposition of two out-of-phase fundamental solitons. The interaction between the two components beside the central lattice site is repulsive due to the fact that the left component is out-of-phase with the right one. In comparison with the solitons in unchirped lattice, the peaks of dipoles in chirped lattices no longer reside in the nearest-to-center lattice sites. Instead, the maxima of solitons are repelled to the second lattice channels beside the center site, as shown in Figs. 4(b) and 4(c). Comparing with that of the fundamental solitons, with the growth of chirp rate α, the peaks of dipole solitons increase with a faster rate.

In contrast to the fundamental solitons, the existence domain of dipole solitons is dramatically reduced [see Fig. 5(a)]. Dipole solitons in both periodic and chirped $\mathscr{P}\mathscr{T}$ symmetric lattices have both lower and upper propagation constant cutoffs. The existence domain of dipoles shrinks with the growth of lattice chirp rate and the upper cutoff of propagation constant moves away from the edge of the semi-infinite gap. The broader profiles and the higher peaks of dipoles [see e.g., Figs. 4(b) and 4(c)] account for the higher power of solitons in chirped lattice [Fig. 5(a)].

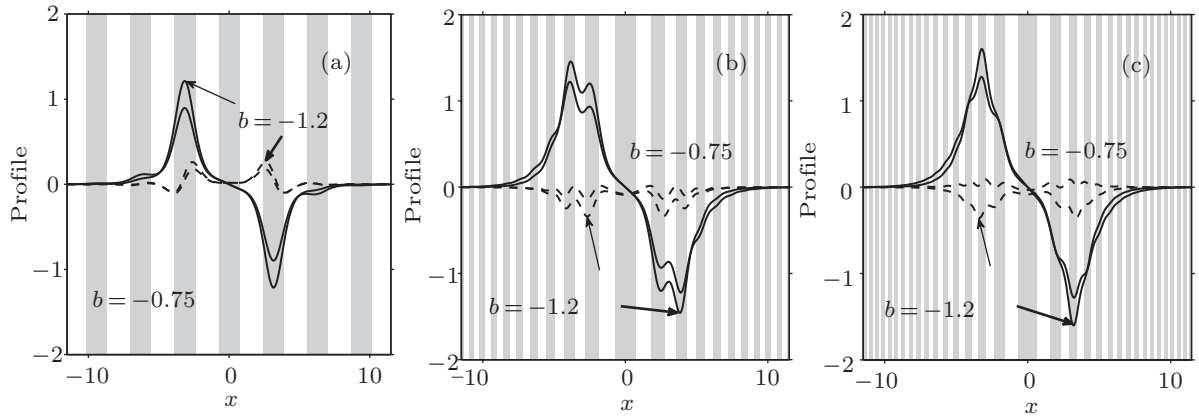


Fig. 4. Profiles of dipole solitons in lattices with (a) $\alpha = 0$, (b) $\alpha = 0.15$, and (c) $\alpha = 0.3$. Solid and dashed curves represent the real and imaginary parts of dipoles.

Although the chirped lattice cannot eliminate the instability of dipole solitons completely, it can dramatically suppress the instability. We show the perturbation growth rates of two families of dipole solitons in chirped and unchirped lattices in Figs. 5(b) and 5(c), respectively. Dipole solitons in periodic $\mathscr{P}\mathscr{T}$ lattice are entirely unstable. In chirped lattices, for dipoles at a higher power or a smaller propagation constant, the corresponding maxima of real parts of λ equal 0, which indicates that such nonlinear modes can propagate stably without any distortion. The linear-stability analysis results discussed above are confirmed by the outcome of the direct numerical integration of Eq. (1) with the input condition

$$\psi(x, z = 0) = \phi(x)[1 + \rho(x)],$$

where $\rho(x)$ is a function describing a noise with a random distribution. Two typical examples of stable and unstable dipole solitons in chirped and unchirped $\mathscr{P}\mathscr{T}$ lattices marked in Figs. 5(b) and 5(c) are displayed in Figs. 5(e) and 5(f), respectively. The dipole soliton in unchirped lattice breaks asymmetrically after a relatively short propagtion distance. It demonstrates again that the frequency modulation of $\mathscr{P}\mathscr{T}$ lattice plays an important role for the stabilization of solitons in it.

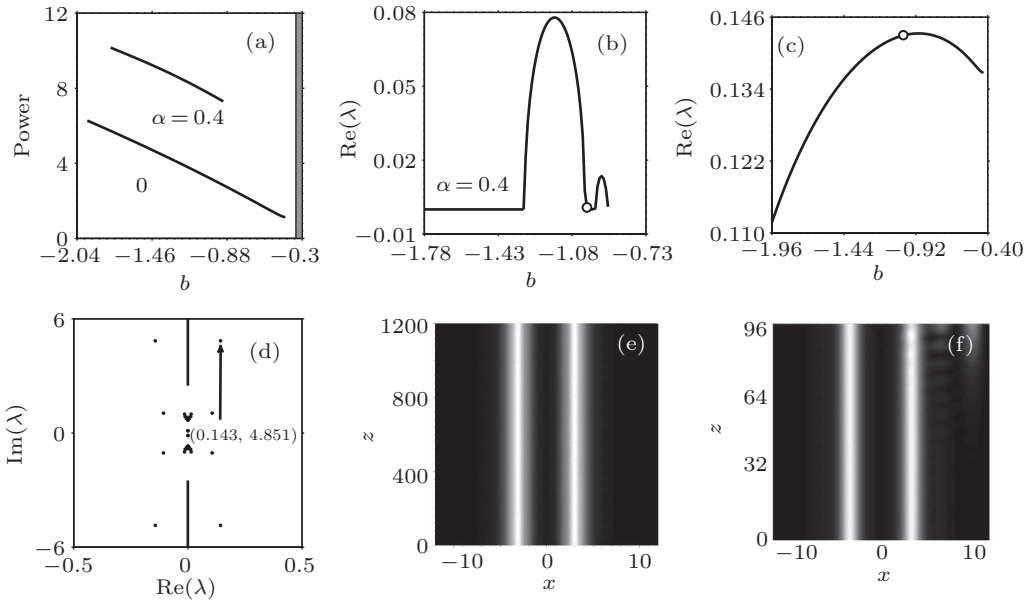


Fig. 5. (a) Dependence of power of dipole solitons on propagation constant. Instability growth rate of dipole solitons versus propagation constant in lattices with (b) $\alpha = 0.4$ and (c) $\alpha = 0$. (d) Spectrum of the linearization operator for dipole at $b = -1.01$ in lattice with $\alpha = 0$. (e) Stable and (f) unstable propagations of dipole solitons at $b = -1.01$ corresponding to (d).

4. Conclusions

In summary, we addressed the dynamics of optical solitons in a $\mathscr{P}\mathscr{T}$-symmetric lattice with a chirped frequency modulation in the transverse direction. The instability of solitons can be dramatically suppressed by the appropriately chirped lattice. The completely unstable fundamental solitons in unchirped lattices become completely stable if the chirp rate exceeds a critical value. Meanwhile, dipole solitons in chirped $\mathscr{P}\mathscr{T}$ lattices can propagate stably in a wide parameter region. Thus, we proposed an effective way to suppress the instability of solitons in $\mathscr{P}\mathscr{T}$-symmetric potentials.

References

[1] Kivshar Y S and Agrawal G P 2003 *Optical Solitons: From Fibers to Photonic Crystals* 1st edn. (San Diego: Academic Press)
[2] Fleischer J W, Segev M, Efremidis N K and Christodoulides D N 2003 *Nature* **422** 147
[3] Chen Z and Yang J 2007 *Nonlinear Optics and Applications* (Kerala: Research Signpost) p. 5
[4] Zhu W, Luo L, He Y and Wang H 2009 *Chin. Phys. B* **18** 4319
[5] Lederer F, Stegeman G I, Christodoulides D N, Assanto G, Segev M and Silberberg Y 2008 *Phys. Rep.* **463** 1
[6] Kartashov Y V, Malomed B A and Torner L 2011 *Rev. Mod. Phys.* **83** 247
[7] Musslimani Z H, Makris K G, El-Ganainy R and Christodoulides D N 2008 *Phys. Rev. Lett.* **100** 030402
[8] Rüter C E, Makris K G, El-Ganainy R, Christodoulides D N, Segev M and Kip D 2010 *Nature Phys.* **6** 192
[9] Zhou K, Guo Z, Wang J and Liu S 2010 *Opt. Lett.* **35** 2928
[10] Wang H and Wang J 2011 *Opt. Express* **19** 4030
[11] Lu Z and Zhang Z 2011 *Opt. Express* **19** 11457
[12] Abdullaev F K, Kartashov Y V, Konotop V V and Zezyulin D A 2011 *Phys. Rev. A* **83** 041805
[13] Zezyulin D A, Kartashov Y V and Konotop V V 2011 *Europhys. Lett.* **96** 64003
[14] He Y, Zhu X, Mihalache D, Liu J and Chen Z 2012 *Phys. Rev. A* **85** 013831
[15] Li C, Liu H and Dong L 2012 *Opt. Express* **20** 16823
[16] Shi Z, Jiang X, Zhu X and Li H 2011 *Phys. Rev. A* **84** 053855
[17] Nixon S, Ge L and Yang J 2012 *Phys. Rev. A* **85** 023822
[18] Hu S and Hu W 2012 *Chin. Phys. B* **21** 024212
[19] Kartashov Y V, Vysloukh V A and Torner L 2005 *J. Opt. Soc. Am. B* **22** 1356
[20] Kartashov Y V, Vysloukh V A and Torner L 2007 *Phys. Rev. A* **76** 013831
[21] Molina M I, Kartashov Y V, Torner L and Kivshar Y S 2007 *Opt. Lett.* **32** 2668
[22] Molina M I, Kartashov Y V, Torner L and Kivshar Y S 2008 *Phys. Rev. A* **77** 053813
[23] Xie J, He Y and Wang H 2010 *J. Opt. Soc. Am. B* **27** 484
[24] Zheng J, Huang C, Zhong S and Dong L 2010 *J. Opt. Soc. Am. B* **27** 2224
[25] Yin G, Zheng J and Dong L 2010 *Opt. Commun.* **283** 583