Preparing quantum vortex states with odd Schrödinger cat states through a coupled waveguide system

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A scheme is proposed for preparing a quantum vortex state with a coupled waveguide, in which a single-mode odd cat state with weak intensity and a single-mode coherent state are inserted in the input ports, respectively. The analytical wavefunction of the resulting state in the quadrature space is derived, and the vortex structure of the output state is analyzed. It is found that the obtained states, which may carry a vortex with topological charge index one, are entangled and nonclassical, depending only on the scaled propagation time and the weak intensity of the input odd cat state instead of the displacement parameter of the input coherent state. The phase distribution, however, in the quadrature space, depends on the displacement parameter of the input coherent state

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1. Introduction

Constructing new multimode quantum states based on the fundamental principles in quantum mechanics has received wide attention in recent years. The motivation to prepare those quantum states originates from some important applications of their multimode entangled properties in quantum communications, for instance, they are used as necessary resources in multiuser quantum communication networks.[1,2] Among multimode quantum states, vortex states of a quantized radiation field in the quadrature space bear strong nonclassical features and are also typical entangled non-Gaussian states. Agarwal *et al.*[3] first proposed a scheme to generate a quantum vortex by using a two-mode squeezed vacuum. They have shown that some special two-mode states experiencing the linear transformation belonging to the SU(2) group could lead to a vortex state.[4] Very recently, it was proposed that a two-mode quantized vortex state could also be prepared through a squeezed vacuum by subtracting a photon,[5] and a quantum elliptic vortex state could be realized with the help of a two-mode squeezed coherent state through a beam splitter or a dual channel directional coupler.[6,7] Moreover, it was suggested that two-mode non-Gaussian states with vortex structures could be generated with pair coherent states or two-mode superposition coherent states through parametric frequency conversion processes.[8–11]

Coupled waveguides are widely used in a large number of well-known interesting processes.[12–17] There are a few designs utilizing coupled waveguides to generate quantum states with entanglement[17] and nonclassical effects.[12] Motivated by the distinguished work mentioned above, in this paper, we propose a scheme to prepare the quantum vortex state with coupled waveguides using the input states involving a single-mode even or odd Schrödinger cat state $N_{\pm}(|\beta\rangle \pm |-\beta\rangle)_a$ in mode a and a single-mode coherent state $|\alpha\rangle$ in mode b and investigate the evolution dynamics of the propagating state through the coupled waveguides. Compared with those schemes proposed in Refs. [3]–[11], the present scheme is more advantageous because the coupled waveguide is an available device and the input quantized field states are also easy to obtain in laboratory. The rest of the paper is organized as follows. In Section 2, the state vector evolution of two-separable state input through coupled waveguides is analytically solved. We derive the explicit wavefunction for the propagating state in the quadrature space and investigate its vortex structure. Then, we study the evolution dynamics of the vortex structure for propagating states through coupled waveguides. In Section 3, we analyze the entanglement and non-classicality for the resulting state by the linear entropy criterion and the Wigner function, respectively. Finally, some conclusions are presented in Section 4.

2. Quantized vortex state realized through a coupled waveguide system

To prepare a quantum vortex state with coupled wave guides, the model of the coupled waveguide introduced by Rai *et al.*[17] is used. The coupled waveguide consists of two single mode waveguides, and the coupling is realized through the nearest-neighbor interaction. The operators a and b, denoting

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the field operators for the modes in each waveguide, satisfying bosonic commutation relations $[a,\ a^\dagger] = [b,\ b^\dagger]=1$. The Hamiltonian describing the evanescent coupling between the waveguide modes in such a system of two coupled waveguides may be derived from the coupled mode theory and has the following form (with $\hbar$=1):

$$H = \omega(a^\dagger a + b^\dagger b) + J(a^\dagger b + a b^\dagger), \tag{1}$$

where the two terms inside the first bracket on the right-hand side represent the free energies of the waveguide modes and the two terms inside the second bracket arise from the evanescent coupling between the waveguide modes with coupling strength J, J depends on the distance between the waveguides. The input to the coupled waveguide system may be in a separable or an entangled state. Under the interaction picture the interaction Hamiltonian becomes $H_{\rm int} = J(a^\dagger b + a b^\dagger)$ and the corresponding evolution operator may be expressed as

$$U(t) = \exp(-{\rm i}H_{\rm int}t). \tag{2}$$

In this paper, we deal with two separable single-mode input states, one is an odd or even Schrödinger cat state $N_\pm(|\beta\rangle \pm |-\beta\rangle)_a$ for mode a and the other is a coherent state $|\alpha\rangle_b$ for mode b. Nowadays, obtaining such a single-mode odd or even cat state with weak intensities seems to be easier than realizing two-mode squeezed states or two-mode pair coherent states with current available technology.[18–20] The input state can be then described as

$$|\psi\rangle_{\rm in}^{\pm} = N_\pm(|\beta\rangle \pm |-\beta\rangle)_a|\alpha\rangle_b \equiv N_\pm(|\beta,\alpha\rangle \pm |-\beta,\alpha\rangle), \tag{3}$$

where $N_\pm = 1/\sqrt{2(1\pm {\rm e}^{-2\beta^2})}$ is the normalized constant, α and β characterize the displacement parameters of the coherent states and, for simplicity, are assumed to be real. In fact, state (3) has been prepared in Ref. [21].

Due to $U(t)|0,0\rangle = |0,0\rangle$, a two-mode coherent state $|\alpha,\beta\rangle$ passing through such a coupled waveguide system will evolve into a new two-mode coherent state

$$U(t)|\alpha,\beta\rangle = |\alpha\cos\tau - {\rm i}\beta\sin\tau, \beta\cos\tau - {\rm i}\alpha\sin\tau\rangle, \tag{4}$$

where $\tau = Jt$ is the dimensionless propagation time. Then, for the input state (3), the state vector of the propagating state turns into

$$\begin{aligned} |\psi\rangle_{\rm out} &= N_\pm U(t)(|\beta,\alpha\rangle \pm |-\beta,\alpha\rangle) \\ &= N_\pm(|\beta\cos\tau - {\rm i}\alpha\sin\tau, \alpha\cos\tau - {\rm i}\beta\sin\tau\rangle \\ &\quad \pm |-\beta\cos\tau - {\rm i}\alpha\sin\tau, \alpha\cos\tau + {\rm i}\beta\sin\tau\rangle), \end{aligned} \tag{5}$$

which is still a superposition state consisting of two-mode coherent states but with new displacement parameters. It is readily seen from Eq. (5) that except for additional phases, the propagating states return to their initial states $|\psi\rangle_{\rm out} = N_\pm(|\delta_k\beta\rangle \pm |\delta_{k+1}\beta\rangle)_a|\delta_k\alpha\rangle_b$ at $\tau = \tau_E \equiv 2k\pi/2$ ($k = 0,1,2,3,\ldots$) and $|\psi\rangle_{\rm out} = N_\pm|{\rm i}\delta_{k+1}\alpha\rangle_a(|{\rm i}\delta_{k+1}\beta\rangle \pm |{\rm i}\delta_k\beta\rangle)_b$ at $\tau = \tau_O \equiv (2k+1)\pi/2$ [where $\delta_k = (-1)^k$], respectively. That is, both single-mode states exchange their initial states (5). All these resulting states are separable.

In what follows, we evaluate the wavefunction $\psi_{\rm out}^{\pm}(x,y) = \langle x,y|\psi\rangle_{\rm out}^{\pm}$ in the quadrature space, where $|x,y\rangle$ is the eigenvector of operators, $x = (a^\dagger + a)/\sqrt{2}$ and $y = (b^\dagger + b)/\sqrt{2}$ corresponding to eigenvalues x and y, respectively. Based on the relation

$$\langle x|\alpha\rangle = \frac{1}{\pi^{1/4}}\exp\left(-\frac{x^2}{2} + \sqrt{2}\alpha x - \frac{\alpha^2}{2} - \frac{|\alpha|^2}{2}\right), \tag{6}$$

wavefunctions $\psi_{\rm out}^{\pm}(x,y,\tau)$ respectively are

$$\begin{aligned} \psi_{\rm out}^{+}(x,y) &= \frac{2N_+}{\sqrt{\pi}}\exp\left(-\frac{x^2+y^2+A}{2}\right) \\ &\quad \times \exp[\sqrt{2}\alpha(y\cos\tau - {\rm i}x\sin\tau)]\cosh\{\sqrt{2}\beta[x\cos\tau \\ &\quad - {\rm i}\sin\tau(y - \sqrt{2}\alpha\cos\tau)]\} \end{aligned} \tag{7}$$

and

$$\begin{aligned} \psi_{\rm out}^{-}(x,y) &= \frac{2N_-}{\sqrt{\pi}}\exp\left(-\frac{x^2+y^2+A}{2}\right) \\ &\quad \times \exp[\sqrt{2}\alpha(y\cos\tau - {\rm i}x\sin\tau)]\sinh\{\sqrt{2}\beta[x\cos\tau \\ &\quad - {\rm i}\sin\tau(y - \sqrt{2}\alpha\cos\tau)]\}, \end{aligned} \tag{8}$$

where $A = (\alpha^2 + \beta^2)(1 + \cos 2\tau)$.

Because

$$\sinh(x + {\rm i}y) = \sinh x\cos y + {\rm i}\cosh x\sin y$$

for real x and y, according to Eq. (8), it is noted that $|\psi_{\rm out}^{-}(x,y)|^2$ may exhibit a dark center located at $(0,\sqrt{2}\alpha\cos\tau)$ in the x–y plane by choosing the appropriate values of α and β, which means that $\psi_{\rm out}^{-}(x,y)$ carries a vortex. By setting the suitable parameters α and β for $\tau = \pi/4$, we plot the probability distribution $|\psi_{\rm out}^{-}(x,y)|^2$ and the corresponding phase distributions of the wavefunction $\psi_{\rm out}^{-}(x,y)$ in Fig. 1. From the figures we can clearly see that when β is sufficiently small or the input odd cat state is appropriately weak, the phase singularity centered at $(0,\sqrt{2}\alpha\cos\tau)$ with one topological charge. For example, to realize a vortex with a perfect ring structure, it is found that β should not exceed 0.25 at which the average photon number $\langle a^\dagger a\rangle_{\rm in}$ of the input odd cat state approximates to 1.0. Besides, larger β will lead to the ring structure deformity. In particular, the phase distributions of $\psi_{\rm out}^{-}(x,y)$ depend on the choice of α but $|\psi_{\rm out}^{-}(x,y)|^2$ does not. And the vortex changes from canonical vortex to non-canonical one with the increase of α because the real and imaginary zero contours for the phase distribution of the wavefunction evolve from orthogonal forms into the non-orthogonal ones at the dark center.

Moreover, for the other values of τ deviating from $\pi/4$, by carefully adjusting parameters α and β, we may still obtain the vortex state with perfect ring structure distributions of $|\psi_{\rm out}^{-}(x,y)|^2$, which can be seen from Fig. 2. For instance, at $\tau = 3\pi/10 > \pi/4$, $\beta \approx 1$ corresponding to $\langle a^{\dagger}a\rangle_{\rm in} \approx 1.3$, $|\psi_{\rm out}^{-}(x,y)|^2$ exhibits the perfect ring distributions and is independent of α.

However, for an even Schrödinger cat state input, it is recognized from Eq. (7) that $\psi_{\rm out}^{+}(x,y)$ does not carry any dark center or vortex due to $\cosh(x+{\rm i}y) = \cosh x\cos y + {\rm i}\sinh x\sin y$. In the next section we will only pay our attention to the state $|\psi\rangle_{\rm out}^{-}$ associated with an odd cat-state input.

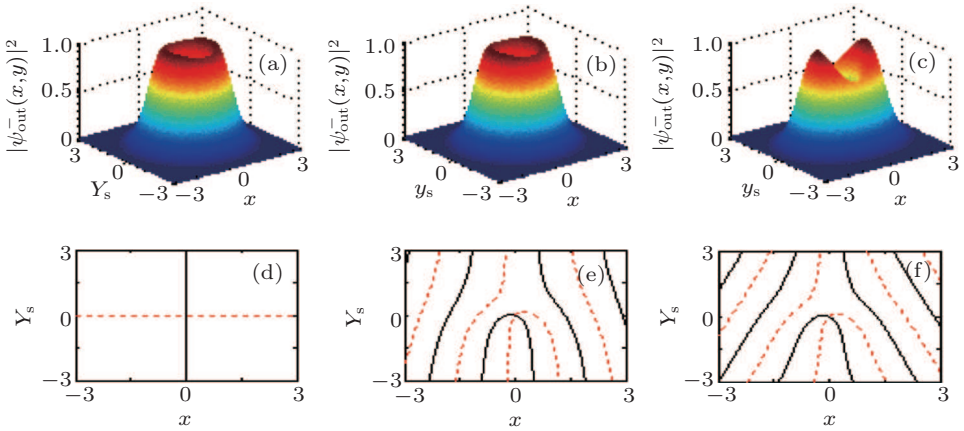


Fig. 1. (color online) Distributions of probability $|\psi_{\rm out}^{-}(x,y)|^2$ (upper), and the corresponding real (solid) and imaginary (dashed) zero contours (bottom) for the phase distributions of the wavefunction $\psi_{\rm out}^{-}(x,y)$ at $\tau = \pi/4$, where $Y_{\rm s} = y - \sqrt{2}\alpha\cos\tau$ and $(\alpha,\ \beta)$ are: (a) and (d) $(0, 0.25)$, (b) and (e) (2, 0.25) (b), (c) and (f) (2, 0.75).

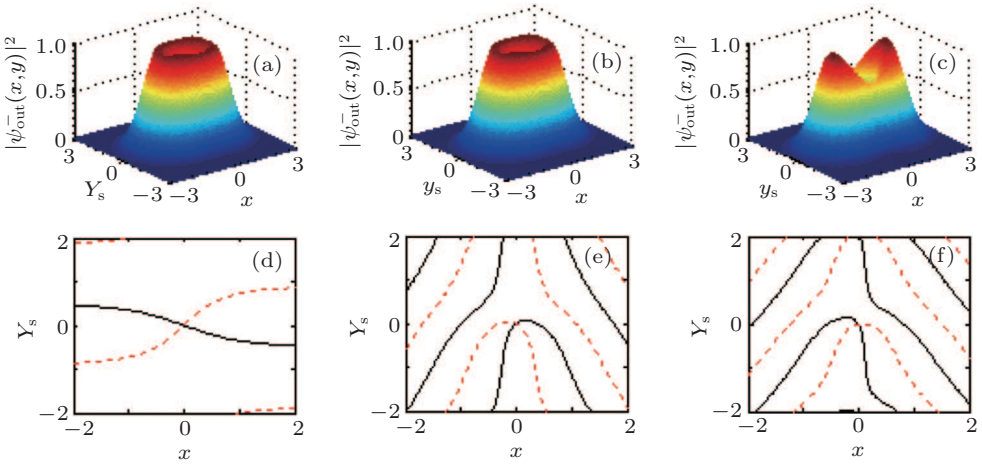


Fig. 2. (color online) Distributions of probability $|\psi_{\rm out}^{-}(x,y)|^2$ (upper), the corresponding real (solid) and imaginary (dashed) zero contours (bottom) for the phase distributions of the wavefunction $\psi_{\rm out}^{-}(x,y)$ at $\tau = 3\pi/10$, where $Y_{\rm s} = y - \sqrt{2}\alpha\cos\tau$ and (α,β) are: (a) and (d) $(0,1)$, (b) and (e) (2, 1), (c) and (f) (2, 1.3).

3. Entanglement and non-classicality for the resulting vortex state $|\psi\rangle_{\rm out}^{-}$

In order to investigate the entanglement of state (5) originating from a coherent state and an odd cat-state passing coupled waveguides, the linear entropy associated with the purity ${\rm Tr}(\rho^2)$ of quantum state ρ is calculated. For the present case the whole system is pure and the linear entropy of the subsystem $k = a$ or b is defined as

$$S_{\rm lin} = 1 - {\rm Tr}(\rho_k^2), \tag{9}$$

where ρ_k is the reduced density operator of the subsystem k. Any non-zero $S_{\rm lin}$ provides an indicator of entanglement occurrence in the state. Using the linear entropy, Agarwal and Biswas[22] analyzed the entanglement of pair coherent states and, based on the similar idea, Marian *et al.*[23] investigated the inseparability of mixed two-mode Gaussian states.

For state (5), the reduced density operator of mode a is

$$\begin{aligned}
\rho_a &= {\rm Tr}_b(|\psi\rangle_{\rm out}^{-}\langle\psi|) \\
&= |N_-|^2\,{\rm Tr}_b[(|\beta_-,\alpha_-\rangle - |-\beta_+,\alpha_+\rangle) \\
&\quad \times(\langle\beta_-,\alpha_-| - \langle-\beta_+,\alpha_+|)] \\
&= |N_-|^2[|\beta_-\rangle\langle\beta_-|\langle\alpha_-|\alpha_-\rangle - |-\beta_+\rangle\langle\beta_-|\langle\alpha-|\alpha_+\rangle \\
&\quad -|\beta_-\rangle\langle-\beta_+|\langle\alpha_+|\alpha_-\rangle + |-\beta_+\rangle\langle-\beta_+|\langle\alpha_+|\alpha_+\rangle] \\
&= |N_-|^2[|\beta_-\rangle\langle\beta_-| + |-\beta_+\rangle\langle-\beta_+| \\
&\quad -|-\beta_+\rangle\langle\beta_-|\langle\alpha_-|\alpha_+\rangle - |\beta_-\rangle\langle-\beta_+|\langle\alpha_+|\alpha_-\rangle].
\end{aligned} \tag{10}$$

From

$$\langle\alpha|\beta\rangle = \exp\left(-\frac{|\alpha|^2+|\beta|^2}{2}+\alpha^*\beta\right) \tag{11}$$

and for state (5), the purity of mode a is

$$\begin{aligned}&\mathrm{Tr}(\rho_a^2)\\ &= \frac{2(1+\mathrm{e}^{-4\beta^2\cos^2\tau}+\mathrm{e}^{-4\beta^2\sin^2\tau}+\mathrm{e}^{-4\beta^2}-4\mathrm{e}^{-2\beta^2})}{|N_-|^4}\\ &= \frac{2[(1+\mathrm{e}^{-4\beta^2\cos^2\tau})(1+\mathrm{e}^{-4\beta^2\sin^2\tau})-4\mathrm{e}^{-2\beta^2}]}{|N_-|^4}.\end{aligned} \tag{12}$$

Very interestingly, the purity is independent of the displacement parameter α, but depends on the displacement parameter β characterizing the odd coherent state. Figure 3 shows the dependences of $S_{\mathrm{lin}} = 1-\mathrm{Tr}(\rho_a^2)$ on β and τ. We can see that at $\tau = (2k+1)\pi/4$ $(k = 0, 1, \ldots)$, S_{lin} always reaches its maximum, which means that the maximum entanglement realizes. However, $S_{\mathrm{lin}} = 0$ at $\tau = k\pi/2$. In practice, as pointed above, the resulting states are always separable at the moment.

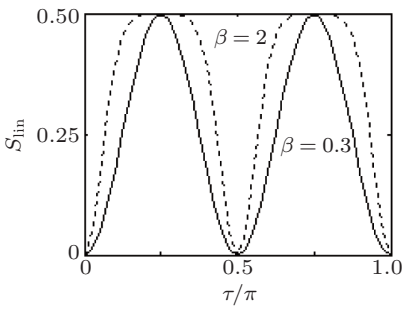


Fig. 3. Variations of linear entropy S_{lin} with propagating time τ for different values of β.

Moreover, the Wigner function is an effective measure for studying the nonclassical features of quantum state. It gives the most analogous descriptions of quantum mechanics in the phase space to classical statistical mechanics of Hamiltonian system. Based on the wavefunction $\psi(x,y)$ of a quantum state in the quadrature space, the Wigner function is expressed as[24]

$$\begin{aligned}&W(x,y,p,q)\\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\mathrm{d}u\mathrm{d}v\psi^*\left(x-\frac{u}{2},y-\frac{v}{2}\right)\\ &\quad\times\psi\left(x+\frac{u}{2},y+\frac{v}{2}\right)\exp(-\mathrm{i}pu-\mathrm{i}qv).\end{aligned} \tag{13}$$

By substituting the resulting wavefunction (8) into Eq. (13), the Wigner function becomes

$$\begin{aligned}&W(x,y,p,q)\\ &= \Omega\int_{-\infty}^{\infty}\mathrm{d}u\mathrm{d}v\exp\left[-\frac{u^2+v^2}{2}-\mathrm{i}(p+\sqrt{2}\alpha\sin\tau)u-\mathrm{i}qv\right]\\ &\quad\times\{|Q|^2\exp[\sqrt{2}\beta(2x\cos\tau-\mathrm{i}v\sin\tau)]\\ &\quad-Q^2\exp[\sqrt{2}\beta(u\cos\tau-2\mathrm{i}y\sin\tau)]\\ &\quad+|Q|^2\exp[-\sqrt{2}\beta(2x\cos\tau-\mathrm{i}v\sin\tau)]\\ &\quad-Q^{*2}\exp[-\sqrt{2}\beta(u\cos\tau-2\mathrm{i}y\sin\tau)]\},\end{aligned} \tag{14}$$

where $\Omega = N_-^2\exp(-A-x^2-y^2+2\sqrt{2}\alpha y\cos\tau)/(2\pi^3)$. Noting that the integral formula

$$\int_{-\infty}^{\infty}\exp(-ax^2+bx)\mathrm{d}x = \sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2}{4a}\right) \tag{15}$$

for $\mathrm{Re}(a) > 0$ and performing the integrations with respect to u and v in Eq. (14), we obtain

$$\begin{aligned}&W(x,y,p,q)\\ &= 4\pi^2\Omega\exp[-(p+\sqrt{2}\alpha\sin\tau)^2-q^2-2\beta^2\sin^2\tau]\\ &\quad\times\{\cosh[2\sqrt{2}\beta(x\cos\tau-q\sin\tau)]\\ &\quad-\exp(4\beta^2)\cos[2\sqrt{2}\beta(y\sin\tau+p\cos\tau)]\}.\end{aligned} \tag{16}$$

Then only when

$$\begin{aligned}&\cosh[2\sqrt{2}\beta(x\cos\tau-q\sin\tau)]\\ &-\exp(4\beta^2)\cos[2\sqrt{2}\beta(y\sin\tau+p\cos\tau)]<0,\end{aligned} \tag{17}$$

the Wigner function will be negative, which is the indicator of the nonclassicality of the quantized state. Clearly, this condition can be satisfied in four-dimensional space spanned by $(x,\ y,\ p,\ q)$ for many values. For example, for $x\cos\tau = q\sin\tau$ and $y\sin\tau + p\cos\tau$=0, condition (16) becomes $1-\exp(\beta^2) < 0$ and always holds true for $\beta \neq 0$.

Additionally, from Eqs. (12) and (15), the entanglement and nonclassicality are independent of the displacement parameter α of the input coherent state.

4. Conclusion

We investigate the vortex structure, entanglement and nonclassicality of light field resulting from a coupled waveguide where a single-mode odd Schrödinger cat state and coherent state are respectively inserted in the input ports. The analytical wavefunction of the resulting state in the quadrature space is derived and the vortex structure of output states is analysed numerically. It is found that for an odd cat state with sufficiently weak intensity, by carefully adjusting the scaled propagation time τ, the resulting states may bear a vortex with topological charge index one. Using the linear entropy method and the Wigner function, the entanglement and nonclassicality of the resulting output fields are investigated. The results reveal that the obtained states may be entangled and nonclassical for some specified values of τ and β. In particular, the vortex, the entanglement and nonclassicality are dependent mainly on the propagating time τ and β characterizing the intensity of the input odd cat state but independent of the input coherent state parameter α. Moreover, when

the odd Schrödinger cat state is replaced with an even one, there appears neither a dark centre nor a vortex. In particular, the scheme designed here is superior to those proposed in Refs. [3]–[11] because the coupled waveguide is an available device and the single-mode quantized field states used here with such properties[18–20] are also quite easy to prepare with currently available technology in a laboratory.

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