Generalized projective synchronization of two coupled complex networks of different sizes*

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We investigate a new generalized projective synchronization between two complex dynamical networks of different sizes. To the best of our knowledge, most of the current studies on projective synchronization have dealt with coupled networks of the same size. By generalized projective synchronization, we mean that the states of the nodes in each network can realize complete synchronization, and the states of a pair of nodes from both networks can achieve projective synchronization. Using the stability theory of the dynamical system, several sufficient conditions for guaranteeing the existence of the generalized projective synchronization under feedback control and adaptive control are obtained. As an example, we use Chua's circuits to demonstrate the effectiveness of our proposed approach.

Keywords: complex network, projective synchronization, adaptive control

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1. Introduction

Complex networks have attracted considerable attention from researchers in various fields recently. In fact, complex networks exist widely in our daily life, such as in ecological and food webs,[1] the World Wide Web,[2] the transmission of diseases,[3–6] neural networks,[7] and social networks.[8] Thus, the study of complex networks is highly relevant to our daily life. In general, a complex network consists of a large number of nodes and connections between them. These nodes can have different meanings in different situations, such as microprocessors, computers, and companies. Several important characteristics of complex networks include the complexity of the topological structure, dynamical evolution, node diversity, and so on.

Among various complex dynamical behaviors, synchronization is a significant and interesting phenomenon.[9–12] The synchronization in coupled chaotic systems has been studied extensively, both in terms of theory and practical applications, since Pecora and Carroll[13] used the master stability function approach to determine the stability of the synchronous state in coupled systems. So far, different types of chaos synchronization have been studied, such as complete synchronization (CS),[14–18] cluster synchronization,[19] lag synchronization (LS),[20] generalized synchronization,[21,22] projective synchronization (PS),[23–25] and anti-synchronization.[26] The synchronization observed in a network is usually called "inner synchronization" as it is a collective behavior within this network. In addition, one may also consider "outer synchronization", which refers to the synchronization between two or more complex networks.[27] However, until now, there have been few studies regarding the coexistence of outer and inner synchronization.[28] Therefore, in this paper we will investigate a new generalized projective synchronization, where the states of the nodes in each network can realize complete synchronization (i.e., inner synchronization) and the states of a pair of nodes from both networks can achieve projective synchronization (i.e., outer synchronization).

In 1999, Mainieri and Rehacek[29] observed projective synchronization in coupled partially linear chaotic systems, where the drive and response vectors synchronize up to a nonzero constant α. Recently, many important studies on the projective synchronization in dynamical networks have been made.[30] However, research dealing with coupled complex networks with different sizes is still in the initial stages. Hence, we will consider the projective synchronization of coupled complex networks with arbitrary sizes.

To realize the generalized projective synchronization between two coupled complex networks with different sizes, linear feedback control and adaptive control are utilized, respectively. This implies that pinning control[31] can be considered a special case in our proposed control method. Based on the Lyapunov stability theory and Lasalle's invariance principle for ordinary differential equations, several sufficient conditions for guaranteeing the existence of the generalized projective synchronization are obtained.

This paper is organized as follows. In Section 2, a general model of two coupled complex dynamical networks is described and some preliminaries are given. In Section 3, the

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generalized projective synchronization between two complex dynamical networks under linear feedback control is investigated, and some synchronization criteria are derived using the Lyapunov stability theorem. In Section 4, we further obtain the synchronization criteria of the two coupled complex dynamical networks under adaptive control by Lasalle's invariance principle. In Section 5, numerical simulations are performed to show the effectiveness and feasibility of the proposed synchronization control scheme, and a conclusion is presented in Section 6.

2. Preliminaries

We consider two coupled complex networks with arbitrary sizes, which can be regarded as a drive-response dynamical network. The drive network can be described as

$$\frac{\mathrm{d}\boldsymbol{x}^i(t)}{\mathrm{d}t} = f\big(\boldsymbol{x}^i(t)\big) + c\sum_{j=1}^{N_1} a_{ij}\boldsymbol{\Gamma}\boldsymbol{x}^j(t), \quad (i=1,2,\ldots,N_1), \quad (1)$$

and the response network is given by

$$\frac{\mathrm{d}\boldsymbol{y}^i(t)}{\mathrm{d}t} = f\big(\boldsymbol{y}^i(t)\big) + c\sum_{j=1}^{N_2} b_{ij}\boldsymbol{\Gamma}\boldsymbol{y}^j(t) + \boldsymbol{u}_i(t), \quad (i=1,2,\ldots,N_2), \quad (2)$$

where $\boldsymbol{u}_i(t)$ are the feedback controllers. For $t>0$, $\boldsymbol{x}^i(t) = (x_1^i, x_2^i, \ldots, x_n^i)^{\mathrm{T}} \in \boldsymbol{R}^n$ and $\boldsymbol{y}^i(t) = (y_1^i, y_2^i, \ldots, y_n^i)^{\mathrm{T}} \in \boldsymbol{R}^n$ are the state vectors of the i-th node in the driving network and the response network, respectively. The function $f: \boldsymbol{R}^n \to \boldsymbol{R}^n$ is continuous, and the constant $c>0$ denotes the coupling strength. The matrix $\boldsymbol{\Gamma} = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is an inner coupling matrix that determines the interaction of the variables. The matrix $\boldsymbol{A} = (a_{ij}) \in \boldsymbol{R}^{N_1\times N_1}$ ($\boldsymbol{A}$ is not necessarily symmetric) is an irreducible matrix with zero-sum rows, where $a_{ii} = -\sum_{j=1, i\neq j}^{N_1} a_{ij}$ and a_{ij} are defined as follows: if there is a connection from node j to node i ($i\neq j$), then $a_{ij}>0$; otherwise $a_{ij}=0$ ($i\neq j$ and $i,j=1,2,\ldots,N_1$). The matrix $\boldsymbol{B} = (b_{ij}) \in \boldsymbol{R}^{N_2\times N_2}$ (B is not necessarily symmetric) is an irreducible matrix with zero-sum rows, where $b_{ii} = -\sum_{j=1, i\neq j}^{N_2} b_{ij}$ and b_{ij} are defined as follows: if there is a connection from node j to node i ($i\neq j$), then $b_{ij}>0$; otherwise $b_{ij}=0$ ($i\neq j$ and $i,j=1,2,\ldots,N_2$). The symmetrical part of a matrix $\boldsymbol{H}$ is denoted by $\boldsymbol{H}^{\mathrm{s}} = (\boldsymbol{H}+\boldsymbol{H}^{\mathrm{T}})/2$.

Now, we introduce the following definitions and assumptions.

Definition 1 Generalized projective synchronization (GPS) between networks (1) and (2) is said to be achieved, if

$$\lim_{t\to\infty}\|\boldsymbol{x}^i(t) - \boldsymbol{x}^j(t)\| = 0, \quad (i,j=1,2,\ldots,N_1),$$
$$\lim_{t\to\infty}\|\boldsymbol{y}^i(t) - \boldsymbol{y}^j(t)\| = 0, \quad (i,j=1,2,\ldots,N_2),$$
$$\lim_{t\to\infty}\|\boldsymbol{y}^i(t) - \alpha\boldsymbol{x}^i(t)\| = 0, \quad (i=1,2,\ldots,N),$$

where $N = \min\{N_1, N_2\}$, and α is the ratio coefficient.

To achieve the goal given by Definition 1, one can intuitively understand the function of the coupling terms and controllers $\boldsymbol{u}_i$ in dynamical networks (1) and (2) in the following way. The coupling terms $c\sum_{j=1}^{N_1} a_{ij}\boldsymbol{\Gamma}\boldsymbol{x}^j(t)$ and $c\sum_{j=1}^{N_2} b_{ij}\boldsymbol{\Gamma}\boldsymbol{y}^j(t)$ are devoted to achieving the complete synchronization in each network (since the self-nonlinear dynamics of all nodes are the same, the coupled terms can vanish asymptotically), while the controllers $\boldsymbol{u}_i$ will be designed to realize projective synchronization between the networks (1) and (2).

Definition 2 The matrix $\boldsymbol{A} = (a_{ij}) \in \boldsymbol{R}^{N_1\times N_1}$ is said to belong to $\boldsymbol{M}$, if the following two conditions are satisfied: (i) $a_{ij}\geq 0$ $(i\neq j), a_{ii} = -\sum_{j=1, i\neq j}^{N_1} a_{ij}$ $(i=1,2,\ldots,N_1)$; (ii) the real parts of all eigenvalues of $\boldsymbol{A}$ are negative except an eigenvalue 0 with a multiplicity of one.

It is clear that $\boldsymbol{A}\in\boldsymbol{M}$ is irreducible. Suppose that $\xi = (\xi_1, \xi_2, \ldots, \xi_{N_1}) \in \boldsymbol{R}^{N_1}$ (without loss of generality, $\sum_{i=1}^{N_1}\xi_i = 1$) is a left eigenvector corresponding to the eigenvalue 0 of $\boldsymbol{A}$. Let $\boldsymbol{\Xi} = \mathrm{diag}\{\xi_1, \xi_2, \ldots, \xi_{N_1}\}$ and $\boldsymbol{U} = \boldsymbol{\Xi} - \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{\xi}$. Then, $\xi_i \geq 0$ holds for $i=1,2,\ldots,N_1$.[32]

Definition 3 Function class QUAD $(\boldsymbol{\Delta}, \boldsymbol{P})$:[32] let $\boldsymbol{P} = \mathrm{diag}\{p_1, p_2, \ldots, p_n\}$ be a positive definite diagonal matrix, and $\Delta = \mathrm{diag}\{\beta_1, \beta_2, \ldots, \beta_n\}$ a diagonal matrix. The QUAD $(\boldsymbol{\Delta}, \boldsymbol{P})$ denotes a class of continuous function $f(x,t): \boldsymbol{R}^n \times [0,\infty) \to \boldsymbol{R}^n$ satisfying: $(\boldsymbol{x}-\boldsymbol{y})^{\mathrm{T}}\boldsymbol{P}\big[f(x,t) - f(y,t) - \boldsymbol{\Delta}(x-y)\big] \leq -\epsilon(\boldsymbol{x}-\boldsymbol{y})^{\mathrm{T}}(\boldsymbol{x}-\boldsymbol{y})$ for $\epsilon>0$, $\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{R}^n$, and $t>0$.

Assumption 1 For all $\boldsymbol{x}_i \in \boldsymbol{R}^n$ and arbitrary constants μ_i $(i=1,2,\ldots,m)$, a positive constant L always exists, such that

$$\|\sum_{i=1}^{m}\mu_i f(\boldsymbol{x}^i)\| \leq L\|\sum_{i=1}^{m}\mu_i\boldsymbol{x}^i\|, \quad (3)$$

where $\|\cdot\|$ denotes the Euclidean vector norm, and $m\in\boldsymbol{Z}_+$.

In fact, there are many systems that can meet the condition in Assumption 1. For example, in Section 5, we will verify that the well-known Chua's circuit can satisfy the inequality condition (3).

3. GPS under linear feedback control

Theorem 1 Suppose that Assumption 1 holds, and $\boldsymbol{A}\in\boldsymbol{M}$. Let $\boldsymbol{P} = \mathrm{diag}\{p_1, p_2, \ldots, p_n\}$ be a positive definite matrix, and $\boldsymbol{\Delta} = \mathrm{diag}\{\beta_1, \beta_2, \ldots, \beta_n\}$ be a diagonal matrix, such that $f(x) \in \mathrm{QUAD}(\Delta, P)$. If there exist constants c, β_j, γ_j $(j=1,2,\ldots,n)$, and matrix $\boldsymbol{K}$, such that

$$\begin{cases} \beta_j\boldsymbol{U} + (c\gamma_j\boldsymbol{\Xi}\boldsymbol{A})^{\mathrm{s}} < 0, \\ \dfrac{p_j^2+L^2}{2}\boldsymbol{I}_{N_2} + cp_j\gamma_j\boldsymbol{B}^{\mathrm{s}} + p_j\gamma_j\boldsymbol{K} < 0, \quad (j=1,2,\ldots,n), \end{cases}$$

where $\boldsymbol{K} = \mathrm{diag}(k_1, k_2, \ldots, k_{N_2})$ and $\boldsymbol{I}_{N_2}$ denote the N_2-th order identity matrix. Then, driving network (1) and response network (2) under linear feedback control

$$\boldsymbol{u}_i(t) = \boldsymbol{\Gamma}k_i\big(\boldsymbol{y}^i(t) - \alpha\boldsymbol{s}(t)\big), \quad (i=1,2,\ldots,N_2), \quad (4)$$

with $s(t) = \sum_{i=1}^{N_1} \xi_i x^i(t)$, will achieve generalized projective synchronization.

Proof According to Ref. [32], we can define the synchronization state of the driving network as $s(t) = \sum_{i=1}^{N_1} \xi_i x^i(t)$. Then, we have the synchronization error of the response system as $\delta y^i(t) = y^i(t) - \alpha s(t)$ for $i = 1, 2, \ldots, N_2$. We construct a Lyapunov function candidate as

$$V(t) = V_1(t) + V_2(t),$$

where

$$V_1(t) = \frac{1}{2} \sum_{i=1}^{N_1} \xi_i \big(\delta x^i(t)\big)^{\mathrm{T}} P \big(\delta x^i(t)\big),$$

and

$$V_2(t) = \frac{1}{2} \sum_{i=1}^{N_2} \big(\delta y^i(t)\big)^{\mathrm{T}} P \big(\delta y^i(t)\big).$$

Since $\xi A = 0$, we have $\sum_{k=1}^{N_1} \xi_k c \sum_{l=1}^{N_1} a_{kl} \Gamma x^l(t) = 0$. Then, we obtain

$$\begin{aligned}
\frac{\mathrm{d}\delta x^i(t)}{\mathrm{d}t} &= \frac{\mathrm{d}x^i(t)}{\mathrm{d}t} - \sum_{k=1}^{N_1} \xi_k \frac{\mathrm{d}x^k(t)}{\mathrm{d}t} \\
&= f\big(x^i(t)\big) + c \sum_{j=1}^{N_1} a_{ij} \Gamma x^j(t) \\
&\quad - \sum_{k=1}^{N_1} \xi_k \Big[f\big(x^k(t)\big) + c \sum_{l=1}^{N_1} a_{kl} \Gamma x^l(t) \Big] \\
&= f\big(x^i(t)\big) - \sum_{k=1}^{N_1} \xi_k f\big(x^k(t)\big) + c \sum_{j=1}^{N_1} a_{ij} \Gamma x^j(t) \\
&= f\big(x^i(t)\big) - f\big(s(t)\big) + c \sum_{j=1}^{N_1} a_{ij} \Gamma x^j(t) + J,
\end{aligned}$$

where

$$J = f\Big(\sum_{i=1}^{N_1} \xi_i x^i(t) \Big) - \sum_{k=1}^{N_1} \xi_k f\big(x^k(t)\big).$$

Differentiating $V_1(t)$ along systems (1) and (2), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}V_1(t)}{\mathrm{d}t} \\
&= \sum_{i=1}^{N_1} \xi_i \big(\delta x^i(t)\big)^{\mathrm{T}} P \frac{\mathrm{d}\delta x^i(t)}{\mathrm{d}t} \\
&= \sum_{i=1}^{N_1} \xi_i \big(\delta x^i(t)\big)^{\mathrm{T}} P \Big(f\big(x^i(t)\big) - f\big(s(t)\big) \\
&\quad + c \sum_{j=1}^{N_1} a_{ij} \Gamma x^j(t) + J \Big) \\
&= \sum_{i=1}^{N_1} \xi_i \big(x^i(t) - s(t)\big)^{\mathrm{T}} P \Big(f\big(x^i(t)\big) - f\big(s(t)\big) \\
&\quad - \Delta\big(x^i(t) - s(t)\big) \Big) \\
&\quad + \sum_{i=1}^{N_1} \xi_i \big(x^i(t) - s(t)\big)^{\mathrm{T}} P \Big(\Delta\big(x^i(t) - s(t)\big) \\
&\quad + c \sum_{j=1}^{N_1} a_{ij} \Gamma x^j(t) + J \Big) \\
&\le \sum_{i=1}^{N_1} \xi_i (-\epsilon) \big(x^i(t) - s(t)\big)^{\mathrm{T}} \big(x^i(t) - s(t)\big) \\
&\quad + \sum_{i=1}^{N_1} \xi_i \big(x^i(t) - s(t)\big)^{\mathrm{T}} P \Big(\Delta\big(x^i(t) - s(t)\big) \\
&\quad + c \sum_{j=1}^{N_1} a_{ij} \Gamma x^j(t) \Big) \\
&= \sum_{i=1}^{N_1} \xi_i (-\epsilon) \big(x^i(t) - s(t)\big)^{\mathrm{T}} \big(x^i(t) - s(t)\big) \\
&\quad + \sum_{i=1}^{N_1} \xi_i \Big(\big(x^i(t)\big)^{\mathrm{T}} P \Delta x^i(t) - \big(x^i(t)\big)^{\mathrm{T}} P \Delta s(t) \\
&\quad - s(t)^{\mathrm{T}} P \Delta x^i(t) + s(t)^{\mathrm{T}} P \Delta s(t) + c \big(x^i(t)\big)^{\mathrm{T}} \\
&\quad \times P \sum_{j=1}^{N_1} a_{ij} \Gamma x^j(t) + c \big(s(t)\big)^{\mathrm{T}} P \sum_{j=1}^{N_1} a_{ij} \Gamma x^j(t) \Big) \\
&\le \frac{2(-\epsilon) V_1}{\max(p_i)} + X^{\mathrm{T}} \Big((\Xi - \xi^{\mathrm{T}} \xi) \otimes P\Delta \Big) X \\
&\quad + c X^{\mathrm{T}} (\Xi A \otimes P\Gamma) X \\
&= \frac{2(-\epsilon) V_1}{\max(p_i)} + \sum_{j=1}^{n} p_j \tilde{x}_j(t)^{\mathrm{T}} (\beta_j U + c\gamma_j \Xi A^{\mathrm{s}}) \tilde{x}_j(t),
\end{aligned}$$

where $\tilde{x}_j(t) = \big(x_j^1(t), x_j^2(t), \ldots, x_j^{N_1}(t)\big)^{\mathrm{T}}$, and $\otimes$ denotes the Kronecter product of two matrices.

With Assumption 1, computing the derivative of $V_2(t)$ along systems (1) and (2), we have

$$\begin{aligned}
&\frac{\mathrm{d}V_2(t)}{\mathrm{d}t} \\
&= \sum_{i=1}^{N_2} \big(\delta y^i(t)\big)^{\mathrm{T}} P \frac{\mathrm{d}\delta y^i(t)}{\mathrm{d}t} \\
&= \sum_{i=1}^{N_2} \big(y^i(t) - \alpha s(t)\big)^{\mathrm{T}} P \Big(f(y^i) - \alpha \frac{\mathrm{d}s}{\mathrm{d}t} \Big) \\
&\quad + \sum_{i=1}^{N_2} \big(y^i(t) - \alpha s(t)\big)^{\mathrm{T}} P \Big(c \sum_{j=1}^{N_2} b_{ij} \Gamma y^j(t) \\
&\quad + k_i \Gamma \big(y^i(t) - \alpha s(t)\big) \Big) \\
&\le \sum_{i=1}^{N_2} \frac{1}{2} \big(y^i(t) - \alpha s(t)\big)^{\mathrm{T}} P^2 \big(y^i(t) - \alpha s(t)\big) \\
&\quad + \sum_{i=1}^{N_2} \frac{1}{2} \Big(f\big(y^i(t)\big) - \alpha \frac{\mathrm{d}s(t)}{\mathrm{d}t} \Big)^{\mathrm{T}} \Big(f\big(y^i(t)\big) - \alpha \frac{\mathrm{d}s(t)}{\mathrm{d}t} \Big) \\
&\quad + \sum_{i=1}^{N_2} \big(y^i(t) - \alpha s(t)\big)^{\mathrm{T}} P \Big(c \sum_{j=1}^{N_2} b_{ij} \Gamma \big(y^j(t) - \alpha s(t)\big) \\
&\quad + k_i \Gamma \big(y^i(t) - \alpha s(t)\big) \Big) \\
&\le \sum_{i=1}^{N_2} \frac{1}{2} \big(\delta y^i(t)\big)^{\mathrm{T}} P^2 \big(\delta y^i(t)\big) \\
&\quad + \sum_{i=1}^{N_2} \frac{1}{2} \Big(f\big(y^i(t)\big) - \alpha \sum_{i=1}^{N_1} \xi_i f\big(x^i(t)\big) \Big)^{\mathrm{T}}
\end{aligned}$$

$$
\begin{aligned}
&\times\Big(f(\boldsymbol{y}^i(t))-\alpha\sum_{i=1}^{N_1}\xi_i f(\boldsymbol{x}^i(t))\Big)\\
&+(\boldsymbol{Y}-\alpha\boldsymbol{S})^{\mathrm{T}}(c\boldsymbol{B}\otimes\boldsymbol{P\Gamma}+K\otimes\boldsymbol{P\Gamma})(\boldsymbol{Y}-\alpha\boldsymbol{S})\\
\le&\sum_{i=1}^{N_2}\frac{1}{2}\big(\delta\boldsymbol{y}^i(t)\big)^{\mathrm{T}}\boldsymbol{P}^2\big(\delta\boldsymbol{y}^i(t)\big)\\
&+\sum_{i=1}^{N_2}\frac{L^2}{2}\Big(\boldsymbol{y}^i(t)-\alpha\sum_{i=1}^{N_1}\xi_i\boldsymbol{x}^i(t)\Big)^{\mathrm{T}}\Big(\boldsymbol{y}^i(t)-\alpha\sum_{i=1}^{N_1}\xi_i\boldsymbol{x}^i(t)\Big)\\
&+(\boldsymbol{Y}-\alpha\boldsymbol{S})^{\mathrm{T}}(c\boldsymbol{B}\otimes\boldsymbol{P\Gamma}+\boldsymbol{K}\otimes\boldsymbol{P\Gamma})(\boldsymbol{Y}-\alpha\boldsymbol{S})\\
\le&\sum_{j=1}^{n}(\delta\tilde{\boldsymbol{y}}_j(t))^{\mathrm{T}}\Big(\frac{\boldsymbol{p}_j^2+\boldsymbol{L}^2}{2}\Big)\boldsymbol{I}_{N_2}(\delta\tilde{\boldsymbol{y}}_j(t))\\
&+(\delta\tilde{\boldsymbol{y}}_j(t))^{\mathrm{T}}(cp_j\gamma_j\boldsymbol{B}^{\mathrm{s}}+p_j\gamma_j\boldsymbol{K})(\delta\tilde{\boldsymbol{y}}_j(t)),
\end{aligned}
$$

where $\boldsymbol{Y}-\alpha\boldsymbol{S}=\big(\boldsymbol{y}^1(t)-\alpha\boldsymbol{s}(t),\ldots,\boldsymbol{y}^{N_2}(t)-\alpha\boldsymbol{s}(t)\big)^{\mathrm{T}}$, and $\delta\tilde{\boldsymbol{y}}_j(t)=(\delta\boldsymbol{y}_j^1(t),\delta\boldsymbol{y}_j^2(t),\ldots,\delta\boldsymbol{y}_j^{N_2}(t))^{\mathrm{T}}$. Using the conditions in this theorem, we can obtain constants c, β_j, γ_j, and matrix $\boldsymbol{K}$ such that

$$
\begin{cases}
\beta_j\boldsymbol{U}+(c\gamma_j\boldsymbol{\Xi}\boldsymbol{A})^{\mathrm{s}}<0,\\
\Big(\frac{{p_j}^2}{2}+\frac{L^2}{2}\Big)\boldsymbol{I}_{N_2}+cp_j\gamma_j\boldsymbol{B}^{\mathrm{s}}+p_j\gamma_j\boldsymbol{K}<0,
\end{cases}
$$

where $j=1,2,\ldots,n$. Then, we obtain

$$
\frac{\mathrm{d}V(t)}{\mathrm{d}t}=\frac{\mathrm{d}V_1(t)}{\mathrm{d}t}+\frac{\mathrm{d}V_2(t)}{\mathrm{d}t}<-\frac{V_1\epsilon}{\max(p_i)}\le 0.
$$

Therefore, the inner complete synchronization of driving network (1), and the outer projective synchronization between driving network (1) and response network (2) are achieved simultaneously. Thus, we complete the proof. If we only discuss the outer synchronization in Theorem 1, without loss of generality, we assume $N_1<N_2$, and derive the following corollary.

Corollary 1 Suppose that Assumption 1 holds, $\boldsymbol{A}\in\boldsymbol{M}$. Let $\boldsymbol{P}=\mathrm{diag}\{p_1,p_2,\ldots,p_n\}$ be a positive definite matrix. If there exist constants γ_j $(j=1,2,\ldots,n)$, matrix $\boldsymbol{K}=\mathrm{diag}\{k_1,k_2,\ldots,k_{N_1}\}$, and N_1-th order identity matrix $\boldsymbol{I}_{N_1}$ such that

$$
\Big(\frac{\boldsymbol{p}_j^2}{2}+\frac{\boldsymbol{L}^2}{2}\Big)\boldsymbol{I}_{N_1}+p_j\gamma_j\boldsymbol{K}<0,\ \ (j=1,2,\ldots,n),
$$

then the two networks (1) and (2) can realize projective synchronization under the following feedback control:

$$
\boldsymbol{u}_i(t)=\begin{cases}
\boldsymbol{\Gamma}k_i\big(\boldsymbol{y}^i(t)-\alpha\boldsymbol{x}^i(t)\big)+\alpha c\sum\limits_{j=1}^{N_1}a_{ij}\boldsymbol{\Gamma}\boldsymbol{x}^j(t)\\
-c\sum\limits_{j=1}^{N_2}b_{ij}\boldsymbol{\Gamma}\boldsymbol{y}^j(t),\ \ (i=1,2,\ldots,N_1),\\
\boldsymbol{\Gamma}k_i\boldsymbol{y}^i(t)+\alpha c\sum\limits_{j=1}^{N_1}a_{ij}\boldsymbol{\Gamma}\boldsymbol{x}^j(t)\\
-c\sum\limits_{j=1}^{N_2}b_{ij}\boldsymbol{\Gamma}\boldsymbol{y}^j(t),\ \ (i=N_1+1,N_1+2,\ldots,N_2),
\end{cases}
$$

where the gain k_i is a constant, and α is the ratio coefficient.

Proof We let the synchronization error of GPS be $\boldsymbol{e}^i(t)=\boldsymbol{y}^i(t)-\alpha\boldsymbol{x}^i(t)$, for $i=1,2,\ldots,N_1$. We construct a Lyapunov function candidate as

$$
V(t)=\frac{1}{2}\sum_{i=1}^{N_1}\big(\boldsymbol{e}^i(t)\big)^{\mathrm{T}}\boldsymbol{P}\big(\boldsymbol{e}^i(t)\big).
$$

Then, with the above control, the derivative of $V(t)$ along systems (1) and (2) is given by

$$
\begin{aligned}
&\frac{\mathrm{d}V(t)}{\mathrm{d}t}\\
=&\sum_{i=1}^{N_1}\big(\boldsymbol{e}^i(t)\big)^{\mathrm{T}}\boldsymbol{P}\Big(f(\boldsymbol{y}^i(t))-\alpha f(\boldsymbol{x}^i(t))\\
&+c\sum_{j=1}^{N_2}b_{ij}\boldsymbol{\Gamma}\boldsymbol{y}^j(t)-\alpha c\sum_{j=1}^{N_1}a_{ij}\boldsymbol{\Gamma}\boldsymbol{x}^j(t)+\boldsymbol{u}_i(t)\Big)\\
=&\sum_{i=1}^{N_1}\big(\boldsymbol{e}^i(t)\big)^{\mathrm{T}}\boldsymbol{P}\Big(f\big(\boldsymbol{y}^i(t)\big)-\alpha f\big(\boldsymbol{x}^i(t)\big)\Big)\\
&+\sum_{i=1}^{N_1}\big(\boldsymbol{e}^i(t)\big)^{\mathrm{T}}\boldsymbol{P}\Big(k_i\boldsymbol{\Gamma}\boldsymbol{e}^i(t)\Big)\\
\le&\sum_{i=1}^{N_1}\frac{1}{2}\big(\boldsymbol{e}^i(t)\big)^{\mathrm{T}}\boldsymbol{P}^2\big(\boldsymbol{e}^i(t)\big)\\
&+\sum_{i=1}^{N_1}\frac{1}{2}\Big(f\big(\boldsymbol{y}^i(t)\big)-\alpha f\big(\boldsymbol{x}^i(t)\big)\Big)^{\mathrm{T}}\Big(f\big(\boldsymbol{y}^i(t)\big)-\alpha f\big(\boldsymbol{x}^i(t)\big)\Big)\\
&+\sum_{i=1}^{N_1}\big(\boldsymbol{e}^i(t)\big)^{\mathrm{T}}\boldsymbol{P}(k_i\boldsymbol{\Gamma}\boldsymbol{e}^i(t))\\
\le&\sum_{i=1}^{N_1}\frac{1}{2}\big(\boldsymbol{e}^i(t)\big)^{\mathrm{T}}\boldsymbol{P}^2\big(\boldsymbol{e}^i(t)\big)+\sum_{i=1}^{N_1}\frac{L^2}{2}\big(\boldsymbol{e}^i(t)\big)^{\mathrm{T}}\big(\boldsymbol{e}^i(t)\big)\\
&+\sum_{i=1}^{N_1}\big(\boldsymbol{e}^i(t)\big)^{\mathrm{T}}\boldsymbol{P}\Big(k_i\boldsymbol{\Gamma}\boldsymbol{e}^i(t)\Big)\\
=&\boldsymbol{E}^{\mathrm{T}}\Big(\boldsymbol{I}_{N_1}\otimes\Big(\frac{\boldsymbol{P}^2}{2}+\frac{L^2}{2}\boldsymbol{I}_n\Big)\Big)\boldsymbol{E}+\boldsymbol{E}^{\mathrm{T}}(K\otimes\boldsymbol{P\Gamma})\boldsymbol{E}\\
=&\sum_{j=1}^{n}\tilde{\boldsymbol{e}}_j(t)^{\mathrm{T}}\Big[\Big(\frac{\boldsymbol{p}_j^2}{2}+\frac{L^2}{2}\Big)\boldsymbol{I}_{N_1}+p_j\gamma_j\boldsymbol{K}\Big]\tilde{\boldsymbol{e}}_j(t).
\end{aligned}
$$

Using the conditions in this theorem, we can obtain constants γ_j, matrix $\boldsymbol{K}$, and N_1-th order identity matrix $\boldsymbol{I}_{N_1}$ such that

$$
\Big(\frac{p_j^2}{2}+\frac{L^2}{2}\Big)\boldsymbol{I}_{N_1}+p_j\gamma_j\boldsymbol{K}<0,\ \ (j=1,2,\ldots,n).
$$

Then, we obtain $(\mathrm{d}V(t))/(\mathrm{d}t)<0$. Therefore, driving network (1) and response network (2) can realize projective synchronization. The proof is completed.

Remark 1 In Corollary 1, if $N_1\ge N_2$, we can take the control

$$
\boldsymbol{u}_i(t)=\begin{cases}
\boldsymbol{\Gamma}k_i\big(\boldsymbol{y}^i(t)-\alpha\boldsymbol{x}^i(t)\big)+\alpha c\sum\limits_{j=1}^{N_1}a_{ij}\boldsymbol{\Gamma}\boldsymbol{x}^j(t)\\
-c\sum\limits_{j=1}^{N_2}b_{ij}\boldsymbol{\Gamma}\boldsymbol{y}^j(t),\ \ (i=1,2,\ldots,N_2),\\
-\boldsymbol{\Gamma}k_i\alpha\boldsymbol{x}^i(t)+\alpha c\sum\limits_{j=1}^{N_1}a_{ij}\boldsymbol{\Gamma}\boldsymbol{x}^j(t)\\
-c\sum\limits_{j=1}^{N_2}b_{ij}\boldsymbol{\Gamma}\boldsymbol{y}^j(t),\ \ (i=N_2+1,N_2+2,\ldots,N_1).
\end{cases}
$$

Then, we can still draw similar conclusions using the above method.

Remark 2 If we take the synchronization state of the driving network as $f(s) = \mathrm{d}s/\mathrm{d}t$, then we can obtain similar results as in Theorem 1, and the conditions can be obtained in reality.

Remark 3 If $N_2 \ll N_1$, then our control method can be considered as a pinning control.[31]

4. GPS under adaptive control

The gains k_i of the controller $\boldsymbol{u}_i(t)$ in Eq. (4) derived above require that the system parameters must be known beforehand. However, in many real applications, it may be difficult to determine exactly the values of the system's parameters. Even if we know them exactly, the derived gains k_i may be so large that it is of no significance in some real applications. To overcome these drawbacks, adaptive gains k_i should be adapted. In this case, the controllers $\boldsymbol{u}_i$ are chosen as

$$\boldsymbol{u}_i(t) = k_i(t)\boldsymbol{\Gamma}(\boldsymbol{y}^i(t) - \alpha\boldsymbol{s}(t)), \quad (i = 1,2,\ldots,N_2), \tag{5}$$

where the time-varying gains $k_i(t)$ are updated according to the following adaptive algorithm:

$$\dot{k}_i(t) = q_i(\boldsymbol{y}^i(t) - \alpha\boldsymbol{s}(t))^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\Gamma}(\boldsymbol{y}^i(t) - \alpha\boldsymbol{s}(t)), \quad (i = 1,2,\ldots,N_2), \tag{6}$$

where $q_i > 0$ and $k_i(0)$ are arbitrary constants. Then, we can have the following results.

Theorem 2 Suppose that Assumption 1 holds and $\boldsymbol{A} \in \boldsymbol{M}$. Let $\boldsymbol{P} = \mathrm{diag}\{p_1, p_2, \ldots, p_n\}$ be a positive definite matrix, and $\boldsymbol{\Delta} = \mathrm{diag}\{\beta_1, \beta_2, \ldots, \beta_n\}$ be a diagonal matrix, such that $f(x) \in \mathrm{QUAD}(\boldsymbol{\Delta}, \boldsymbol{P})$. Under the adaptive controllers (5) and the time-varying gains (6), two coupled complex networks (1) and (2) will achieve the generalized projective synchronization.

Proof We construct a Lyapunov function candidate as

$$V(t) = V_1(t) + V_3(t),$$

$$V_1(t) = \frac{1}{2}\sum_{i=1}^{N_1}\xi_i\big(\delta\boldsymbol{x}^i(t)\big)^{\mathrm{T}}\boldsymbol{P}\big(\delta\boldsymbol{x}^i(t)\big),$$

$$V_3(t) = \frac{1}{2}\sum_{i=1}^{N_2}\big(\delta\boldsymbol{y}^i(t)\big)^{\mathrm{T}}\boldsymbol{P}\big(\delta\boldsymbol{y}^i(t)\big) + \frac{1}{2q_i}\sum_{i=1}^{N_2}\big(k_i^* - k_i(t)\big)^2,$$

where k_i^* is a constant. Differentiating $V_1(t)$ along the system (1), according to the Proof in Theorem 1, we have

$$\dot{V}_1(t) \le \frac{(-\epsilon)V_1}{\max(p_i)} + \sum_{j=1}^{n} p_j\tilde{\boldsymbol{x}}_j(t)^{\mathrm{T}}(\beta_j\boldsymbol{U} + c\gamma_j\boldsymbol{\Xi}\boldsymbol{A}^{\mathrm{s}})\tilde{\boldsymbol{x}}_j(t).$$

Evaluating the time derivative of $V_3(t)$ along systems (1) and (2), we have

$$\begin{aligned}
&\frac{\mathrm{d}V_3(t)}{\mathrm{d}t}\\
&= \sum_{i=1}^{N_2}\big(\delta\boldsymbol{y}^i(t)\big)^{\mathrm{T}}\boldsymbol{P}\frac{\mathrm{d}\delta\boldsymbol{y}^i(t)}{\mathrm{d}t}\\
&\quad + \sum_{i=1}^{N_2}(k_i^* - k_i(t))\big(\delta\boldsymbol{y}^i(t)\big)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\Gamma}(\delta\boldsymbol{y}^i(t))\\
&= \sum_{i=1}^{N_2}\big(\boldsymbol{y}^i(t) - \alpha\boldsymbol{s}(t)\big)^{\mathrm{T}}\boldsymbol{P}\Big(f(\boldsymbol{y}^i) - \alpha\frac{\mathrm{d}s}{\mathrm{d}t}\Big)\\
&\quad + \sum_{i=1}^{N_2}\big(\boldsymbol{y}^i(t) - \alpha\boldsymbol{s}(t)\big)^{\mathrm{T}}\boldsymbol{P}\Big(c\sum_{j=1}^{N_2}b_{ij}\boldsymbol{\Gamma}\boldsymbol{y}^j(t)\Big)\\
&\quad + \sum_{i=1}^{N_2}k_i^*\big(\delta\boldsymbol{y}^i(t)\big)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\Gamma}(\delta\boldsymbol{y}^i(t))\\
&\le \sum_{i=1}^{N_2}\frac{1}{2}\big(\boldsymbol{y}^i(t) - \alpha\boldsymbol{s}(t)\big)^{\mathrm{T}}\boldsymbol{P}^2\big(\boldsymbol{y}^i(t) - \alpha\boldsymbol{s}(t)\big)\\
&\quad + \sum_{i=1}^{N_2}\frac{1}{2}\Big(f\big(\boldsymbol{y}^i(t)\big) - \alpha\frac{\mathrm{d}\boldsymbol{s}(t)}{\mathrm{d}t}\Big)^{\mathrm{T}}\Big(f\big(\boldsymbol{y}^i(t)\big) - \alpha\frac{\mathrm{d}\boldsymbol{s}(t)}{\mathrm{d}t}\Big)\\
&\quad + \sum_{i=1}^{N_2}\big(\boldsymbol{y}^i(t) - \alpha\boldsymbol{s}(t)\big)^{\mathrm{T}}\boldsymbol{P}\Big(c\sum_{j=1}^{N_2}b_{ij}\boldsymbol{\Gamma}\big(\boldsymbol{y}^j(t) - \alpha\boldsymbol{s}(t)\big)\Big)\\
&\quad + \sum_{i=1}^{N_2}k_i^*\big(\delta\boldsymbol{y}^i(t)\big)^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\Gamma}(\delta\boldsymbol{y}^i(t))\\
&\le \sum_{i=1}^{N_2}\frac{1}{2}\big(\delta\boldsymbol{y}^i(t)\big)^{\mathrm{T}}\boldsymbol{P}^2\big(\delta\boldsymbol{y}^i(t)\big)\\
&\quad + \sum_{i=1}^{N_2}\frac{1}{2}\Big(f\big(\boldsymbol{y}^i(t)\big) - \alpha\sum_{i=1}^{N_1}\xi_i f\big(\boldsymbol{x}^i(t)\big)\Big)^{\mathrm{T}}\\
&\quad \times\Big(f\big(\boldsymbol{y}^i(t)\big) - \alpha\sum_{i=1}^{N_1}\xi_i f\big(\boldsymbol{x}^i(t)\big)\Big)\\
&\quad + (\boldsymbol{Y} - \alpha\boldsymbol{S})^{\mathrm{T}}(c\boldsymbol{B}\otimes\boldsymbol{P}\boldsymbol{\Gamma} + K^*\otimes\boldsymbol{P}\boldsymbol{\Gamma})(\boldsymbol{Y} - \alpha\boldsymbol{S})\\
&\le \sum_{j=1}^{n}(\delta\tilde{\boldsymbol{y}}_j(t))^{\mathrm{T}}\Big(\frac{p_j^2 + L^2}{2}\Big)\boldsymbol{I}_{N_2}(\delta\tilde{\boldsymbol{y}}_j(t))\\
&\quad + (\delta\tilde{\boldsymbol{y}}_j(t))^{\mathrm{T}}(cp_j\gamma_j\boldsymbol{B}^{\mathrm{s}} + p_j\gamma_j\boldsymbol{K}^*)(\delta\tilde{\boldsymbol{y}}_j(t)),
\end{aligned}$$

where $\boldsymbol{Y} - \alpha\boldsymbol{S} = \big(\boldsymbol{y}^1(t) - \alpha\boldsymbol{s}(t), \boldsymbol{y}^2(t) - \alpha\boldsymbol{s}(t), \ldots, \boldsymbol{y}^{N_2}(t) - \alpha\boldsymbol{s}(t)\big)^{\mathrm{T}}$, $\delta\tilde{\boldsymbol{y}}_j(t) = (\delta\boldsymbol{y}_j^1(t), \cdots, \delta\boldsymbol{y}_j^{N_2}(t))^{\mathrm{T}}$, and $\boldsymbol{K}^* = \mathrm{diag}\{k_1^*, k_2^*, \cdots, k_{N_2}^*\}$. Therefore,

$$\begin{aligned}
\dot{V}(t) &= \frac{\mathrm{d}V_1(t)}{\mathrm{d}t} + \frac{\mathrm{d}V_3(t)}{\mathrm{d}t}\\
&\le \frac{(-\epsilon)V_1}{\max(p_i)} + \sum_{j=1}^{n}p_j\tilde{\boldsymbol{x}}_j(t)^{\mathrm{T}}(\beta_j\boldsymbol{U} + c\gamma_j\boldsymbol{\Xi}\boldsymbol{A}^{\mathrm{s}})\tilde{\boldsymbol{x}}_j(t)\\
&\quad + \sum_{j=1}^{n}(\delta\tilde{\boldsymbol{y}}_j(t))^{\mathrm{T}}\Big(\frac{p_j^2 + L^2}{2}\Big)\boldsymbol{I}_{N_2}(\delta\tilde{\boldsymbol{y}}_j(t))\\
&\quad + (\delta\tilde{\boldsymbol{y}}_j(t))^{\mathrm{T}}(cp_j\gamma_j\boldsymbol{B}^{\mathrm{s}} + p_j\gamma_j\boldsymbol{K}^*)(\delta\tilde{\boldsymbol{y}}_j(t)).
\end{aligned}$$

Then, we can choose suitable c, β_j, γ_j, $\boldsymbol{K}$, and $\boldsymbol{K}^*$ $(j = 1,2,\ldots,n)$ such that

$$\begin{cases}
\beta_j\boldsymbol{U} + (c\gamma_j\boldsymbol{\Xi}A)^{\mathrm{s}} < 0,\\
\dfrac{p_j^2 + L^2}{2}\boldsymbol{I}_{N_2} + cp_j\gamma_j\boldsymbol{B}^{\mathrm{s}} + p_j\gamma_jK^* < 0, \quad (j = 1,2,\ldots,n),
\end{cases}$$

Thus, we have

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} < \frac{(-\epsilon)V_1}{\max(p_i)} \le 0.$$

It is obvious that the largest invariant set contained in set

$\boldsymbol{E} = \{\mathrm{d}V(t)/\mathrm{d}t = 0\} = \{\delta x^i = 0, \delta y^i = 0\}$ is $\boldsymbol{M} = \{\delta x^i = 0, \delta y^i = 0, k_i(t) = k_i^*, i = 1,2,\ldots,N_2\}$. According to Lasalle's invariance principle, starting with the arbitrary initial values given by Eqs. (1), (2), (5), and (6), the trajectory asymptotically converges to the set $\boldsymbol{M}$, i.e., as $t \to \infty, \boldsymbol{x}^i(t) \to \boldsymbol{s}(t), \boldsymbol{y}^i(t) - \alpha \boldsymbol{x}^i(t) \to 0$, and $k_i(t) \to k_i^*$. This indicates that the inner complete synchronization of the driving network (1) and the outer projective synchronization between the driving network (1) and the response network (2) with the adaptive controllers (5) and the time-varying gains (6) are realized globally. Therefore, the GPS is realized, and we complete the proof.

5. Numerical simulations

In this section, numerical simulations are performed to verify the theorems obtained above. The local dynamics of the nodes is controlled by the chaotic Chua's circuit[28] which is described by

$$\begin{cases} \dot{x_1} = p\big[x_2 - x_1 - g(x_1)\big], \\ \dot{x_2} = x_1 - x_2 + x_3, \\ \dot{x_3} = -q x_2, \end{cases}$$

where $g(x) = m_0 x + (m_1 - m_0)(|x+1| - |x-1|)/2$, $p = 9$, $q = 100/7$, $m_0 = -5/7$, and $m_1 = -8/7$. Then, we have

$$\begin{aligned} & \|\sum_{i=1}^{n} \mu_i f(x^i)\|^2 \\ = & \Big(\sum_{i=1}^{n} \mu_i p\big(x_2^i - x_1^i - g(x_1^i)\big)\Big)^2 \\ & + \Big(\sum_{i=1}^{n} \mu_i (x_1^i - x_2^i + x_3^i)\Big)^2 + \Big(\sum_{i=1}^{n} -q\mu_i x_2^i\Big)^2 \\ \le & (3p^2+3)\Big(\sum_{i=1}^{n} \mu_i x_1^i\Big)^2 + (3p^2+3+q^2)\Big(\sum_{i=1}^{n} \mu_i x_2^i\Big)^2 \\ & + 3p^2\Big(\sum_{i=1}^{n} \mu_i g\big(x_1^i\big)\Big)^2 + 3\Big(\sum_{i=1}^{n} \mu_i x_3^i\Big)^2 \\ \le & \big(3p^2+3+6p^2m_0^2+6p^2(m_1-m_0)^2\big)\Big(\sum_{i=1}^{n} \mu_i x_1^i\Big)^2 \\ & + (3p^2+3+q^2)\Big(\sum_{i=1}^{n} \mu_i x_2^i\Big)^2 + 3\Big(\sum_{i=1}^{n} \mu_i x_3^i\Big)^2. \end{aligned}$$

If $\boldsymbol{L}^2 \ge \max\{3p^2+3+6p^2m_0^2+6p^2(m_1-m_0)^2, 3p^2+3+q^2, 3\}$, then Assumption 1 holds. Thus, we can select $L = 25$.

We consider two coupled complex dynamical networks, both with a small-world topological structure,[5] which can be given by

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{x}^i(t)}{\mathrm{d}t} = f(\boldsymbol{x}^i(t)) + c\sum\limits_{j=1}^{N_1} a_{ij}\boldsymbol{\Gamma}\boldsymbol{x}^j(t), \quad (i = 1,2,\ldots,N_1), \\ \dfrac{\mathrm{d}\boldsymbol{y}^i(t)}{\mathrm{d}t} = f(\boldsymbol{y}^i(t)) + c\sum\limits_{j=1}^{N_2} b_{ij}\boldsymbol{\Gamma}\boldsymbol{y}^j(t) + k_i\boldsymbol{\Gamma}\big(\boldsymbol{y}^i(t) - \alpha\boldsymbol{s}(t)\big), \\ \qquad\qquad (i = 1,2,\ldots,N_2), \end{cases} \tag{7}$$

where $\boldsymbol{\Gamma} = \boldsymbol{I}_3$ and $\boldsymbol{s}(t) = \sum\limits_{i=1}^{N_1} \xi_i \boldsymbol{x}^i(t)$. For simplicity, we consider the two networks with $N_1 = 30$, $N_2 = 20$, and $\alpha = 2$. It is clear that $\boldsymbol{A}$ belongs to $\boldsymbol{M}$. If we assume $\boldsymbol{P} = 100\boldsymbol{I}_3$ and $\boldsymbol{\Delta} = 3\boldsymbol{I}_3$, then Assumption 1 holds. If we choose $c = 0.4$ and $\boldsymbol{K} = -4\boldsymbol{I}_{N_2}$, then the eigenvalues of $\boldsymbol{\Delta}_j\boldsymbol{U} + (c\gamma_j\boldsymbol{\Xi}\boldsymbol{A})^{\mathrm{s}}$ and $[(p_j^2+L^2)/2]\boldsymbol{I}_{N_2} + cp_j\gamma_j\boldsymbol{B}^{\mathrm{s}} + p_j\gamma_j\boldsymbol{K}$ are all negative. In this case, the conditions of Theorem 1 are satisfied, and the two coupled networks (7) can achieve projective synchronization. The simulation results are plotted in Fig. 1. Figures 1(a)–1(c) show the states of the variables, figure 1(d) shows the inner synchronization error

$$E_1(t) = \sum_{i=1}^{N_1}\sum_{j=1}^{N_1} \frac{1}{2}\|(\boldsymbol{x}^i(t) - \boldsymbol{x}^j(t))\|$$

of the driving network, and figure 1(e) shows the outer projective synchronization error

$$E_2(t) = \sum_{i=1}^{N} \|(\boldsymbol{y}^i(t) - \alpha\boldsymbol{x}^i(t))\| (N = \min\{N_1, N_2\})$$

between the driving network and the response network. Obviously, the GPS is achieved.

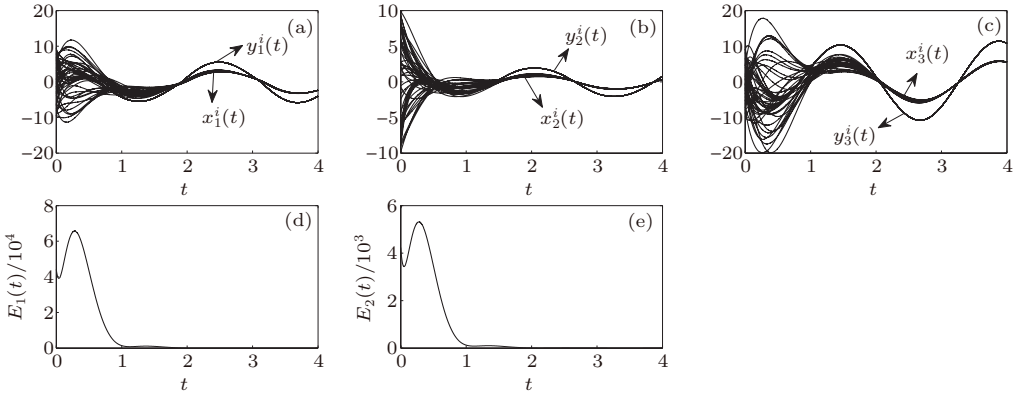


Fig. 1. The waveforms of variables (a) $x_1^i(t)$, $y_1^i(t)$, (b) $x_2^i(t)$, $y_2^i(t)$, and (c) $x_3^i(t)$, $y_3^i(t)$. (d) The inner synchronization error $E_1(t)$ of driving network (1). (e) The outer projective synchronization error $E_2(t)$ between driving network (1) and response network (2) with linear feedback controllers (4).

In order to illustrate how to apply Theorem 2 to judge whether two coupled networks can achieve the GPS with adaptive controls (5) and (6), we select suitable parameters as $\boldsymbol{P} = \boldsymbol{\Gamma} = I_3$, $c = 0.4$, $(k_1(0), k_2(0), \ldots, k_{N_2}(0)) = (-9, -9, \ldots, -9)$, and $q_i = 0.05$. The simulation results are plotted in Fig. 2. Figures 2(a)–2(c) show the states of the variables, while figure 2(d) shows the inner synchronization error $E_1(t)$ of the driving network, and figure 2(e) shows the outer projective synchronization error $E_2(t)$ between the driving network and the response network. Figure 2(f) illustrates the evolution of the time-varying gains $k_i(t)$. It is clear that two coupled complex networks with adaptive controllers can realize the GPS.

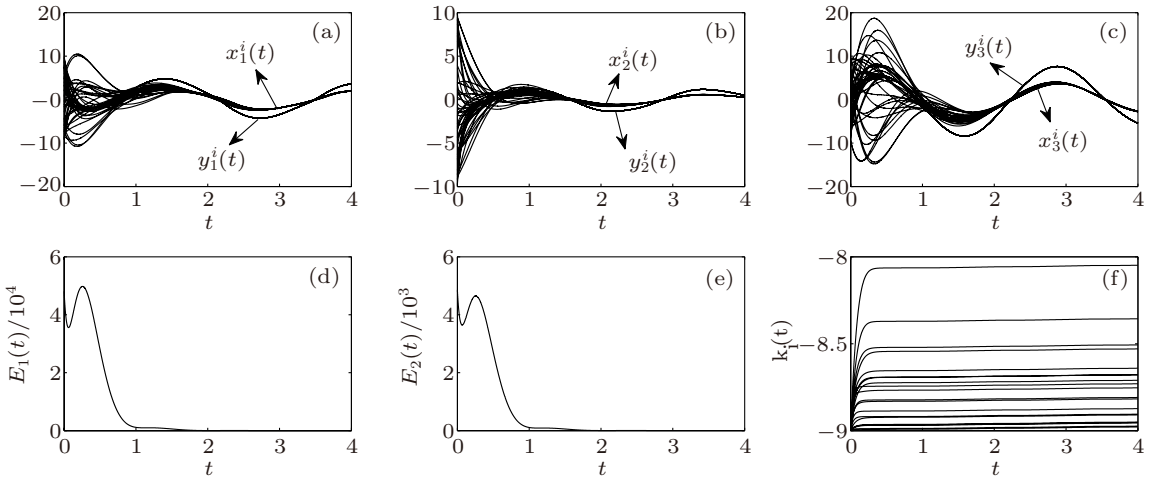


Fig. 2. The waveforms of variables (a) $x_1^i(t)$, $y_1^i(t)$, (b) $x_2^i(t)$, $y_2^i(t)$, and (c) $x_3^i(t)$, $y_3^i(t)$. (d) The inner synchronization error $E_1(t)$ of driving network (1). (e) The outer projective synchronization error $E_2(t)$ between driving network (1) and response network (2) with linear feedback controllers (5) and (6). (f) The evolution of the time-varying gains $k_i(t)$.

6. Conclusions and remarks

In this paper, we investigated a new generalized projective synchronization phenomenon in a system of two coupled complex networks with different sizes via linear feedback control and the adaptive control method. Using stability analysis, we obtained some of the criteria of the generalized projective synchronization of the two coupled networks. Analytical results show that under suitable conditions, the two complex networks can realize generalized projective synchronization with different sizes, i.e., the outer projective synchronization between the driving network and the response network is achieved, and the inner complete synchronization in the driving network and the response network is also realized. Numerical simulations demonstrate the effectiveness of the proposed generalized projective synchronization scheme.

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