Phase synchronization and synchronization frequency of two-coupled van der Pol oscillators with delayed coupling

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In this paper, phase synchronization and the frequency of two synchronized van der Pol oscillators with delay coupling are studied. The dynamics of such a system are obtained using the describing function method, and the necessary conditions for phase synchronization are also achieved. Finding the vicinity of the synchronization frequency is the major advantage of the describing function method over other traditional methods. The equations obtained based on this method justify the phenomenon of the synchronization of coupled oscillators on a frequency either higher, between, or lower than the highest, in between, or lowest natural frequency of the aggregate oscillators. Several numerical examples simulate the different cases versus the various synchronization frequency delays.

Keywords: van der Pol oscillator, delay coupling, phase synchronization, synchronization frequency

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1. Introduction

The study of synchronization among oscillators has a long history.[1–7] In most works, the effect of oscillators on each other has been assumed to be rapid and without delay, which is an ideal assumption. However, we know that the mutual interaction of oscillators in nature occurs with some delays. For instance, microwave oscillators work at a frequency as high as 10 GHz. In other words, their periods are about 0.6 ns. Since signals propagate through space at the speed of light, the signal will move about 18 cm during one full period of the oscillators. If the oscillators are separated by, for example, 4 cm, we can see that each oscillator will go through about one-fourth period before it is reached by the signal of the other oscillator. Thus, the time for a signal to travel from one oscillator to the other represents a substantial portion of the period of the uncoupled oscillator. This immediately leads to the inclusion of delay effects in the coupling terms.[8] In another example, we refer to heart pacemakers, where the effect of the sino-atrial node signal reaches the atrio-ventricular node after about 0.2 seconds.

The study of coupled oscillators with delay coupling is a relatively recent and complicated issue. Generally, two methods have been used to analyze such systems. In the first method, the nonlinear system is linearized around the fixed point, then the linear systems analyze tools such as the systems characteristics equation, and its eigenvalues are employed for analysis.[8–10] The second method uses averaging (one of the perturbation methods). The slow-flow dynamics are obtained first using averaging, and then applying first-degree tailor expansion, the delay equations are approximated by equations without delay.[11–13] The advantage of the aforementioned methods is that they provide a close form equation for analysis of the bifurcation points, stability and synchronization areas. However, they have two major disadvantages: first, the oscillators have to be identical; and second, they cannot give information about the synchronization frequency. In this paper, the two oscillators are assumed to be completely non-identical, and the aim is to gain the necessary conditions for the synchronization and the vicinity of the oscillators' frequency if they are phase synchronized. To achieve these goals, the describing function is used.[14] The advantage of this method over other methods, in addition to achieving the synchronization frequency, is that delay coupling appears in the equations as a phase lag, and the phase of equations is only affected by delay. Therefore, the analysis is easier than in the other methods. This advantage is similar that of the Nyquist method over other methods in the stability analysis of time delay linear systems, in which the time delay only causes the rotation of the Nyquist diagram around the origin.

Since the van der Pol oscillator can model many phenomena in nature, such as the normal state and blocking arrhythmias in the heart,[15] microwave oscillators and lasers,[8] the behavior of a pair of mutual coupled van der Pol with delay coupling is studied in this paper.

The paper is organized as follows: phase synchronization is investigated and the necessary conditions are achieved in Section 2. In Section 3, the switching synchronization frequency versus time delay variations are studied. The various cases of switching are simulated in Section 4, and the conclu-

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sion is given in Section 5.

2. Phase synchronization

Assume two coupled van der Pol oscillators with delayed coupling as follows:

$$\begin{aligned} &\ddot{x}(t)-\mu_1\left[1-x^2(t)\right]\dot{x}(t)+\omega_1^2 x(t)\\ &=\alpha_1\left[y(t-\tau)-x(t)\right], \qquad (1)\\ &\ddot{y}(t)-\mu_2\left[1-y^2(t)\right]\dot{y}(t)+\omega_2^2 y(t)\\ &=\alpha_2\left[x\left(t-\tau'\right)-y(t)\right], \qquad (2)\end{aligned}$$

where μ_1 and μ_2 are damping coefficients, ω_1 and ω_2 are natural frequencies, α_1 and α_2 are coupling intensities, and τ and τ' are delay-time couplings. Equations

$$\ddot{x}(t)-\mu_1\left[1-x^2(t)\right]\dot{x}(t)+\omega_1'^2 x(t)=\alpha_1 y(t-\tau), \quad (3)$$

$$\ddot{y}(t)-\mu_2\left[1-y^2(t)\right]\dot{y}(t)+\omega'^2_2 y(t)=\alpha_2 x\left(t-\tau'\right) \quad (4)$$

are rearranged for analysis, where $\omega_1'^2=\omega_1^2+\alpha_1$ and $\omega'^2_2=\omega_2^2+\alpha_2$. To use the describing function, we decompose the equations into linear and nonlinear parts, that is

$$\ddot{x}-\mu_2\dot{x}+\mu_2\dot{f}(x)+\omega_1'^2 x=\alpha_1 y_\tau, \quad (5)$$

$$\ddot{y}-\mu_2\dot{y}+\mu_2\dot{f}_2(y)+\omega'^2_2 y=\alpha_2 x'_\tau, \quad (6)$$

where $y_\tau=y(t-\tau)$, $x'_\tau=x(t-\tau')$, and the nonlinear function $f(\cdot)$ is

$$f(\cdot)=\frac{1}{3}(\cdot)^3. \quad (7)$$

Suppose that the synchronization frequency is ω'', therefore, the signal that effects x has ω'_2 and ω'' frequencies. On the other hand, the signal that effects y has ω'_1 and ω'' frequencies. According to the natural and imposed frequencies for x and y, and taking into account the first harmonics, we obtain

$$\begin{aligned} x(t)\approx\ & x_1\sin\left(\omega'_1 t+\varphi'_1\right)+x_2\sin\left(\omega'' t+\varphi_x\right)\\ &+x_3\sin\left(\omega'_2 t+\varphi''_1\right), \qquad (8)\\ y(t)\approx\ & y_1\sin\left(\omega'_2 t+\varphi'_2\right)+y_2\sin\left(\omega'' t+\varphi_y\right)\\ &+y_3\sin\left(\omega'_1 t+\varphi''_2\right), \qquad (9)\end{aligned}$$

where $\varphi=\left[\begin{array}{cccccc}\varphi'_1 & \varphi_x & \varphi''_1 & \varphi'_2 & \varphi_y & \varphi''_2\end{array}\right]$ is the phase vector of the signals. One can consider one phase (for instance φ'_1) as the basis for the other phases. A block diagram of the two coupled van der Pol oscillator system is illustrated in Fig. 1, where the transfer function $G_x(p)$ is

$$G_x(p)=\frac{\mu_1 p}{p^2-\mu_1 p+\omega_1'^2}. \quad (10)$$

One can see from Eq. (10) that the transfer function is a band pass filter which attenuates high and low frequencies, and we can overlook signals with frequencies $\omega'_1\pm\omega'_2$ and higher order harmonics. It is clear that we cannot ignore frequencies $\omega'_1-\omega'_2$ if one of the transfer function zeros is not in the origin.

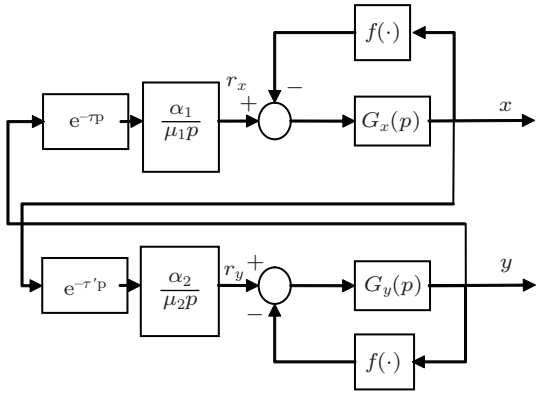


Fig. 1. Block diagram of the two bidirectional coupled van der Pol oscillator system.

According to Eq. (5), Eq. (6), and the transfer function Eq. (10), we obtain the input r_x

$$\begin{aligned} r_x(t)\approx\ & -\frac{\alpha_1 y_1}{\mu_1\omega'_2}\cos\left(\omega'_2 t+\varphi'_2-\omega'_2\tau\right)\\ &-\frac{\alpha_1 y_2}{\mu_1\omega''}\cos\left(\omega'' t+\varphi_y-\omega''\tau\right)\\ &-\frac{\alpha_1 y_3}{\mu_1\omega'_1}\cos\left(\omega'_1 t+\varphi''_2-\omega'_1\tau\right). \qquad (11)\end{aligned}$$

Signals x and y [Eqs. (8) and (9)] are phase synchronized if the amplitude of the synchronization frequency (ω'') is very high in comparison with the amplitudes of the other frequencies. The input signal (11) can be approximated as shown if the two oscillators are phase synchronized

$$r_x(t)\approx-\frac{\alpha_1 y_2}{\mu_1\omega''}\cos\left(\omega'' t+\varphi_y-\varphi_\tau\right), \quad \varphi_\tau=\omega''\tau. \quad (12)$$

For the nonlinear function, we also have

$$\begin{aligned} f(x)=\ & \frac{1}{3}x^3\approx N_1 x_1\sin\left(\omega'_1 t+\varphi'_1\right)+N_{2x}x_2\sin\left(\omega'' t+\varphi_x\right)\\ &+N_{3x}x_3\sin\left(\omega'_2 t+\varphi''_1\right),\\ N_{1x}=\ & \frac{1}{4}\left(x_1^2+2x_2^2+2x_3^2\right),\\ N_{2x}=\ & \frac{1}{4}\left(2x_2^2+x_2^2+2x_3^2\right),\\ N_{3x}=\ & \frac{1}{4}\left(2x_1^2+2x_2^2+x_3^2\right). \qquad (13)\end{aligned}$$

The nonlinear function as $f(x)=x^3/3$ instead of $f(x,\dot{x})=x^2\dot{x}$ has two important advantages.

(i) The nonlinear part is static, therefore, the describing function is independent of frequency.

(ii) The function $f(x)$ is odd; therefore, it is not necessary to consider a bias for consistency of input–output.

According to the block diagram (1), we have

$$G_x^{-1}(\mathrm{j}\omega)x+f(x)=r_x, \quad (14)$$

where

$$G_x^{-1}(\mathrm{j}\omega) = \frac{j}{\mu_1}\left(\omega - \frac{\omega_1'^2}{\omega}\right) - 1. \tag{15}$$

By substituting Eqs. (12) and (13) in Eq. (14) and rewriting it, we have[15]

$$\left[G_x^{-1}(\mathrm{j}\omega_1') + N_{1x}\right] x_1 \approx 0, \tag{16}$$

$$\left[G_x^{-1}(\mathrm{j}\omega'') + N_{2x}\right] x_2 \approx \frac{\alpha_1 y_2}{\mu_1 \omega''} \times \exp\left\{-\mathrm{j}\left[\frac{\pi}{2} + (\varphi_x - \varphi_y) + \varphi_\tau\right]\right\}, \tag{17}$$

$$\left[G_x^{-1}(\mathrm{j}\omega_2') + N_{3x}\right] x_3 \approx 0, \tag{18}$$

Equations (16) and (18) state that

$$-1 + \frac{1}{4}\left(x_1^2 + 2x_2^2 + 2x_3^2\right) \approx 0,$$
$$-1 + \frac{1}{4}\left(2x_1^2 + 2x_2^2 + x_3^2\right) \approx 0. \tag{19}$$

According to Eq. (19), the amplitudes x_1 and x_3 will tend to zero if the amplitude of x_2 becomes greater than $\sqrt{2}$. Since the amplitudes of x_1 and x_3 must be very low in comparison with the amplitude x_1, it would therefore be best if one takes the condition below for phase synchronization

$$x_2 > \sqrt{2}. \tag{20}$$

Accordingly, for the second oscillator

$$G_y(p) = \frac{\mu_2 p}{p^2 - \mu_2 p + \omega_2'^2}, \tag{21}$$

$$r_y(t) \approx -\frac{\alpha_2 x_2}{\mu_2 \omega''}\cos\left(\omega'' t + \varphi_x - \varphi_\tau'\right), \quad \varphi_\tau' = \omega'' \tau'. \tag{22}$$

$$f(y) = \frac{1}{3}y^3 \approx N_1 y_1 \sin\left(\omega_2' t + \varphi_2'\right) + N_2 y_2 \sin\left(\omega'' t + \varphi_y\right) + N_3 y_3 \sin\left(\omega_1' t + \varphi_2''\right),$$
$$N_{1y} = \frac{1}{4}\left(y_1^2 + 2y_2^2 + 2y_3^2\right),$$
$$N_{2y} = \frac{1}{4}\left(2y_2^2 + y_2^2 + 2y_3^2\right),$$
$$N_{3y} = \frac{1}{4}\left(2y_1^2 + 2y_2^2 + y_3^2\right). \tag{23}$$

And similar to Eq. (14)

$$G_y^{-1}(\mathrm{j}\omega)\, y + f(y) = r_y, \tag{24}$$

where

$$G_y^{-1}(\mathrm{j}\omega) = \frac{\mathrm{j}}{\mu_2}\left(\omega - \frac{\omega_2'^2}{\omega}\right) - 1. \tag{25}$$

By substituting Eqs. (22) and (23) in Eq. (25) and rewriting it, we have

$$\left(G_y^{-1}(\mathrm{j}\omega_1') + N_{1y}\right) y_1 \approx 0, \tag{26}$$

$$\left(G_y^{-1}(\mathrm{j}\omega'') + N_{2y}\right) y_2 \approx \frac{\alpha_2 x_2}{\mu_2 \omega''} \exp\left\{-\mathrm{j}\left[\frac{\pi}{2} - (\varphi_x - \varphi_y) + \varphi_\tau'\right]\right\}, \tag{27}$$

$$\left[G_y^{-1}(\mathrm{j}\omega_2') + N_{3y}\right] y_3 \approx 0. \tag{28}$$

Equations (26) and (28) demonstrate

$$-1 + \frac{1}{4}\left(y_1^2 + 2y_2^2 + 2y_3^2\right) \approx 0,$$
$$-1 + \frac{1}{4}\left(2y_1^2 + 2y_2^2 + y_3^2\right) \approx 0. \tag{29}$$

Similar to Eq. (19), amplitudes y_1 and y_3 will tend to zero if the amplitude of y_2 becomes greater than $\sqrt{2}$. Therefore, the other necessary condition is

$$y_2 > \sqrt{2}. \tag{30}$$

3. Synchronization frequency

Assuming $\theta = \varphi_x - \varphi_y$ as the phase difference of two oscillators and rearranging Eqs. (17) and (27), we can obtain

$$\left[\frac{\mathrm{j}}{\mu_1}\left(\omega'' - \frac{\omega_1'^2}{\omega''}\right) - 1 + \frac{1}{4}x_2^2\right] x_2 \approx \frac{\alpha_1 y_2}{\mu_1 \omega''} \times \exp\left[-\mathrm{j}\left(\frac{\pi}{2} + \theta + \varphi_\tau\right)\right], \tag{31}$$

$$\left[\frac{\mathrm{j}}{\mu_2}\left(\omega'' - \frac{\omega_2'^2}{\omega''}\right) - 1 + \frac{1}{4}y_2^2\right] y_2 \approx \frac{\alpha_2 x_2}{\mu_2 \omega''} \times \exp\left[-\mathrm{j}\left(\frac{\pi}{2} - \theta + \varphi_\tau'\right)\right]. \tag{32}$$

The set of Eqs. (31) and (32) is composed of four equations according to the real and imaginary parts and has four unknown parameters ω'', θ, x_2, y_2. Suppose that two oscillators are phase synchronized, and without losing the generality, assume $\tau = \tau' \Rightarrow \varphi_\tau = \varphi_\tau'$. The real parts of Eqs. (31) and (32) are

$$\left(-1 + \frac{1}{4}x_2^2\right) x_2 = -\frac{\alpha_1 y_2}{\mu_1 \omega''}\sin(\theta + \varphi_\tau), \tag{33}$$

$$\left(-1 + \frac{1}{4}y_2^2\right) y_2 = \frac{\alpha_2 x_2}{\mu_2 \omega''}\sin(\theta - \varphi_\tau) \tag{34}$$

and the imaginary part

$$x_2\left(\omega'' - \frac{\omega_1'^2}{\omega''}\right) = -\frac{\alpha_1 y_2}{\omega''}\cos(\theta + \varphi_\tau), \tag{35}$$

$$y_2\left(\omega'' - \frac{\omega_2'^2}{\omega''}\right) = -\frac{\alpha_2 x_2}{\omega''}\cos(\theta - \varphi_\tau). \tag{36}$$

The real parts discuss the amplitudes and the imaginary parts cover the synchronization frequency versus time delay.

If the time delay is so small that $0 < \theta \pm \varphi_\tau < \pi/2$, the system will act as though it were without delay. In this case, only the amplitudes of the oscillators are distorted. Case 1 in the simulation section exemplifies this case. Figure 2 shows the effect of a small time delay coupling.

By increasing the time delay, the system first becomes synchronized. With the increase in the time delay, the system goes back to the synchronization situation, but the synchronization frequency will increase abruptly because according to Eqs. (35) and (36), the synchronization frequency switches from $\omega'' \leq \min(\omega_1', \omega_2')$ to $\omega'' \geq \max(\omega_1', \omega_2')$. Since the time delay is not very large in this case, the phase $\theta \pm \varphi_\tau$ is situated in the third quadrant of the plane. Therefore, according to Eqs. (33) and (34), the amplitude x_2 will be greater than 2 and the amplitude y_2 will be smaller than 2. Figure 3 in the simulation section justifies this, and explains the phenomenon that has been observed in the cells of chicken heart pacemakers, where some of the pacemaker cells discharge at a frequency higher than the highest frequency.[17]

Further increasing the time delay will cause an increase in phase delay φ_τ, so the phase $\theta + \varphi_\tau$ is situated in the third quadrant and the phase $\theta - \varphi_\tau$ in the second quadrant. In this case, the amplitudes of x_2 and y_2 become greater than 2 according to Eqs. (33) and (34), and the synchronization frequency is situated between two ω_1' and ω_2' frequencies according to Eqs. (35) and (36). Figure 4 demonstrates the results with a large time delay.

If the time delay increase is high, the phase $\theta + \varphi_\tau$ will be situated in the fourth quadrant, and phase $\theta - \varphi_\tau$ in the first quadrant. In this case, the amplitudes of x_2 and y_2 become greater than 2 according to Eqs. (33) and (34), but the synchronization frequency drops to a frequency lower than the minimum frequency of two oscillators $[\omega'' \leq \min(\omega_1', \omega_2')]$. Figure 5 demonstrates the result with a very large time delay (Case 3).

It is stated in Ref. [17] that some pacemaker cells discharge at a frequency lower than the lowest frequency. This case explains the phenomenon.

4. Simulation

In this section, the effects of time delay on the amplitude and synchronization frequency are simulated in four different cases.

4.1. Case 1: small delay

Assume the system parameters are $\omega_1 = 1.6$, $\mu_1 = 2$, $\alpha_1 = 1.6$, $\omega_2 = 1$, $\mu_2 = 1$, $\alpha_2 = 1.6$ ($\because \omega_1' = 2.0396$, $\omega_2' = 1.6125$) and the time delay $\tau = 0.0483$. According to Fig. 2, the synchronization frequency and phase delay are $\tau = 0.0483, \omega'' = 1.2483$, which lead to $\phi_\tau = 3.5°$. Choosing coupling intensities larger than $\alpha_1 = \alpha_2 = 1.6$ causes a very low amplitude distortion.

As mentioned in the previous section, because $0 < \theta \pm \varphi_\tau < \pi/2$, we expect $x_2 < \sqrt{2}$ and $y_2 > \sqrt{2}$ from Eqs. (33) and (34), and $\min(\omega_1', \omega_2') \leq \omega'' \leq \max(\omega_1', \omega_2')$ from Eqs. (35) and (36). Figure 2 confirms the results obtained in Section 2. The solid and dashed lines illustrate the outputs of the first and second oscillators, respectively. It can be seen that the output of the second oscillator is distorted a little.

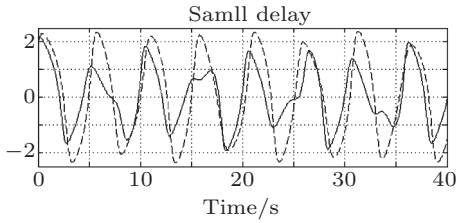


Fig. 2. The effect of a small delay on the oscillators. The solid and dashed lines illustrate the outputs of the first and second oscillators.

4.2. Case 2: medium delay

In this case, the parameters are similar to those in Case 1, except for the delay which is larger than that in Case 1 ($\tau = 0.0785$). The parameters are $x_2 = 2.18$, $y_2 = 1.669$, $\omega'' = 2.1256 > \max(\omega_1', \omega_2')$, $\theta = 226.5°$, $\varphi_\tau = 9.57°$, $180° < \theta \pm \varphi_\tau < 270°$. It is expected that the frequency suddenly increases (in this example from $\omega'' = 1.2483$ to $\omega'' = 2.1256$) with a medium delay. It is also expected that $x_2 > \sqrt{2}$ and $y_2 < \sqrt{2}$ for $180° < \theta \pm \varphi_\tau < 270°$ from Eqs. (33) and (34), and $\omega'' \geq \max(\omega_1', \omega_2')$ from Eqs. (35) and (36), which can be seen in Fig. 3.

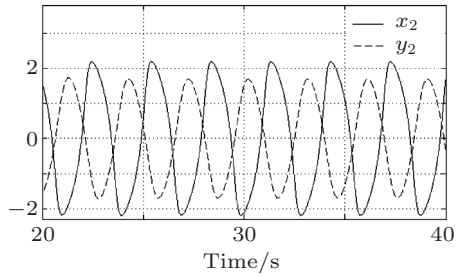


Fig. 3. Switching the synchronization frequency to a high frequency.

4.3. Case 3: large delay

By increasing the time delay to $\tau = 0.4189$, the third case occurs. As mentioned in the previous section, the $\theta + \varphi_\tau$ phase is situated in the third quadrant and the $\theta - \varphi_\tau$ phase in the second quadrant. In this case, the synchronization frequency is situated between two ω_1' and ω_2' frequencies, according to Eqs. (35) and (36). Moreover, according to Eqs. (33) and (34),

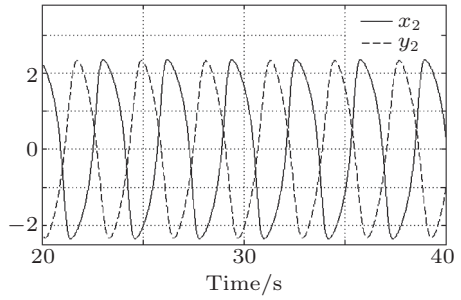


Fig. 4. The large time delay. Both amplitudes are greater than 2 in this case.

the amplitudes of x_2 and y_2 become greater than 2 in this case. This is confirmed in Fig. 4.

The results of the simulation are

$$x_2 = 2.23,\quad y_2 = 2.3,\quad \omega_2' < \omega'' = 1.98 < \omega_1',\quad \theta = 210.6^\circ,$$
$$\varphi_\tau = 47.4^\circ,\quad \theta + \varphi_\tau = 258^\circ,\quad \theta - \varphi_\tau = 163.2^\circ.$$

One can see that the amplitude is greater than 2 and the synchronization frequency situated between the two frequencies. This result confirms Eqs. (33)–(36).

4.4. Case 4: very large time delay

By increasing the time delay to $\tau = 1.5708$, the frequency drops abruptly according to Eqs. (34) and (35). The numerical results are

$$x_2 = 2.39,\quad y_2 = 2.76,\quad \omega'' = 1.38 < \min(\omega_1',\ \omega_2'),$$
$$\theta = 212.5^\circ,\quad \varphi_\tau = 124^\circ,\quad \theta + \varphi_\tau = 336.5^\circ,\quad \theta - \varphi_\tau = 88.5^\circ.$$

According to $\theta + \varphi_\tau$ and Eq. (33), we deduce that $x_2 > \sqrt{2}$, and, $\theta - \varphi_\tau$ and equation (34), $y_2 > \sqrt{2}$. In addition, Eqs. (35) and (36) deduce that $\omega'' < \min(\omega_1', \omega_2')$. All of our expectances are approved by Fig. 5.

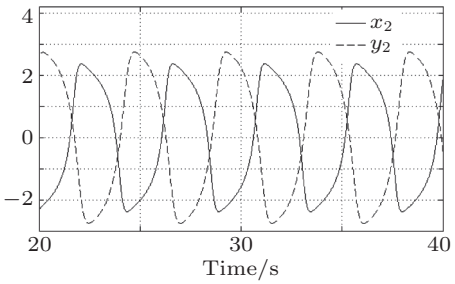


Fig. 5. The effect of a very large time delay. The synchronization drops to a low frequency.

5. Conclusion

The describing function method was used to study the behavior of two coupled van der Pol oscillators with delayed coupling. The necessary conditions and the boundaries of the synchronization frequency were obtained. The major advantage of this method is its capability in confronting delay, since the delay appears only in the phase of equations. Another advantage is finding the boundaries of the synchronization frequency in relation to various time delays. When the delay is small, the synchronization frequency is also small. By increasing the time delay, the synchronization frequency increases abruptly. A further increase in the time delay leads to a middle frequency. When the delay increase is very high, the frequency again switches to a small frequency. According to the achieved equations, the boundaries of the oscillator amplitudes were also obtained.

To use the describing function method properly, the system is decomposed into linear and nonlinear parts, such that the transfer function of the linear part acts as a band pass filter. This attenuates the high and low frequencies and makes the nonlinear part in the form of a static odd function, which results in the describing function becoming independent of bias and frequency.

References

[1] Appleton E V 1922 *Proc. Cambridge Phil. Soc.* (Math. and Phys. Sci.) **21** 231
[2] Li X, Ji J C and Hansen C H 2006 *Mech. Res. Comm.* **33** 614
[3] Bu S L, Zhang Y W and Wang B H *Chin. Phys. Lett.* 2006 **23** 2909
[4] Fang J A, Tang Y, Miao Q Y and Wong W K 2011 *Chin. Phys. B* **20** 040513
[5] Sun W, Chen Z and Chen S H 2012 *Chin. Phys. B* **21** 050509
[6] Feng C F, Xu X J, Wu Z X and Wang Y H 2008 *Chin. Phys. B* **17** 1951
[7] Guo L X, Xu Z Y and Hu M F 2008 *Chin. Phys. B* **17** 836
[8] Wirkus S and Rand R 2002 *Nonlinear Dynamics* **30** 205
[9] Reddy D V R, Sen A and Johnston G L 1998 *Phys. Rev. Lett.* **80** 5109
[10] Sen1 A, Dodla R and Johnston G L 2005 *PRAMANA Journal of Physics* **64** 465
[11] Huailei W, Zaihua W and Haiyan H 2004 *Acta Mech. Sin.* **20** 426
[12] Li X, Ji J C, Hansenb C H and Tan C 2006 *J. Sound Vibr.* **291** 644
[13] Shirahama H, Choe C U and Fukushima K 2008 *The 23rd Interational Technical Conference on Circuts/Systems Computers and Communications* 549
[14] Rompala K, Rand R and Howland H 2007 *Communications in Nonlinear Science and Numerical Simulation* **12** 794
[15] Cook P A 1986 *Nonlinear Dynamical Systems* (Prentice Hall International)
[16] Gholizade-Narm H, Azemi A, Khademi M and Karimi-Ghartemani M 2008 *J. App. Sci.* **8** 3175
[17] Dehaan R L and Hirakow R 1972 *Exp. Cell. Res.* **70** 214