Non-Markovian decoherent quantum walks*

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Quantum walks act in obviously different ways from their classical counterparts, but decoherence will lessen and close this gap between them. To understand this process, it is necessary to investigate the evolution of quantum walks under different decoherence situations. In this article, we study a non-Markovian decoherent quantum walk on a line. In a short time regime, the behavior of the walk deviates from both ideal quantum walks and classical random walks. The position variance as a measure of the quantum walk collapses and revives for a short time, and tends to have a linear relation with time. That is, the walker's behavior shows a diffusive spread over a long time limit, which is caused by non-Markovian dephasing affecting the quantum correlations between the quantum walker and his coin. We also study both quantum discord and measurement-induced disturbance as measures of the quantum correlations, and observe both collapse and revival in the short time regime, and the tendency to be zero in the long time limit. Therefore, quantum walks with non-Markovian decoherence tend to have diffusive spreading behavior over long time limits, while in the short time regime they oscillate between ballistic and diffusive spreading behavior, and the quantum correlation collapses and revives due to the memory effect.

Keywords: non-Markovian decoherence, quantum walks, quantum correlations

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1. Introduction

The quantum walk (QW) is one of the most important developments in theoretical quantum information science.[1–3] It provides an intriguing generalization of the ubiquitous random walk (RW) in physics[4,5] and exponential algorithmic speedups.[6] QWs have been demonstrated using nuclear magnetic resonance,[7,8] trapped ions,[9] single photons in bulk[10] and fibre optics,[11] and the scattering of light in waveguide arrays.[12–14] With decoherence, the spreading of the QW is transited from ballistic to diffusive scaling,[15,16] and with quantum loss, the QW is transited to RW.[17–22]

An open quantum system loses its quantumness when the information about the quantum state leaks into the environment around it. The unidirectional flow of information in which decoherence and noise act consistently characterizes a Markovian process.[23] However, some systems, such as soft- and condensed-matter systems, strongly couple to the environment, and this coupling leads to a different regime where the information also flows back into the system from the surroundings, characterizing a non-Markovian process.[24] During a Markovian process, the distinguishability tends to monotonically decrease for the pair of states, and in this paper, this is the quantum walker+coin state.[15–21] Memory effects caused by information flowing back into the system during a non-Markovian process can temporarily increase it for the walker+coin states.

Dephasing for some kinds of qubits is the major source of decoherence and in the strong-coupling regime, non-Markovian dephasing has to be considered instead of Markovian decoherence. For some physical systems such as photons and atoms, which have often been used to realize QWs, coin qubits, such as the polarization states of photons and two-level atomic states, suffer major decoherence by dephasing. The realistic systems are not only impacted by a Markovian white noise reservoir, but also by non-Markovian noise due to the strong coupling between the systems and the environment. Therefore, it is very important to study QWs in a non-Markovian environment, which provides good motivation for our work.

In this paper, we use quantum correlation between a quantum walker and his coin as a distinguishability to present the behavior of a QW on a line with a non-Markovian dephasing coin. The quantum correlation between the quantum walker and his coin decays quickly but is interrupted by revivals. Correspondingly, in the short time regime, the position variance of a non-Markovian decoherent QW deviates from both those of the ideal QW and RW. In the long time limit, the quantum correlation tends to be zero and the position variance shows a diffusive spread.

This paper is organized as follows: in Section 2, we introduce an ideal QW on a line. In Section 3, non-Markovian dephasing on the coin state–two-level system is shown. We

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study the effects of non-Markovian dephasing on the coin, and comparing ideal QWs with RWs, the non-Markovian decoherent QWs show different behaviors in the short time regime. In Section 4, the quantum correlation between the quantum walker and his non-Markovian dephasing coin is studied via quantum discord (QD) and measurement-induced disturbance (MID). It is observed that both QD and MID collapse and revive in the short time regime, and tend to be zero in the long time limit. In Section 5, we analyze the behavior of non-Markovian decoherent QWs in the long time limit and observe that because of the disappearance of the quantum correlation between the quantum walker and his coin, the non-Markovian decoherent QW in the long time regime shows a diffusive behavior. Finally, we briefly summarize the paper.

2. Ideal QW on a line

For an ideal QW on a line,[25,26] the Hilbert space is

$$\mathscr{H} = \mathscr{H}_{\rm w} \otimes \mathscr{H}_{\rm c} \tag{1}$$

with the walker Hilbert space $\mathscr{H}_{\rm w}$ spanned by the position vectors $\{|x\rangle\}$, and the coin space $\mathscr{H}_{\rm c}$ spanned by two orthogonal vectors, which we denote as $|\pm 1\rangle$. Each step of the walker is affected by two subsequent unitary operators: one is the coin-flip operator

$$C = H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{2}$$

where H is the Hadamard matrix, and the other is the conditional-translation operator

$$F = S \otimes |1\rangle\langle 1| + S^{\dagger} \otimes |-1\rangle\langle -1|, \tag{3}$$

where $S|x\rangle = |x+1\rangle$ and $S^{\dagger}|x\rangle = |x-1\rangle$. The resultant step operator is $U = F(I \otimes C)$ with I the identity operator on $\mathscr{H}_{\rm w}$.

The choice of the initial state $\big|\psi(t=0)\big\rangle$ is important in studies of QWs because the interference features sensitively depend on the choice of the initial state. This dependence is persistent because the dynamics are unitary, and hence, it does not die out. On the other hand, the general properties of QWs do not depend on the choice of the initial state, so this is not crucial, providing that the focus is on such characterization.

As we are interested in the general properties and the initial state is not important, we choose the initial state with the walker at the origin of a line and holding a coin in an equal superposition of the $+1$ and -1 states

$$\big|\psi(t=0)\big\rangle = \frac{1}{\sqrt{2}}|0\rangle(|1\rangle + {\rm i}|-1\rangle). \tag{4}$$

After t steps, the final state of the walker+coin system is

$$|\psi(t)\rangle = U^t|\psi(0)\rangle. \tag{5}$$

We now follow the evolution by performing a Fourier transform of the evolution operator to the "momentum" k space.

The eigenvectors

$$|k\rangle = \sum_x {\rm e}^{{\rm i}kx}|x\rangle \tag{6}$$

of S and $S^{\dagger}$ in Eq. (3) have the eigenrelations

$$S|k\rangle = {\rm e}^{-{\rm i}k}|k\rangle, \quad S^{\dagger}|k\rangle = {\rm e}^{{\rm i}k}|k\rangle, \tag{7}$$

where k is a continuous real quantity. The inverse transformation is

$$|x\rangle = \int_{-\pi}^{\pi} \frac{{\rm d}k}{2\pi} {\rm e}^{-{\rm i}kx}|k\rangle. \tag{8}$$

The walker is initialized at the origin of a line, hence, the initial state of the walker is

$$|0\rangle = \int_{-\pi}^{\pi} \frac{{\rm d}k}{2\pi}|k\rangle. \tag{9}$$

On the basis of the walker, the evolution operator becomes

$$U|k\rangle \otimes |\Phi\rangle_{\rm c} = |k\rangle \otimes U_k|\Phi\rangle_{\rm c} \tag{10}$$

with $|\Phi\rangle_{\rm c}$ the coin state and

$$U_k = \frac{1}{\sqrt{2}} \begin{pmatrix} {\rm e}^{-{\rm i}k} & {\rm e}^{-{\rm i}k} \\ {\rm e}^{{\rm i}k} & -{\rm e}^{{\rm i}k} \end{pmatrix}. \tag{11}$$

The general density operator for the initial state of the system in the k basis can be expressed as

$$\rho(0) = \int_{-\pi}^{\pi} \frac{{\rm d}k}{2\pi} \int_{-\pi}^{\pi} \frac{{\rm d}k'}{2\pi} |k\rangle\langle k'| \otimes |\Phi_0\rangle\langle\Phi_0|. \tag{12}$$

The final state after t steps is

$$\rho(t) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} {\rm d}k \int_{-\pi}^{\pi} {\rm d}k' |k\rangle\langle k'| \otimes U_k^t|\Phi_0\rangle\langle\Phi_0|(U_k^{\dagger})^t. \tag{13}$$

In terms of the superoperator $\mathscr{L}_{kk'}\hat{O} = U_k\hat{O}U_k^{\dagger}$,

$$\rho(t) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} {\rm d}k \int_{-\pi}^{\pi} {\rm d}k' |k\rangle\langle k'| \otimes \mathscr{L}_{kk'}^t|\Phi_0\rangle\langle\Phi_0|. \tag{14}$$

The walker's position is on a line labeled x, with the initial position localized at 0. Measurement of the walker's position corresponds to the projection-valued measure $\{|x\rangle\langle x| \otimes I; x \in \mathbb{Z}\}$ on the walker reduced state (after tracing out the coin).

The probability $P(x;t)$ that the walker will be found at position x is

$$P(x;t) = {\rm Tr}\{|x\rangle\langle x| \otimes I_{\rm c}\rho(t)\}, \tag{15}$$

which is obtained by tracing over the coin of the walker+coin state, and then measuring the walker's position. We can characterize $P(x;t)$ by the moments of this position distribution $\langle x^m\rangle$. The mean $\langle x\rangle$ and the variance

$${\rm var} = \langle x^2\rangle - \langle x\rangle^2 \tag{16}$$

can be used as the measure of QWs and show the signature of QWs compared with RWs. For an RW, var $\sim t$, which is characteristic of diffusive motion, whereas, for a QW, a quadratic enhancement is achieved: var $\sim t^2$.[15,16,25,26]

3. A QW on a line with a non-Markovian dephasing coin in the short time regime

We now generalize to allow for decoherence. An open quantum system loses its quantumness when the information about the state leaks into its surroundings. The unidirectional flow of information characterizes a Markovian process,[23] whereas for a non-Markovian process,[6] information also flows back into the system. Recently, though it is not yet well understood, non-Markovian processes with memory have become of central importance in the study of open systems both theoretically[27–31] and experimentally.[32–37]

If we suppose that before each unitary flip of the coin, a completely positive map is performed on the coin, we consider a pure dephasing quantum process for which the density matrix $\rho_{\rm c}(t) = {\rm Tr}_{\rm w}\rho(t)$ of the open system evolves according to the master equation

$$\frac{\rm d}{{\rm d}t}\rho_{\rm c}(t) = -{\rm i}\frac{\varepsilon(t)}{2}\left[\sigma_z, \rho_{\rm c}(t)\right] + \frac{\gamma(t)}{2}\left[\sigma_z\rho_{\rm c}(t)\sigma_z, \rho_{\rm c}(t)\right]. \quad (17)$$

Here, $\varepsilon(t)$ represents the time-dependent energy shift and $\gamma(t)$ is the time-dependent decay rate of the channel described by the Pauli operator σ_z. The dephasing process influences the coherence between the walker and his coin, and the evolution can be described by the time-dependent function

$$\kappa(t) = \exp\left\{-\int_0^t {\rm d}t'\left[\gamma(t') + {\rm i}\varepsilon(t')\right]\right\}, \quad (18)$$

which is connected to the energy shift and the decay rate of the master Eq. (17) by the relations

$$\varepsilon(t) = -{\rm Im}\left[\frac{\dot{\kappa}(t)}{\kappa(t)}\right], \quad \gamma(t) = -{\rm Re}\left[\frac{\dot{\kappa}(t)}{\kappa(t)}\right]. \quad (19)$$

If the relaxation rates are positive, namely $\gamma(t) \geq 0$, then the generator in Eq. (17) is in Lindblad form for each fixed $t \geq 0$. Such a process with $\gamma(t) \geq 0$ may be called time-dependent Markovian. The generalized master equation involving a certain memory kernel can be described as a non-Markovian process, in which the $\gamma(t)$ rate must have negative values for some time interval. In the Markovian regime, information about the state of the coin leaks into its surroundings, and $|\kappa(t)|$ is a monotonically decreasing function of time. In the non-Markovian environment, in contrast, information also flows back into the system of the coin state and a revival of the distinguishability can be observed in the time evolution. With the definition of non-Markovianity, we see that an increase in $|\kappa(t)|$ leads to a negative rate $\gamma(t)$ in the generator of the master Eq. (17). As an example, we consider the case of a Lorentzian reservoir spectral density, which is on resonance with the coin qubit transition frequency and leads to an exponential two-point correlation function

$$f(t) = g_0\eta\,{\rm e}^{-\eta|t|}/2, \quad (20)$$

where g_0 describes the coupling strength between the coin qubit and the environment, and η denotes the spectral width (here, we consider the coin qubit coupled to a damped Jaynes-Cummings model). The time-dependent function $\kappa(t)$ is defined as the solution of the integrodifferential equation

$$\frac{\rm d}{{\rm d}t}\kappa(t) = -\int_0^t {\rm d}t' f\left(t-t'\right)\kappa\left(t'\right) \quad (21)$$

with an initial condition $\kappa(0) = 1$. We obtain

$$\kappa(t) = {\rm e}^{-\eta t/2}\left[\cosh\left(\frac{dt}{2}\right) + \frac{\eta}{d}\sinh\left(\frac{dt}{2}\right)\right], \quad (22)$$

where $d = \sqrt{\eta^2 - 2g_0\eta}$. We see that for weak couplings, $g_0 < \eta/2$, $|\kappa(t)|$ decreases monotonically. The correlation function $f(t)$ tends to $\delta(t)$ when η increases. For $\eta \to \infty$, $f(t) = \delta(t)$ and the different time noise is not correlated. Whereas in the strong coupling regime $g_0 > \eta/2$, d turns out to be a complex number and $\kappa(t)$ can be rewritten as

$$\kappa(t) = {\rm e}^{-\eta t/2}\left[\cos\left(\frac{d't}{2}\right) + \frac{\eta}{d'}\sin\left(\frac{d't}{2}\right)\right] \quad (23)$$

with $d' = {\rm Im}(d)$ and starts to oscillate, showing non-Markovian behavior. The non-Markovian decoherence model [Eqs. (17)–(23)] was presented first in Ref. [30].

Plots of $\kappa(t)$ are shown in Fig. 1. In the Markovian regime, for example $g_0 = 1$ and $\eta = 10$, the flow of information goes only from the system into the environment, and $\kappa(t)$ decreases monotonically. However, in the non-Markovian regime, such as $g_0 = 1$ and $\eta = 0.01, 0.05, 0.1$, respectively, the memory effects can temporarily increase it. The $\kappa(t)$ function decreases faster at $t = 0$, but will be interrupted by revivals, which are due to the memory effects on the quantum correlations between the quantum walker and his coin.

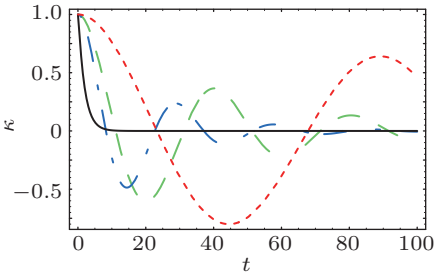


Fig. 1. (color online) The time-dependent function κ as a function of time t with $g_0 = 1$ and different η: $\eta = 0.01$ (red dotted line); $\eta = 0.05$ (green dashed line); $\eta = 0.1$ (blue dashed-dotted line), presenting non-Markovian dephasing; and $\eta = 10$ (black solid line), presenting Markovian dephasing.

The corresponding dynamical mapping, which maps the coin state $\rho_{\rm c}(0)$ to the state $\rho_{\rm c}(t)$ at time t is

$$\rho_{\rm c}^{00} = \rho_{\rm c}^{00}, \quad \rho_{\rm c}^{01} = \kappa^*(t)\rho_{\rm c}^{01},$$
$$\rho_{\rm c}^{10} = \kappa(t)\rho_{\rm c}^{10}, \quad \rho_{\rm c}^{11} = \rho_{\rm c}^{11}. \quad (24)$$

This map can also be written by a set of operators $\{A_1, A_2\}$ on the coin degree of freedom, which satisfies

$$\sum_{n=1,2} A_n^\dagger A_n = I, \tag{25}$$

where

$$A_1 = \frac{\left(1-\sqrt{1-|\kappa(t)|^2}\right)|\kappa(t)|}{\kappa^*(t)\sqrt{|\kappa(t)|^2+\left(1-\sqrt{1-|\kappa(t)|^2}\right)^2}}|0\rangle\langle 0| + \frac{|\kappa(t)|}{\sqrt{|\kappa(t)|^2+\left(1-\sqrt{1-|\kappa(t)|^2}\right)}}|1\rangle\langle 1| \tag{26}$$

and

$$A_2 = \frac{|\kappa(t)|}{\sqrt{|\kappa(t)|^2+\left(1-\sqrt{1-|\kappa(t)|^2}\right)}}|0\rangle\langle 0| + \frac{\left(1-\sqrt{1-|\kappa(t)|^2}\right)|\kappa(t)|}{\kappa(t)\sqrt{|\kappa(t)|^2+\left(1-\sqrt{1-|\kappa(t)|^2}\right)^2}}|1\rangle\langle 1|. \tag{27}$$

If the initial state is

$$\rho(0) = \int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}k'}{2\pi}|k\rangle\langle k'|\otimes|\Phi_0\rangle\langle\Phi_0| \tag{28}$$

and the QW proceeds for t steps, then the system evolves to

$$\begin{aligned}\rho(t) &= \int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}k'}{2\pi}|k\rangle\langle k'|\otimes U_k\sum_{j=1,2}A_j\cdots \\ &\quad U_k\sum_{i=1,2}A_i|\Phi_0\rangle\langle\Phi_0|A_i^\dagger U_{k'}^\dagger\cdots A_j^\dagger U_{k'}^\dagger \\ &= \int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}k'}{2\pi}|k\rangle\langle k'| \\ &\quad \otimes\mathscr{L}_{kk'}\left[\mathscr{L}_{kk'}\cdots\mathscr{L}_{kk'}\left(\mathscr{L}_{kk'}|\Phi_0\rangle\langle\Phi_0|\right)\right] \\ &= \int_{-\pi}^{\pi}\frac{\mathrm{d}k}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}k'}{2\pi}|k\rangle\langle k'|\otimes\mathscr{L}_{kk'}^t|\Phi_0\rangle\langle\Phi_0|. \end{aligned}\tag{29}$$

Note that this superoperator $\mathscr{L}_{kk}^t$ preserves the trace for $k=k'$. This implies that

$$\mathrm{Tr}\left(\mathscr{L}_{kk}^t\hat{O}\right) = \mathrm{Tr}\left(\hat{O}\right) \tag{30}$$

for any operator $\hat{O}$.

Now we consider the effect of non-Markovian dephasing on the coin. As we know, without decoherence, the variance of the position distribution of a QW is quadratically dependent on time t. With Markovian dephasing on the coin, the transition from a ballistic to a diffusion spreading behavior is observed in the long time limit. In this paper, we will show the behavior of a QW with a non-Markovian dephasing coin. The variance is used as a signature of the QW compared with an RW. In the short time regime, we can calculate the variance of the position distribution numerically. In Fig. 2, from the numerical results, we can see the variance as a function of time collapses and revivals for a short time, and the period of oscillations depends on coefficient d. Compared with that of an ideal QW, in the short time regime ($t \lesssim 100$), the variance of the QW with a non-Markovian dephasing coin deviates from a ballistic spreading behavior. However, in comparison with an RW, the oscillating variance never shows a diffusion behavior, which is due to the effect on the quantum correlations between the quantum walker and his coin from non-Markovian decoherence. From the numerical calculations, the position variance of an RW distribution is the lower bound of that of the QW with a non-Markovian dephasing coin.

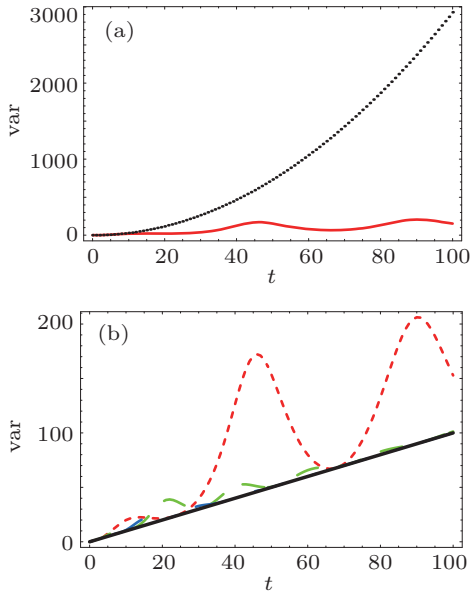


Fig. 2. (color online) Variances for a QW on a line after $t = 100$ steps with a perfect coin (dotted line) and with a non-Markovian dephasing coin with parameters $g_0 = 1$ and $\eta = 0.01$ (red solid line), $\eta = 0.05$ (green dashed line), $\eta = 0.1$ (blue dashed-dotted line), and $\eta = 10$ (black solid line). Here, we choose $(|0\rangle + \mathrm{i}|1\rangle)/\sqrt{2}$ as an initial coin state for the calculations. (a) Comparison of the variances between the RW and QW with a non-Markovian dephasing coin with parameters $g_0 = 1$ and $\eta = 0.01$. (b) Comparison of the variances between the QW with a non-Markovian dephasing coin with $g_0 = 1$ and $\eta = 0.01$, $\eta = 0.05$, $\eta = 0.1$ and $\eta = 10$.

In Fig. 3, we plot the position distribution for a QW with a non-Markovian dephasing coin ($g_0 = 1$ and $\eta = 0.01$) at the 38th, 42nd, 50th, and 100th steps, which are combinations of the position distribution of a standard QW and that of an RW. At the 38th step, the position distribution shows an RW behavior, that is, the probabilities of the positions around the original point are larger. And at the 42nd step, there are two small peaks higher than the others nearby in the position distribution. This shows that the memory effect from the non-Markovian decoherence draws the QW behavior back slightly. The small peaks disappear at the 50th step. With increasing time, the position distribution is quite close to a Gaussian distribution, as is shown at the 100th step. The position distribution plots show that the behavior of a QW with a non-Markovian dephasing coin oscillates between the quantum (ballistic) and classical (diffusive) behaviors.

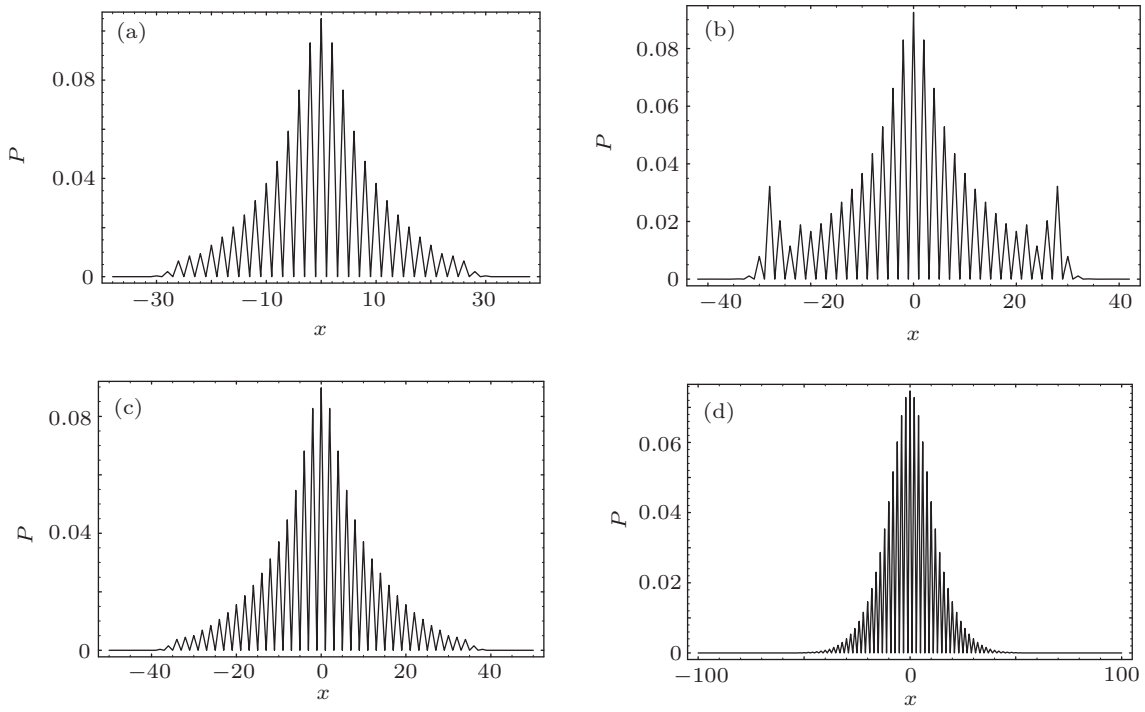


Fig. 3. position distributions for a QW on a line with a non-Markovian dephasing coin with parameters $g_0 = 1$ and $\eta = 0.01$ at different steps: (a) 38th step, (b) 42nd step, (c) 50th step, and (d) 100th step.

4. The dynamics of quantum correlations in a non-Markovian environment

The signature of a decoherent discrete-time QW on a line can be demonstrated in various ways, such as the position probability distribution becoming increasingly Gaussian, with a concomitant fall in the standard deviation. Neither of these recover the correlations between the quantum walker and his coin, which is known as the reason for the quantum behavior of QWs. In this section, we study the quantum correlation of a decoherent QW by two popular measures: measurement-induced disturbance (MID)[38–40] and quantum discord (QD).[41,42]

Given the walker+coin state $\rho(t)$, let the density matrix for the system be diagonalized to

$$\rho_i = \sum_j p_i^j \Pi_i^j \tag{31}$$

for $i = w, c$, where $\{\Pi_i^j\}$ is a complete projection-valued measure (i.e., using von Neumann measurements) for the walker w or coin c. Summing over the joint projections on the state yields the diagonalized state

$$\Pi\rho(t) = \sum_k I_{\rm w} \otimes \Pi_{\rm c}^k \rho(t) I_{\rm w} \otimes \Pi_{\rm c}^k. \tag{32}$$

The leftmost Π is an operator on the density matrix that diagonalizes the density matrix in the spectral basis corresponding to Π_i^j projective measurements. The operator Π can also be described as a "local measurement strategy".

Correlations between the walker state $\rho_{\rm w}$ and the coin state $\rho_{\rm c}$ are regarded as classical if there is a unique local measurement strategy Π leaving $\Pi\rho(t)$ unaltered from the original walker+coin state $\rho(t)$.[38–40] We ascertain whether the walker+coin state is "quantum" by determining whether a local measurement strategy exists that leaves the state unchanged.

The degree of quantumness is given by MID[38–40]

$$Q[\rho(t)] = \mathscr{I}[\rho(t)] - \mathscr{I}[\Pi\rho(t)], \tag{33}$$

where $\mathscr{I}[\rho(t)] = S[\rho_{\rm w}(t)] + S[\rho_{\rm c}(t)] - [\rho(t)]$ is the quantum mutual information and $S[\rho(t)] = -{\rm Tr}[\rho(t)\log_2\rho(t)]$ denotes the von Neumann entropy. By construction, $\Pi\rho(t)$ is classical. Thus, MID is the difference between the quantum and classical mutual information, which quantifies the total and classical correlations, respectively. Accordingly, equation (33) is interpreted as the difference between the total and classical correlations, which are represented by the quantum mutual information and the mutual information.

QD[41,42] is defined by

$$D[\rho(t)] = \mathscr{I}[\rho(t)] - \sup_{\Pi} \mathscr{I}[\Pi\rho(t)]. \tag{34}$$

MID has an operational definition. Unlike QD, it also tends to overestimate non-classicality because of the lack of optimization over local measurements. Applied to QWs, we find that MID, while acting as a loose upper bound on QD, still tends to reflect the trends in the behavior of the latter well.

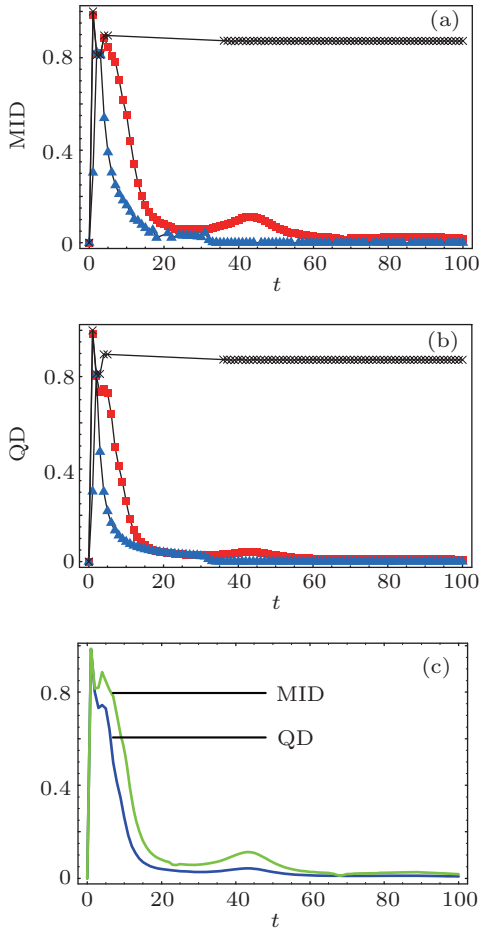


Fig. 4. (color online) The (a) MID and (b) QD for a QW on a line after $t = 100$ steps with perfect (black crosses), non-Markovian (red boxes), and Markovian (blue triangles) dephasing coins, respectively. Here, we choose $g_0 = 1$, $\eta = 0.01$, and $\eta = 10$, respectively, for the calculations. (c) Comparisons of MID (red) and QD (green) for a QW on a line with the same parameters as above.

We numerically calculated the MID and QD for a QW with perfect, Markovian and non-Markovian dephasing coins in the short time regime. The quantum correlations between the quantum walker and his coin presented by MID and QD are oscillating for the first several steps ($\lesssim 100$) shown in Figs. 4(a) and 4(b). Without decoherence, there is no difference between the MID and QD. Memory effects caused by the non-Markovianity of the environment deviate the MID and the QD from the monotonical behavior as that during the Markovian process. Both the MID and QD decay steeply and are interrupted by revivals. The oscillating period is the same as that of the corresponding position variance, which proves that the quantum correlation between the quantum walker and his coin results in the behavior of the QW. Both the MID and the QD tend to zero with increasing time, whereas for ideal QWs, neither the MID nor the QD decrease with time, as shown in Refs. [43] and [44]. From the numerical results, one can see that the quantum correlation in the case of Markovian decoherence is the lower bound of that with non-Markovian decoherence, which explains why the position variance of an RW distribution provides the lower bound of that of the QW with a non-Markovian dephasing coin. In comparing MID and QD, as shown in Fig. 4(c), both can show quantum correlation between the quantum walker and his coin changing with time, and MID oscillates with a larger amplitude than QD, which makes MID a better signature for observing this case.

The position variance as a measure of QWs collapses and revives for a short time and tends to be linear with time. That is, the walker's behavior shows a diffusive spread in the long time limit, which is caused by non-Markovian dephasing affecting the quantum correlations between the quantum walker and his coin. Since the quantum correlations tend to disappear in the long time limit, one can only see the collapses and revivals for a short time (let us say, less than 50 steps). Due to the period we choose in this paper, one can see the revivals of quantum correlations around $t = 20 \sim 30$.

5. QW on a line with a non-Markovian dephasing coin in the long time limit

In the above sections, we show that the behavior of QW on a line with a non-Markovian dephasing coin results in the quantum correlation between the quantum walker and his coin, which is presented by MID and QD. We observed that both MID and QD decrease with time and tend to zero in the long time limit. This means that the quantum correlation will disappear after a long enough time. In this section, we will show the behavior of a non-Markovian decoherent QW in the long time regime.

The probability of the walker reaching a point x at time t is

$$\begin{aligned} P(x;t) &= \mathrm{Tr}[|x\rangle\langle x|\rho(t)] \\ &= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\mathrm{d}k\int_{-\pi}^{\pi}\mathrm{d}k'\,\mathrm{e}^{-\mathrm{i}(k-k')x} \\ &\quad\times\mathrm{Tr}[\mathscr{L}^t_{kk'}|\Phi_0\rangle\langle\Phi_0|]. \end{aligned} \tag{35}$$

Thus, we can calculate the moments of this distribution

$$\begin{aligned} \langle x^m\rangle &= \sum_x x^m P(x;t) \\ &= \frac{(-1)^m}{(2\pi)^2}\int_{-\pi}^{\pi}\mathrm{d}k\int_{-\pi}^{\pi}\mathrm{d}k'\,\delta^{(m)}(k-k') \\ &\quad\times\mathrm{Tr}[\mathscr{L}^t_{kk'}|\Phi_0\rangle\langle\Phi_0|]. \end{aligned} \tag{36}$$

We can then invert the order of operations and sum x first. This sum can be carried out exactly in terms of the derivatives of the δ function

$$\frac{1}{2\pi}\sum_x x^m\,\mathrm{e}^{-\mathrm{i}x(k-k')} = (-\mathrm{i})^m\delta^{(m)}(k-k'). \tag{37}$$

Inserting this result back into the expression for $\langle x\rangle$, we get the first moment

$$\langle x\rangle = \frac{-\mathrm{i}}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k\frac{\mathrm{d}}{\mathrm{d}k}\mathrm{Tr}[\mathscr{L}_k^t|\Phi_0\rangle\langle\Phi_0|] = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k\sum_{j=1}^{t}\mathrm{Tr}[\sigma_z\mathscr{L}_k^j|\Phi_0\rangle\langle\Phi_0|]. \quad (38)$$

We can carry out a similar integration in parts to get the second moment

$$\langle x^2\rangle = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k\Big\{\sum_{j=1}^{t}\sum_{j'=1}^{j}\mathrm{Tr}\left[\sigma_z\mathscr{L}_k^{j-j'}(\sigma_z\mathscr{L}_k^{j'}|\Phi_0\rangle\langle\Phi_0|)\right] + \sum_{j=1}^{t}\sum_{j'=1}^{j-1}\mathrm{Tr}\left\{\sigma_z\mathscr{L}_{kj}^{j-j'}\left[(\mathscr{L}_k^{j'}|\Phi_0\rangle\langle\Phi_0|)\sigma_z\right]\right\}\Big\}. \quad (39)$$

Because $\mathscr{L}_k$ is a linear transformation, we can represent it as a matrix acting on the space of 2×2 operators. We choose the representation as

$$\hat{O} = r_1 I + r_2\sigma_x + r_3\sigma_y + r_4\sigma_z. \quad (40)$$

The actions of $\mathscr{L}_k$ on $\hat{O}$ are given by the matrix

$$\mathscr{L}_k\hat{O} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \sin 2k\mathrm{Im}[\kappa(t)] & \sin 2k\mathrm{Re}[\kappa(t)] & \cos 2k\\ 0 & -\cos 2k\mathrm{Im}[\kappa(t)] & -\cos 2k\mathrm{Re}[\kappa(t)] & \sin 2k\\ 0 & \mathrm{Re}[\kappa(t)] & -\mathrm{Im}[\kappa(t)] & 0 \end{pmatrix}\begin{pmatrix} r_1\\ r_2\\ r_3\\ r_4\end{pmatrix}. \quad (41)$$

Because $r_1 = \mathrm{Tr}(\hat{O}) = \mathrm{Tr}(\mathscr{L}_k\hat{O})$. The only nontrivial dynamics results from the 3×3 submatrix

$$M_k = \begin{pmatrix} \sin 2k\mathrm{Im}[\kappa(t)] & \sin 2k\mathrm{Re}[\kappa(t)] & \cos 2k\\ -\cos 2k\mathrm{Im}[\kappa(t)] & -\cos 2k\mathrm{Re}[\kappa(t)] & \sin 2k\\ \mathrm{Re}[\kappa(t)] & -\mathrm{Im}[\kappa(t)] & 0\end{pmatrix}. \quad (42)$$

We also need to know the effects of left and right multiplying by σ_z. These are given by the two matrices

$$Z_{\mathrm{L}} = \begin{pmatrix} 0&0&0&1\\ 0&0&\mathrm{i}&0\\ 0&-\mathrm{i}&0&0\\ 1&0&0&0\end{pmatrix},\quad Z_R = \begin{pmatrix} 0&0&0&1\\ 0&0&-\mathrm{i}&0\\ 0&\mathrm{i}&0&0\\ 1&0&0&0\end{pmatrix}. \quad (43)$$

Let us take these expressions and apply them to Eq. (38) for the first moment. In the integrand, the initial density matrix for the coin is multiplied j times by $\mathscr{L}_k$, and then left multiplied by σ_z, and finally the trace is taken. Given the above expression for Z_{L}, we see that this is the same as multiplying the 3-vector (r_2, r_3, r_4) j times by M_k and then keeping only the r_4 component of the result. This gives us the new expression for the first moment

$$\langle x\rangle = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k\begin{pmatrix}0&0&1\end{pmatrix}\left(\sum_{j=1}^{t}M_k^j\right)\begin{pmatrix}r_2\\r_3\\r_4\end{pmatrix} = -\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k\begin{pmatrix}0&0&1\end{pmatrix}\times\left[(1-M_k)^{-1}\left(M_k - M_k^{t+1}\right)\right]\begin{pmatrix}r_2\\r_3\\r_4\end{pmatrix}. \quad (44)$$

The eigenvalues of M_k satisfy $0<|\lambda|<1$, and then in the long time limit, therefore, $M_k^{t+1}\to 0$ and the first moment becomes approximately

$$\langle x\rangle \approx -\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k\begin{pmatrix}0&0&1\end{pmatrix}\left[(1-M_k)^{-1}M_k\right]\begin{pmatrix}r_2\\r_3\\r_4\end{pmatrix} = \frac{r_2\mathrm{Re}[\kappa(t)] - r_3\mathrm{Im}[\kappa(t)] + r_4|\kappa(t)|^2}{-1+|\kappa(t)|^2}. \quad (45)$$

In the long time limit, the second moment can also be calculated as

$$\begin{aligned}\langle x^2\rangle &= t - \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k\begin{pmatrix}1&0&0&0\end{pmatrix}\times\left[Z_{\mathrm{L}}\sum_{j=1}^{t}\sum_{j'=1}^{j-1}\mathscr{L}_k^{j-j'}(Z_{\mathrm{L}}+Z_R)\mathscr{L}_k^{j'}\right]\begin{pmatrix}1\\r_2\\r_3\\r_4\end{pmatrix}\\ &= t - \sum_{j=1}^{t}\sum_{j'=1}^{j-1}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k\begin{pmatrix}0&0&1\end{pmatrix}M_k^{j-j'}\begin{pmatrix}0\\0\\2\end{pmatrix}\\ &= t - \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}k\begin{pmatrix}0&0&1\end{pmatrix}\times\left[t-(1-M_k)^{-1}M_k + (1-M_k)^{-1}M_k^t\right]\times(1-M_k)^{-1}M_k\begin{pmatrix}0\\0\\2\end{pmatrix}\\ &= t\left(1+\frac{2|\kappa(t)|^2}{-1+|\kappa(t)|^2}\right) + \frac{2|\kappa(t)|^4+5|\kappa(t)|^2}{(-1+|\kappa(t)|^2)^2}, \end{aligned} \quad (46)$$

which is independent of the initial coin state.

The position variance is obtained as

$$\begin{aligned}\mathrm{var} &= \langle x^2\rangle - \langle x\rangle^2\\ &= t\left(1+\frac{2|\kappa(t)|^2}{-1+|\kappa(t)|^2}\right) + \frac{2|\kappa(t)|^4+5|\kappa(t)|^2}{(-1+|\kappa(t)|^2)^2} - \frac{r_2\mathrm{Re}[\kappa(t)] - r_3\mathrm{Im}[\kappa(t)] + r_4|\kappa(t)|^2}{-1+|\kappa(t)|^2}. \end{aligned} \quad (47)$$

In the long time regime, $\kappa(t)$ in Eq. (22) tends to zero. Then the position variance is linear dependent on time t. Thus in the case of the non-Markovian dephasing coin, the variance grows linearly with time and a QW with a non-Markovian dephasing coin shows a diffusive spread in the long time limit.

In the long time limit, we can obtain the analytical results of the position variance of a non-Markovian decoherent QW.

In Fig. 5, we can see that for the first several steps, the numerical and analytical results agree with each other. Whereas for the long time, the variance becomes linearly dependent on time t, showing a diffusive spread, just as in the classical case, though the rate of growth will be greater than that for the RW.

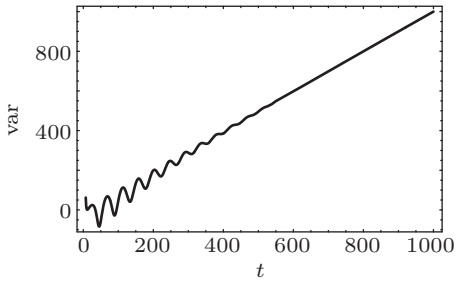


Fig. 5. Analytical result of the variance of position distribution for a QW on a line with a non-Markovian dephasing coin in the long time regime. Here, we choose $g_0 = 1$ and $\eta = 0.01$ for the calculations.

6. Conclusion

In this paper, we presented how non-Markovian decoherence can influence the evolution of a quantum walker on a line, and showed the different behavior from the ballistic and diffusive spreads. In addition, the correlation between the walker and the QW coin also varies in a different way under non-Markovian decoherence. We used MID and QD as measures of the quantum correlations, and analyzed the dynamics and quantum correlation of the quantum walker and his coin. QD will be a valuable measurement method, and MID also suffices to show the pure quantum correlation induced by having non-Markovian dephasing on the coin state. The memory effect caused by information flowing back to the system during a non-Markovian process decreases the quantum correlation quickly, which is later interrupted by revivals. In the long time limit, the quantum correlation disappears with $\kappa(t)$ tending to zero, which results in a transition from a QW with ballistic scaling to an RW with diffusive scaling.

We draw the conclusions as follows: in the short time regime, the behavior of a QW with a non-Markovian dephasing coin oscillates between ballistic and diffusive spreading behavior, and the quantum correlation between the walker and the coin collapses and revives due to the memory effect. Whereas in the long time limit, a QW with non-Markovian decoherence does not differ too much from that with Markovian decoherence. Thus, our analysis is quite valuable in that we characterize this QW on a line with a non-Markovian dephasing coin carefully and devise appropriate, meaningful QD and MID as measures to study the quantum correlation dynamics for this system.

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