Multi-symplectic scheme for the coupled Schrödinger–Boussinesq equations*

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(Received 10 March 2013; revised manuscript received 8 April 2013)

In this paper, a multi-symplectic Hamiltonian formulation is presented for the coupled Schrödinger–Boussinesq equations (CSBE). Then, a multi-symplectic scheme of the CSBE is derived. The discrete conservation laws of the Langmuir plasmon number and total perturbed number density are also proved. Numerical experiments show that the multi-symplectic scheme simulates the solitary waves for a long time, and preserves the conservation laws well.

Keywords: coupled Schrödinger–Boussinesq equations, multi-symplectic scheme, conservation laws, numerical experiments

PACS: 02.60.Cb, 02.70.Bf, 02.30.Jr **DOI:** 10.1088/1674-1056/22/7/070201

1. Introduction

This work is aimed at developing a multi-symplectic scheme for the coupled Schrödinger–Boussinesq equations (CSBE) as

$$\begin{cases} \mathrm{i}u_t - u_{xx} + uv = 0, \quad x \in \mathbf{R}, \quad t > 0, & (1) \\ v_{tt} - v_{xx} + \alpha(v^2)_{xx} + \beta v_{xxxx} = (|u|^2)_{xx}, \\ \quad x \in \mathbf{R}, \quad t > 0, & (2) \end{cases}$$

where $\mathrm{i} = \sqrt{-1}$, $u(x,t)$ represents the complex short-wave amplitude, $v(x,t)$ the long-wave amplitude, and α and β are real parameters. Equations (1)–(2) were considered as a model of the interactions between short and intermediate long waves, and were originated in describing the dynamics of Langmuir soliton formation, the interaction in a plasma[1–4] and the diatomic lattice system,[5] etc.

The exact solutions of the CSBE have been considered by several authors (see Refs. [2], [3], [5]–[8] and references therein). However, only a few numerical methods exist in the literature for solving the CSBE. Zhang *et al.*[9] studied a conservative difference scheme to solve the CSBE, and Bai *et al.*[10,11] proposed the time-splitting Fourier spectral method and the quadratic B-spline finite element method of the CSBE.

Numerical methods are important tools in investigating the special characters or structures of Hamiltonian PDEs. In recent decades, multi-symplectic methods,[12] which preserve multi-symplectic geometric structures under numerical discretizations, have been widely developed and studied by many researchers to solve Hamiltonian PDEs, such as nonlinear Schrödinger equation,[13–19] the Zakharov system,[20] the Klein–Gordon–Zakharov equations,[21,22] and the Zakharov–Kuznetsov equation,[23]. Until now, the methods that could preserve the multi-symplectic geometric structure for solving the CSBE have not been found. Therefore, we intend to study the multi-symplectic scheme for the CSBE.

An outline of this paper is as follows. In the next section, we give the initial-boundary value problem and some of the properties of the CSBE. In Section 3, a new multi-symplectic scheme is constructed for the CSBE by discretizing the Hamiltonian formulation. In Section 4, we study the discrete conservation laws of the CSBE, and some numerical experiments are displayed in Section 5. In the last section, we give the conclusions of this paper.

2. Initial-boundary value problem

In this section, we consider the CSBE with initial-boundary value conditions as

$$\begin{cases} \mathrm{i}u_t - u_{xx} + uv = 0, \quad a < x < b, \quad t > 0, & (3) \\ v_{tt} - v_{xx} + \alpha(v^2)_{xx} + \beta v_{xxxx} = (|u|^2)_{xx}, \\ \quad a < x < b, \quad t > 0, & (4) \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \\ v_t(x,0) = v_1(x), \quad a \le x \le b, & (5) \\ u(a,t) = u(b,t) = 0, \quad v(a,t) = v(b,t) = 0, \quad t \ge 0, & (6) \\ u_x(a,t) = u_x(b,t) = 0, \quad v_x(a,t) = v_x(b,t) = 0, \\ \quad t \ge 0, & (7) \end{cases}$$

where $u_0(x), v_0(x)$, and $v_1(x)$ are given smooth functions. Moreover, we supplement Eqs. (3)–(7) by imposing the com-

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11271171, 11001072, and 11101381), the Natural Science Foundation of Fujian Province, China (Grant No. 2011J01010), the Fundamental Research Funds for the Central Universities, China, and the Natural Science Foundation of Huaqiao University, China (Grant No. 10QZR21).

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$$u_0(a)=u_0(b),\quad v_0(a)=v_0(b),$$
$$v_1(a)=v_1(b),\quad \int_a^b v_1(x)\,\mathrm{d}x=0. \tag{8}$$

From Eqs. (3) and (4), it is easy to verify that[11]

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b |u(x,t)|^2\,\mathrm{d}x=0,\quad \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b v(x,t)\,\mathrm{d}x=0. \tag{9}$$

Thus, we introduce a new potential function $q(x,t)$, which satisfies

$$q_x(x,t)=\int_a^x v_t(y,t)\,\mathrm{d}y,\quad t\geq 0, \tag{10}$$

and then the problem (3)–(7) is equivalent to the following form:

$$\begin{cases}
\mathrm{i}u_t-u_{xx}+uv=0,\quad a<x<b,\ t>0, & (11)\\
v_t=q_{xx},\quad a<x<b,\ t>0, & (12)\\
q_t-v+\alpha v^2+\beta v_{xx}=|u|^2,\quad a<x<b,\ t>0, & (13)\\
u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\\
v_t(x,0)=v_1(x),\quad a\leq x\leq b, & (14)\\
u(a,t)=u(b,t)=0,\ v(a,t)=v(b,t)=0,\\
q(a,t)=q(b,t)=0,\quad t\geq 0, & (15)\\
u_x(a,t)=u_x(b,t)=0,\ v_x(a,t)=v_x(b,t)=0,\\
q_x(a,t)=q_x(b,t)=0,\quad t\geq 0. & (16)
\end{cases}$$

For the initial-boundary value problem (11)–(16), there are at least three conservation laws.[6]

(i) The Langmuir plasmon number

$$I_u(t)=\int_a^b |u|^2\,\mathrm{d}x=\int_a^b |u_0|^2\,\mathrm{d}x=I_u(0),\quad t\geq 0. \tag{17}$$

(ii) The total perturbed number density

$$I_v(t)=\int_a^b v\,\mathrm{d}x=\int_a^b v_0\,\mathrm{d}x=I_v(0),\quad t\geq 0. \tag{18}$$

(iii) The total energy

$$I_{\mathrm{H}}(t)=\int_a^b\left[(u_x)^2+\frac{1}{2}(q_x)^2+\frac{1}{2}v^2+\frac{\beta}{2}(v_x)^2-\frac{\alpha}{3}v^3+v|u|^2\right]\mathrm{d}x=I_{\mathrm{H}}(0),\quad t\geq 0. \tag{19}$$

Physically, these conserved laws play major roles in all physical theories, and can be useful tools for qualitative analysis.

3. Numerical scheme of the CSBE

Introducing the canonical momenta $v_t=p_x$, $q_x=p$, $v_x=w$, $a_x=c$, $b_x=e$, we can obtain the multi-symplectic Hamiltonian formulation of the CSBE as

$$\boldsymbol{M}\boldsymbol{z}_t+\boldsymbol{K}\boldsymbol{z}_x=\nabla_{\boldsymbol{z}}S(\boldsymbol{z}),\quad \boldsymbol{z}\in\mathbf{R}^n,\ (x,t)\in\mathbf{R}^2, \tag{20}$$

where

$$\boldsymbol{M}=\begin{bmatrix}
0&2&0&0&0&0&0&0\\
-2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0\\
0&0&0&0&-1&0&0&0\\
0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0
\end{bmatrix},$$

$$\boldsymbol{K}=\begin{bmatrix}
0&0&2&0&0&0&0&0\\
0&0&0&2&0&0&0&0\\
-2&0&0&0&0&0&0&0\\
0&-2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&\beta\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&-1&0&0\\
0&0&0&0&-\beta&0&0&0
\end{bmatrix}$$

with state variable $\boldsymbol{z}=(a,b,c,e,v,q,p,w)^{\mathrm{T}}$, and the Hamiltonian

$$S=(a^2+b^2)v-(c^2+e^2)-\frac{\alpha}{3}v^3+\frac{1}{2}v^2-\frac{1}{2}p^2-\frac{1}{2}\beta w^2.$$

According to the multi-symplectic geometry theory, the Hamiltonian formulation (20) satisfies the following local conservation laws.

(i) Multi-symplectic conservation law

$$\frac{\partial}{\partial t}\omega+\frac{\partial}{\partial x}\kappa=0, \tag{21}$$

where ω and κ are the pre-symplectic two-forms

$$\omega=2\mathrm{d}a\wedge\mathrm{d}b+\mathrm{d}v\wedge\mathrm{d}q,$$
$$\kappa=2\mathrm{d}a\wedge\mathrm{d}c+2\mathrm{d}b\wedge\mathrm{d}e+\beta\mathrm{d}v\wedge\mathrm{d}w+\mathrm{d}q\wedge\mathrm{d}p.$$

(ii) Local energy conservation law

$$\frac{\partial}{\partial t}E(\boldsymbol{z})+\frac{\partial}{\partial x}F(\boldsymbol{z})=0, \tag{22}$$

where the energy density $E(\boldsymbol{z})$ and the energy flux $F(\boldsymbol{z})$ are

$$E(\boldsymbol{z})=(a^2+b^2)v-\frac{\alpha}{3}v^3+\frac{1}{2}v^2-\frac{1}{2}p^2-ac_x-be_x-\frac{\beta}{2}vw_x,$$
$$F(\boldsymbol{z})=ac_t+be_t-ca_t-eb_t-\frac{\beta}{2}v_tw+\frac{\beta}{2}vw_t.$$

(iii) Local momentum conservation law

$$\frac{\partial}{\partial t}I(\boldsymbol{z})+\frac{\partial}{\partial x}G(\boldsymbol{z})=0, \tag{23}$$

where the momentum density $I(\boldsymbol{z})$ and the momentum flux $G(\boldsymbol{z})$ are

$$I(\boldsymbol{z})=ae-bc-\frac{1}{2}qw+\frac{1}{2}vp,$$
$$G(\boldsymbol{z})=(a^2+b^2)v-(c^2+e^2)-\frac{\alpha}{3}v^3+\frac{1}{2}v^2$$

$$-\frac{1}{2}p^2-\frac{\beta}{2}w^2+ba_t-ab_t+\frac{1}{2}qv_t-\frac{1}{2}vq_t.$$

As usual, we introduce some notations: $x_j=jh$, $t_n=n\tau$, $j=0,1,2,\cdots,J$, $n=0,1,2,\cdots$, where $h=(b-a)/J$ and τ are the space and time step length, respectively. The approximation of the value of the function $u(x,t)$ at the node (x_j,t_n) is denoted by u_j^n, etc.

Using the central box scheme to discretize the Hamiltonian formulation (20), we obtain the following multi-symplectic scheme:[12]

$$\begin{aligned}&\frac{1}{\tau}\boldsymbol{M}(\boldsymbol{z}_{j+1/2}^{n+1}-\boldsymbol{z}_{j+1/2}^{n})+\frac{1}{h}\boldsymbol{K}(\boldsymbol{z}_{j+1}^{n+1/2}-\boldsymbol{z}_{j}^{n+1/2})\\&=\boldsymbol{\nabla}_{\boldsymbol{z}}S(\boldsymbol{z}_{j+1/2}^{n+1/2}),\end{aligned}\tag{24}$$

where

$$\begin{aligned}\boldsymbol{z}_{j+1/2}^{n+1/2}&=\frac{1}{2}(\boldsymbol{z}_{j+1/2}^{n+1}+\boldsymbol{z}_{j+1/2}^{n})=\frac{1}{2}(\boldsymbol{z}_{j+1}^{n+1/2}+\boldsymbol{z}_{j}^{n+1/2})\\&=\frac{1}{4}(\boldsymbol{z}_{j}^{n}+\boldsymbol{z}_{j}^{n+1}+\boldsymbol{z}_{j+1}^{n}+\boldsymbol{z}_{j+1}^{n+1}).\end{aligned}$$

Scheme (24) has the following discrete multi-symplectic conservation law:

$$\begin{aligned}&\frac{1}{\tau}\Big[2(\mathrm{d}a_{j+1/2}^{n+1}\wedge\mathrm{d}b_{j+1/2}^{n+1}-\mathrm{d}a_{j+1/2}^{n}\wedge\mathrm{d}b_{j+1/2}^{n})\\&+(\mathrm{d}v_{j+1/2}^{n+1}\wedge\mathrm{d}q_{j+1/2}^{n+1}-\mathrm{d}v_{j+1/2}^{n}\wedge\mathrm{d}q_{j+1/2}^{n})\Big]\\&+\frac{1}{h}\Big[2(\mathrm{d}a_{j+1}^{n+1/2}\wedge\mathrm{d}c_{j+1}^{n+1/2}-\mathrm{d}a_{j}^{n+1/2}\wedge\mathrm{d}c_{j}^{n+1/2})\\&+2(\mathrm{d}b_{j+1}^{n+1/2}\wedge\mathrm{d}e_{j+1}^{n+1/2}-\mathrm{d}b_{j}^{n+1/2}\wedge\mathrm{d}e_{j}^{n+1/2})\\&+\beta(\mathrm{d}v_{j+1}^{n+1/2}\wedge\mathrm{d}w_{j+1}^{n+1/2}-\mathrm{d}v_{j}^{n+1/2}\wedge\mathrm{d}w_{j}^{n+1/2})\\&+(\mathrm{d}q_{j+1}^{n+1/2}\wedge\mathrm{d}p_{j+1}^{n+1/2}-\mathrm{d}q_{j}^{n+1/2}\wedge\mathrm{d}p_{j}^{n+1/2})\Big]=0.\end{aligned}\tag{25}$$

The component formulation of the multi-symplectic scheme (24) is

$$\begin{cases}\dfrac{2}{\tau}(b_{j+1/2}^{n+1}-b_{j+1/2}^{n})+\dfrac{2}{h}(c_{j+1}^{n+1/2}-c_{j}^{n+1/2})\\=2v_{j+1/2}^{n+1/2}a_{j+1/2}^{n+1/2},&(26)\\-\dfrac{2}{\tau}(a_{j+1/2}^{n+1}-a_{j+1/2}^{n})+\dfrac{2}{h}(e_{j+1}^{n+1/2}-e_{j}^{n+1/2})\\=2v_{j+1/2}^{n+1/2}b_{j+1/2}^{n+1/2},&(27)\\-\dfrac{2}{h}(a_{j+1}^{n+1/2}-a_{j}^{n+1/2})=-2c_{j+1/2}^{n+1/2},&(28)\\-\dfrac{2}{h}(b_{j+1}^{n+1/2}-b_{j}^{n+1/2})=-2e_{j+1/2}^{n+1/2},&(29)\\\dfrac{1}{\tau}(q_{j+1/2}^{n+1}-q_{j+1/2}^{n})+\dfrac{\beta}{h}(w_{j+1}^{n+1/2}-w_{j}^{n+1/2})\\=-\alpha(v_{j+1/2}^{n+1/2})^2+v_{j+1/2}^{n+1/2}+(a_{j+1/2}^{n+1/2})^2+(b_{j+1/2}^{n+1/2})^2,&(30)\\-\dfrac{1}{\tau}(v_{j+1/2}^{n+1}-v_{j+1/2}^{n})+\dfrac{1}{h}(p_{j+1}^{n+1/2}-p_{j}^{n+1/2})=0,&(31)\\-\dfrac{1}{h}(q_{j+1}^{n+1/2}-q_{j}^{n+1/2})=-p_{j+1/2}^{n+1/2},&(32)\\-\dfrac{\beta}{h}(v_{j+1}^{n+1/2}-v_{j}^{n+1/2})=-\beta w_{j+1/2}^{n+1/2}.&(33)\end{cases}$$

In the numerical calculation process, we only use the original variables u and v. Therefore, by removing the intermediate variables of Eqs. (26)–(33), we can obtain an equivalent multi-symplectic scheme

$$\begin{cases}\mathrm{i}\dfrac{\delta_t u_{j-1}^n+2\delta_t u_j^n+\delta_t u_{j+1}^n}{4\tau}-\dfrac{\delta_x^2 u_j^{n+1}+\delta_x^2 u_j^n}{2h^2}+\dfrac{1}{2}\Big[u_{j-1/2}^{n+1/2}\cdot v_{j-1/2}^{n+1/2}+u_{j+1/2}^{n+1/2}\cdot v_{j+1/2}^{n+1/2}\Big]=0,&(34)\\\dfrac{\delta_t^2 v_{j-2}^n+4\delta_t^2 v_{j-1}^n+6\delta_t^2 v_j^n+4\delta_t^2 v_{j+1}^n+\delta_t^2 v_{j+2}^n}{16\tau^2}+\beta\dfrac{\delta_x^4 v_j^{n-1}+2\delta_x^4 v_j^n+\delta_x^4 v_j^{n+1}}{4h^4}\\-\dfrac{(v_{j-2}^{n-1}-2v_j^{n-1}+v_{j+2}^{n-1})+2(v_{j-2}^{n}-2v_j^{n}+v_{j+2}^{n})+(v_{j-2}^{n+1}-2v_j^{n+1}+v_{j+2}^{n+1})}{16h^2}\\+\dfrac{\alpha}{4h^2}\Big[(v_{j-3/2}^{n-1/2})^2-(v_{j-1/2}^{n-1/2})^2-(v_{j+1/2}^{n-1/2})^2+(v_{j+3/2}^{n-1/2})^2+(v_{j-3/2}^{n+1/2})^2-(v_{j-1/2}^{n+1/2})^2-(v_{j+1/2}^{n+1/2})^2+(v_{j+3/2}^{n+1/2})^2\Big]\\=\dfrac{1}{4h^2}\Big[|u_{j-3/2}^{n-1/2}|^2-|u_{j-1/2}^{n-1/2}|^2-|u_{j+1/2}^{n-1/2}|^2+|u_{j+3/2}^{n-1/2}|^2+|u_{j-3/2}^{n+1/2}|^2-|u_{j-1/2}^{n+1/2}|^2-|u_{j+1/2}^{n+1/2}|^2+|u_{j+3/2}^{n+1/2}|^2\Big],&(35)\end{cases}$$

where $\delta_t u_j^n=u_j^{n+1}-u_j^n$, $\delta_t^2 u_j^n=u_j^{n+1}-2u_j^n+u_j^{n-1}$, $\delta_x^2 u_j^n=u_{j-1}^n-2u_j^n+u_{j+1}^n$, $\delta_x^4 u_j^n=u_{j-2}^n-4u_{j-1}^n+6u_j^n-4u_{j+1}^n+u_{j+2}^n$, etc.

4. Properties of the multi-symplectic scheme

In this section, we consider the discrete analogues of the conservation laws (17) and (18).

Theorem 1 Under the periodic boundary conditions, the multi-symplectic schemes (34) and (35) have the following discrete Langmuir plasmon number:

$$h\sum_{j=1}^{J}|u_{j+1/2}^{n+1}|^2=h\sum_{j=1}^{J}|u_{j+1/2}^{n}|^2.\tag{36}$$

Proof Equation (34) can be rewritten as

$$\begin{aligned}&\frac{\mathrm{i}}{2\tau}\Big[(u_{j+1/2}^{n+1}-u_{j+1/2}^{n})+(u_{j-1/2}^{n+1}-u_{j-1/2}^{n})\Big]\\&-\frac{1}{h^2}\Big[(u_{j+1}^{n+1/2}+u_{j-1}^{n+1/2})-2u_j^{n+1/2}\Big]\\&+\frac{1}{2}\Big[u_{j+1/2}^{n+1/2}v_{j+1/2}^{n+1/2}+u_{j-1/2}^{n+1/2}v_{j-1/2}^{n+1/2}\Big]=0.\end{aligned}\tag{37}$$

Multiplying Eq. (37) by $\overline{u_j^{n+1/2}}$ and summing over j, we have

$$I - II + III = 0. \tag{38}$$

Considering the periodic boundary conditions, each term of Eq. (38) can be computed as

$$\begin{aligned} I &= \frac{\mathrm{i}}{2\tau}\sum_{j=1}^{J}\Big[(u_{j+1/2}^{n+1}-u_{j+1/2}^{n})\overline{u_j^{n+1/2}} \\ &\quad +(u_{j-1/2}^{n+1}-u_{j-1/2}^{n})\overline{u_j^{n+1/2}}\Big] \\ &= \frac{\mathrm{i}}{2\tau}\Bigg[\sum_{j=1}^{J}(u_{j+1/2}^{n+1}-u_{j+1/2}^{n})\overline{u_j^{n+1/2}} \\ &\quad +\sum_{j=1}^{J}(u_{j+1/2}^{n+1}-u_{j+1/2}^{n})\overline{u_{j+1}^{n+1/2}} \\ &\quad +(u_{\frac{1}{2}}^{n+1}-u_{\frac{1}{2}}^{n})\overline{u_1^{n+1/2}}-(u_{J+\frac{1}{2}}^{n+1}-u_{J+\frac{1}{2}}^{n})\overline{u_{J+1}^{n+1/2}}\Bigg] \\ &= \frac{\mathrm{i}}{2\tau}\sum_{j=1}^{J}(u_{j+1/2}^{n+1}-u_{j+1/2}^{n})(\overline{u_j^{n+1/2}}+\overline{u_{j+1}^{n+1/2}}) \\ &= \frac{\mathrm{i}}{2\tau}\sum_{j=1}^{J}\left(u_{j+1/2}^{n+1}-u_{j+1/2}^{n}\right)\left(\overline{u_{j+1/2}^{n+1}}+\overline{u_{j+1/2}^{n}}\right) \\ &= \frac{\mathrm{i}}{2\tau}\sum_{j=1}^{J}\Big[(|u_{j+1/2}^{n+1}|^2-|u_{j+1/2}^{n}|^2) \\ &\quad +\left(u_{j+1/2}^{n+1}\overline{u_{j+1/2}^{n}}-u_{j+1/2}^{n}\overline{u_{j+1/2}^{n+1}}\right)\Big], \end{aligned} \tag{39}$$

$$\begin{aligned} II &= \frac{1}{h^2}\sum_{j=1}^{J}\left[(u_{j+1}^{n+1/2}+u_{j-1}^{n+1/2})\overline{u_j^{n+1/2}}-2u_j^{n+1/2}\overline{u_j^{n+1/2}}\right] \\ &= \frac{1}{h^2}\Bigg[\sum_{j=1}^{J}(u_{j+1}^{n+1/2}\overline{u_j^{n+1/2}} \\ &\quad +u_j^{n+1/2}\overline{u_{j+1}^{n+1/2}})-2\sum_{j=1}^{J}|u_j^{n+1/2}|^2\Bigg], \end{aligned} \tag{40}$$

$$\begin{aligned} III &= \frac{1}{2}\sum_{j=1}^{J}\left[u_{j+1/2}^{n+1/2}v_{j+1/2}^{n+1/2}\overline{u_j^{n+1/2}}+u_{j-1/2}^{n+1/2}v_{j-1/2}^{n+1/2}\overline{u_j^{n+1/2}}\right] \\ &= \frac{1}{2}\sum_{j=1}^{J}v_{j+1/2}^{n+1/2}u_{j+1/2}^{n+1/2}(\overline{u_j^{n+1/2}}+\overline{u_{j+1}^{n+1/2}}) \\ &= \sum_{j=1}^{J}v_{j+1/2}^{n+1/2}|u_{j+1/2}^{n+1/2}|^2. \end{aligned} \tag{41}$$

Substituting Eqs. (39)–(41) into Eq. (38) and taking the imaginary part, we obtain Eq. (36).

Theorem 2 Under the periodic boundary conditions, the multi-symplectic schemes (34) and (35) have the following discrete total perturbed number density:

$$h\sum_{j=1}^{J}v_{j+1/2}^{n+1}=h\sum_{j=1}^{J}v_{j+1/2}^{n}. \tag{42}$$

Proof Summing Eq. (31) over j, we obtain

$$\sum_{j=1}^{J}\frac{v_{j+1/2}^{n+1}-v_{j+1/2}^{n}}{\tau}=\sum_{j=1}^{J}\frac{p_{j+1}^{n+1/2}-p_j^{n+1/2}}{h}=0, \tag{43}$$

where we have used the periodic boundary conditions. From Eq. (43), we can easily obtain conclusion (42).

5. Numerical experiments

In this section, we investigate some of the numerical results of the multi-symplectic scheme presented in the previous section. We mainly focus on the issues of long-time simulations and conservation laws.

We take $\alpha=-3$, $\beta=1$ in Eq. (2), and consider the following initial conditions:[8]

$$u(x,0)=u_0(x)=A\,\mathrm{sech}\,(Bx)\exp\left(\mathrm{i}\frac{1}{2}\gamma x\right), \tag{44}$$

$$v(x,0)=v_0(x)=-2B^2\,\mathrm{sech}^2\,(Bx), \tag{45}$$

$$v_t(x,0)=v_1(x)=-2\gamma B^2\,\mathrm{sech}^2\,(Bx)\tanh\,(Bx), \tag{46}$$

where γ is an arbitrary parameter, and $A=\sqrt{2B^2(1-\gamma^2-4\beta B^2)}$.

In the process of numerical experiments, we choose $\gamma=9/10$, $B=1/30$, and select a sufficiently large interval [–300, 300] such that the solutions decay rapidly to zero as $|x|\to\infty$. In such a case, the periodic boundary conditions do not introduce a significant error relative to the whole space problem.

5.1. Long-time simulation test

To study the long-time simulated situation of the multi-symplectic scheme, we take $\tau=1/10$, $h=1/10$, and evaluate until $t=500$. Figure 1 shows the evolution of the solitary waves of $|u|$ and v with time. From the figures, we find that the motion of the solitary waves is well simulated after long time evaluation, and the oscillatory phenomena has not appeared.

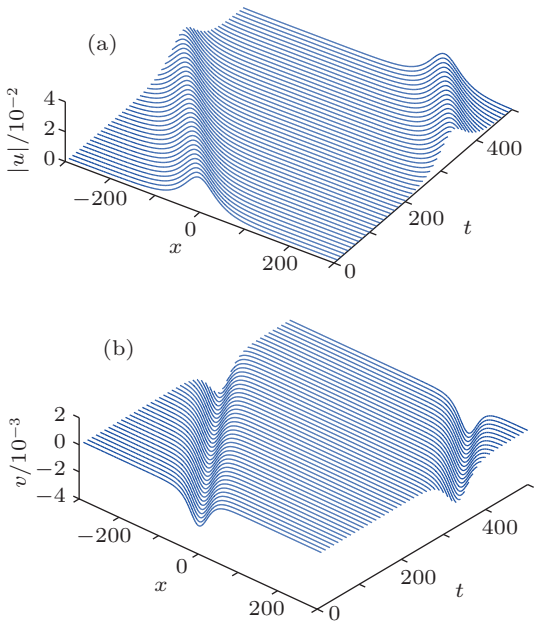


Fig. 1. (color online) The evolution of the solitary waves: (a) $|u|$, (b) v.

5.2. Conservation law test

To examine the conservation laws of the multi-symplectic scheme, we measure the discretization errors for the Langmuir plasmon number, the total perturbed number density, and the total energy by

$$\mathrm{Error}_u^n = I_u^n - I_u^0, \quad \mathrm{Error}_v^n = I_v^n - I_v^0, \quad \mathrm{Error}_{\mathrm{H}}^n = I_{\mathrm{H}}^n - I_{\mathrm{H}}^0,$$

respectively, where

$$\begin{aligned} I_u^n &= h\sum_{j=1}^{J} |u_{j+1/2}^n|^2, \\ I_v^n &= h\sum_{j=1}^{J} v_{j+1/2}^n, \\ I_{\mathrm{H}}^n &= h\sum_{j=1}^{J} \left[\left(\frac{u_{j+1}^n - u_j^n}{h} \right)^2 \right. \\ &\quad + \frac{h^2}{8}\left(\frac{v_j^n - v_j^{n-1}}{\tau} + \frac{v_{j-1}^n - v_{j-1}^{n-1}}{\tau} \right)^2 \\ &\quad + \frac{1}{2}(v_{j+1/2}^n)^2 + \frac{\beta}{2}\left(\frac{v_{j+1}^n - v_j^n}{h} \right)^2 \\ &\quad \left. - \frac{\alpha}{3}(v_{j+1/2}^n)^3 + v_{j+1/2}^n |u_{j+1/2}^n|^2 \right]. \end{aligned}$$

We also take $\tau = 1/10$, $h = 1/10$, and evaluate $t = 500$. In Fig. 2, we give the discretization errors of the conservation laws with the evolution of time. It can be seen from the experiments that the multi-symplectic scheme remains stable for the Langmuir plasmon number, the total perturbed number density, and the total energy. The round-off error increases because of the increment in the computational cost.

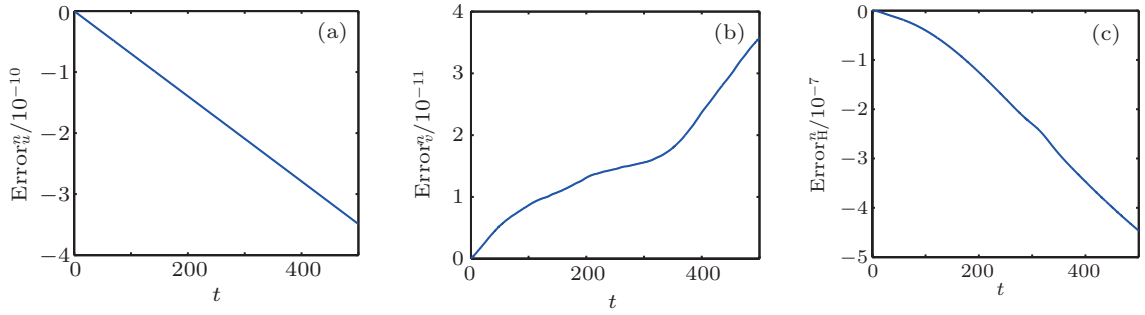


Fig. 2. (color online) The discretization errors of the conservation laws: (a) the Langmuir plasmon number, (b) the total perturbed number density, and (c) the total energy.

6. Conclusions

In this paper, for the CSBE with periodic boundary conditions or in the whole space with the initial data decaying to zero sufficiently fast as $|x| \to \infty$, which can be approximated in a bounded domain with periodic boundary conditions, we obtain a multi-symplectic scheme using the center discrete method. Two discrete conservation laws for the scheme are proved. The numerical results indicate that the proposed scheme can keep the invariants well, which is coincident with the theoretical results.

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