Spin-correlation function of the fully frustrated Ising model and $\pm J$ Ising spin glass on a square lattice*

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In this work we study the correlation function of the ground state of a two-dimensional fully frustrated Ising model as well as spin glass. The Pfaffian method is used to calculate free energy and entropy as well as the correlation function. We estimate the exponent of spin correlation function for the fully frustrated model and spin glass. In this paper an overview of the latest results on the spin correlation function is presented.

Keywords: spin glass, correlation function, exponent, square lattice

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1. Introduction

The purpose of this paper is to present some computational results for the correlation function brought about in a two-dimensional fully frustrated Ising system by introducing ferromagnetic defects. In this model, we replace antiferromagnetic bonds by ferromagnetic bonds in each unit cell randomly. We find that some results are different as the antiferromagnetic bonds increase in our model. These results are relevant to a particular line of research on spin glass. Spin glass is characterized by frustration and randomness. There have been a number of studies that have sought to introduce randomness into an initially non-random frustrated system by means of defects. The starting model is fully frustrated with a highly degenerate ground state and the replacement of magnetic sites with a non-magnetic species reduces frustration but inserts randomness.

The canonical model with $p = 0.5$ on a square lattice has been studied the most, and it is well received that spin glass behavior occurs at zero temperature[1–6] and persists down to a critical probability p_c of about 0.11.[7–18] Good agreement for the values of ground state energy and entropy has been obtained by many researchers. However, up to the present there has been very little development in extending the method to a direct calculation of the spin correlation at zero temperature, and most of the results have been obtained by extrapolating the zero-temperature results.

The present authors have developed an approach[12,19–21] based on a Pfaffian matrix, which appears to capture the essence of the physics of the $\pm J$ system. The procedure enables certain quantities such as the ground state free energy and entropy to be calculated for very large lattice. Spin correlations for a fully frustrated model and the bimodal Ising spin glass are expected to decay algebraically according to

$$[\langle S_0, S_R\rangle^2]_{\text{ave}} \sim R^{-\eta} \tag{1}$$

at the critical temperature.

The present work is to develop the formalism[20] introduced earlier to calculate the ground state energy and entropy for this system. We give an overview of that formalism in Section 2 and results and discussion in Section 3. The conclusion is then drawn in Section 4 finally.

2. Formalism

2.1. Model

One of the simple models used extensively in the study of frustration is the square lattice nearest-neighbor Ising model with staggered interaction on one direction. When interactions are of the same absolute value, the model is said to be fully frustrated.

The two-dimensional fully frustrated Ising model, or Villain model,[22] consists of Ising spins on a square lattice with nearest-neighbor bonds $|J_{ij}| = 1$ and boundary conditions are periodic; in the x direction all bonds are ferromagnetic, while in the y direction columns of bonds are alternately ferromagnetic and antiferromagnetic.

The Hamiltonian is thus given by

$$H = -\sum_{\langle ij\rangle} J_{ij} S_i S_j, \tag{2}$$

where $S_i = \pm 1$ represent Ising spins on a square lattice with $N = L^2$ spins. The system is fully frustrated, i.e., the product of the bonds around each plaquette of the system is negative,

$$\prod J_{ij} = -1. \tag{3}$$

There are many different ways to do this and most of their gauges are invariant. The exchange interactions J_{ij} are un-

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correlated quenched random variables, taking values $\pm J$ with probability distribution

$$P(J_{ij}) = p\delta(J_{ij} - J) + (1-p)\delta(J_{ij} + J). \tag{4}$$

In the following we set $J = 1$ without loss of generality. For $p = 0.0$ we recover the standard Ising model, while for $p = 0.1$, 0.2, and 0.5 we obtain the bimodal Ising spin-glass model.

2.2. Method

The method used in the calculation is based on the combinatorial expansion of the two-dimensional Ising model. The combinatorial, or Pfaffian method[23] is particularly well suited to study the disordered planar Ising system. Essentially, each lattice site is decorated with four fermions which have interactions across bonds and intrasite interactions.

The partition function for a disorder planar Ising model can be expressed in the form of

$$Z = 2^N \left[\prod_{\langle ij \rangle} \cosh(J_{ij}/kT)\right] (\det D)^{1/2}, \tag{5}$$

where the product runs over all nearest neighbor bonds J_{ij} on the N site lattice and D is a skew-symmetric $(4N \times 4N)$ matrix. The square root of the determinant of D is the major feature and precisely the Pfaffian.[23]

In terms of determinants, the correlation function can be expressed as

$$\langle S_0, S_R \rangle = \frac{\det C}{\det D} \prod_{\text{path}} t_{ij}^2, \tag{6}$$

where the matrix C is the same as D except for the reciprocal defects and $t_{ij} = \tanh(J_{ij}/kT)$ for bonds on the path. At zero temperature, D is a singular matrix with zero eigenvalues whose number is equal to the number of frustrated plaquettes. Theses eigenvalues (which occur in pairs) approach to zero at some power of $\exp(-2J/kT)$.

$$\varepsilon = \pm \frac{1}{2} X \exp(-2Jr/kT), \tag{7}$$

where r is an integer and X is a real number that is independent of T but depends only on the configuration of frustrated plaquettes. The ground state energy and entropy can be expressed exactly as

$$F = -2J + 2J \sum_d r_d, \tag{8}$$

$$M = k \sum_d \ln X_d. \tag{9}$$

The summation runs over all zero eigenstates. A method based on degenerate state perturbation theory was developed[20] to evaluate exactly the r and X, and hence the ground state energy and entropy.

In Ref. [20] $\det D$ was proven to be proportional to the ground state degeneracy. We can generalize this by considering $\det C$ as being proportional to an effective degeneracy for the reciprocal defect system. In fact we can reasonably expect that the first cumulant approximation

$$\langle S_0, S_R \rangle^2 \sim A(R) \exp(-B(R)/L) \tag{10}$$

is valid if L is sufficiently large. This is of course in accordance with finite size scaling theory, where it is expected that $A(R) \sim R^{-\eta}$ and $B(R) \sim R$ if R is sufficiently large.

3. Results and discussion

The computational requirements of the method depend only on the number and distribution of frustrated plaquettes, and this opens up possibilities for studying quite large samples. The spin–spin correlation function of the fully frustrated model and spin glass at zero temperature decays as $R^{-\eta}$ at large distance. We have performed the estimation of correlation on $L \times L$ lattice. The lattices are cylindrically wound but the reciprocal defect path is around the circumference of the cylinder as far as possible from the nested boundaries. We first estimate the correlation function with paths of length R. The big advantage of the approach is that there is no need to extrapolate from finite temperature results as in Monte Carlo methods.

For a non-random fully frustrated model we take one sample for $p = 0.0$. The number of random samples taken is 10^4 for $p = 0.1$ and $p = 0.2$, whereas for $p = 0.5$ it is 10^5. The author in Ref. [24] has found theoretically the result $\eta = 1$ of the fully frustrated model. Instead we adopt the strategy of plotting the data against $1/L$ as displayed in Figs. 1, 3, 5, and 7 for $p = 0.0$, $p = 0.1$, $p = 0.2$, and $p = 0.5$ respectively, effectively fitting the form of Eq. (10). The data extrapolated to $L \to \infty$ are used to extract a value of η. From Figs. 2, 4, 6, and 8, the values of η found are 1.067 ± 0.085, 0.179 ± 0.061, 0.174 ± 0.038, and 0.132 ± 0.009 respectively, which is reasonable agreement with the values found above. The calculation on the $L \times L$ lattice yields a figure that agrees within error bars. Interestingly an earlier calculation[25] reports 0.132 ± 0.009 for $p = 0.5$, which is in agreement with current calculation.

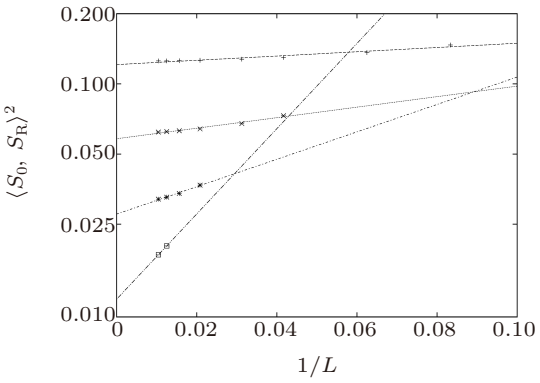


Fig. 1. Squared spin correlations on $L \times L$ lattices with $R = 4$ (pluses), 8 (crosses), 16 (stars), and 32 (squares) for $p = 0.0$. The lines are the best fits to Eq. (10).

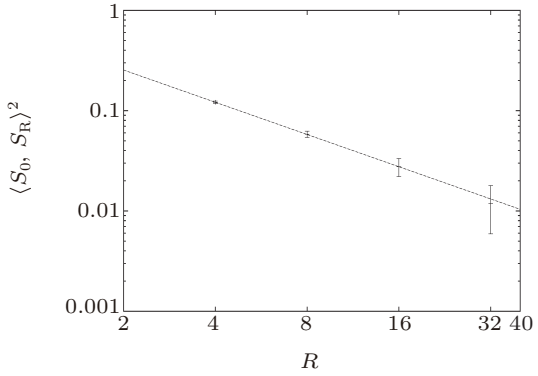


Fig. 2. Fitting function $A(R) \sim R^{-\eta}$ to obtain the value of η for $p = 0.0$.

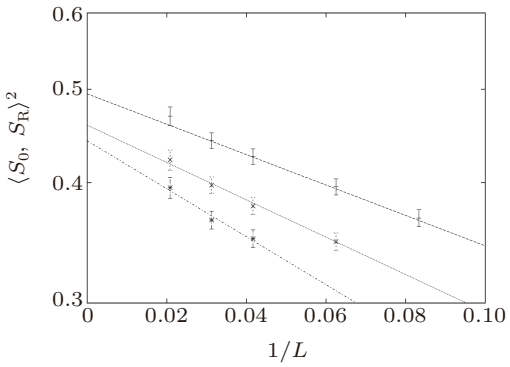


Fig. 3. Squared spin correlations on $L \times L$ lattices with $R = 4$ (pluses), 6 (crosses), and 8 (stars) for $p = 0.1$. The lines are the best fits to Eq. (10).

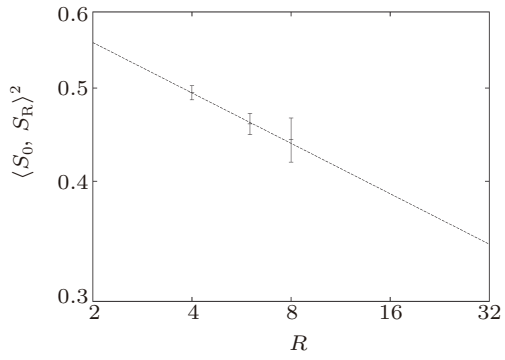


Fig. 4. Fitting function $A(R) \sim R^{-\eta}$ to obtain the value of η for $p = 0.1$.

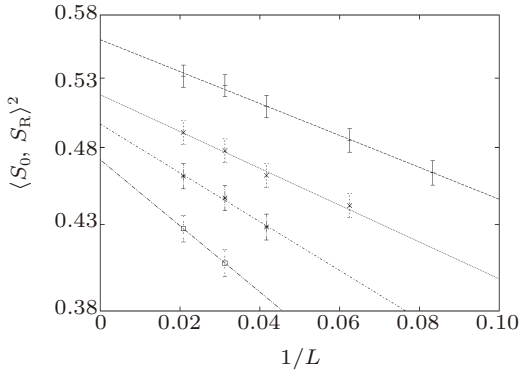


Fig. 5. Squared spin correlations on $L \times L$ lattices with $R = 4$ (plusses), 6 (crosses), 8 (stars), and 12 (squares) for $p = 0.2$. The lines are the best fits to Eq. (10).

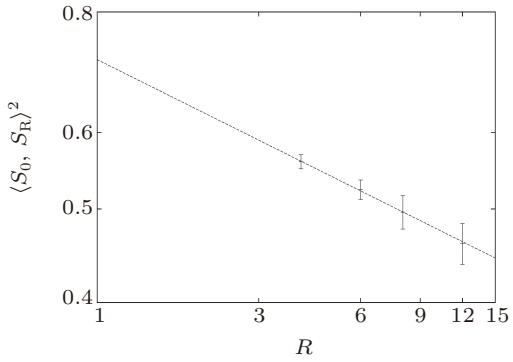


Fig. 6. Fitting function $A(R) \sim R^{-\eta}$ to obtain the value of η for $p = 0.2$.

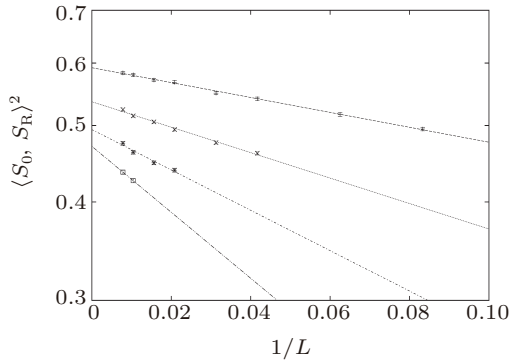


Fig. 7. Squared spin correlation on $L \times L$ lattices with $R = 4$ (plusses), 8 (crosses), 16 (stars), and 32 (squares) for $p = 0.5$. The lines are the best fits to Eq. (10).

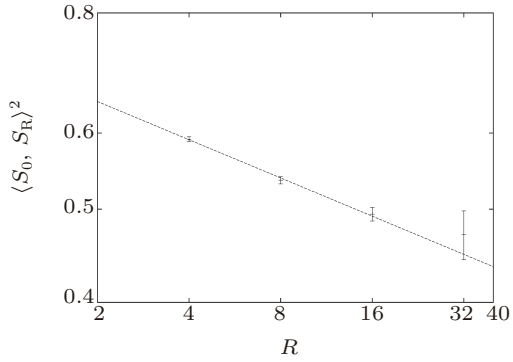


Fig. 8. Fitting function $A(R) \sim R^{-\eta}$ to obtain the value of η for $p = 0.5$.

4. Conclusion

In this paper we numerically studied the spin correlation for a fully frustrated model at zero temperature, showing that there is a mechanism that produces spin glass. Our numerical calculation results show that the exponents are $\eta = 1.067 \pm 0.085$ for the fully frustrated system and $\eta < 1$ for the spin glass.

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