Crystalline undulator radiation and sub-harmonic bifurcation of system

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Looking for new light sources, especially short wavelength laser light sources has attracted widespread attention. This paper analytically describes the radiation of a crystalline undulator field by the sine-squared potential. In the classical mechanics and the dipole approximation, the motion equation of a particle is reduced to a generalized pendulum equation with a damping term and a forcing term. The bifurcation behavior of periodic orbits is analyzed by using the Melnikov method and the numerical method, and the stability of the system is discussed. The results show that, in principle, the stability of the system relates to its parameters, and only by adjusting these parameters appropriately can the occurrence of bifurcation be avoided or suppressed.

Keywords: crystalline undulator radiation, short wavelength laser, Melnikov method, bifurcation, stability

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1. Introduction

One of the development directions of the free-electron laser (FEL) is to obtain a short-wavelength and high power free electron laser. However, attempts to promote free electron lasers to shorter wavelength by using traditional methods has encountered serious challenges in technology. In the 1970s, the Soviet scientist Kumakhov discovered channelling radiation (CR).[1] In the late 1990s it was proposed that a periodically bent crystal obtained by the acoustic method is expected to achieve this goal.[2–8] Charged particles moving in a bent crystal will produce coherent crystalline undulator radiation (CUR) as well as spontaneous CR. The CUR has all the characteristics of the FEL. For the FEL electrons move in a vacuum, but for the CUR electrons move in the lattice field. Because a lattice field is very strong, the energy of the CUR may enter into the X-energy region. However, the radiation encountered more complex problems than the FEL. It was thought that main reason is a technical problem, that is to say, reason why the X-laser or γ-laser cannot be found is attributed to the present technique level. However, the authors believe that the stability of the system needs further consideration. In fact, in order to ensure the stability of the CUR, the motion of the particles must be stable. However, many factors may affect the stability of the system, such as crystal defects, motion damping, radiation loss, lattice thermal vibration, electron multiple scattering, nonlinearity of the system, etc. In the present paper we further consider the effects of damping and nonlinearity on electron motion. To this end, by introducing sine-squared potential, the motion equation of particles is reduced to a pendulum equation with a damping term and a forcing term in classical mechanics and the dipole approximation. By using Jacobian elliptic functions and elliptic integrals, the periodic solution of the system is obtained analytically; by using the Melnikov method and a numerical method the bifurcation behavior of the periodic orbits is analyzed. The results show that the stability of the system relates to its parameters and only by adjusting these parameters appropriately can the occurrence of bifurcation be avoided or suppressed.

The rest of the present paper is organized as follows. In Section 2 the motion equation of particles is derived, and the CUR is discussed. in Section 3 system stability is analyzed by the Melnikov method and the numerical method, and the critical conditions are derived. In Section 4 the physical significance of the critical condition is discussed, and it is pointed out that selection of the amplitude and frequency of the periodic bent crystal must satisfy the stability criteria. In Section 5 it is emphasized that the stability of the system relates to its parameters and only by adjusting these parameters appropriately can the occurrence of bifurcation be avoided or suppressed.

2. Motion equation and the CUR

Assume that the (y, z) plane is the channelling midplane, the x axis is perpendicular to the plane, the y axis and the z axis are located in the this plane, and the z axis is along the beam direction. Strictly speaking, the channelling problem is a three-dimensional problem. Taking into account the transla-

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tional symmetry of the crystal, the coordinate y will not appear. If the energy of the particles in the x direction is not too large, then the energy in the z direction can be regarded as being conserved, so that the three-dimensional problem is reduced into a one-dimensional problem. Assume that the crystal is bent into the following form by acoustic method:

$$x(z) = a\sin(2\pi z/\lambda_u), \tag{1}$$

where a is the "amplitude" and λ_u is a "wavelength" of the crystalline undulator (see Fig. 1).

Two sets of black dots in Fig. 1 represent two adjacent crystal faces of crystal after bending. Note that the particles moving along the z direction will transversely oscillate in the x direction of the lattice field, the charged particles in the lattice field will continually radiate energy, such radiation is called channelling radiation; at the same time, due to the periodic bending of crystal the particles will also produce CUR.

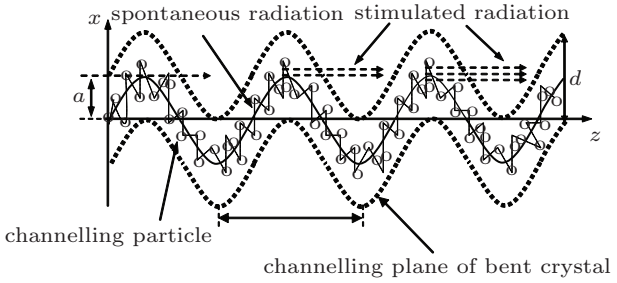


Fig. 1. Schematic representation of a periotically bent crystal, the stimulated radiation (CUR) and the spontaneous radiation (CR).[5]

In the dipole approximation the transverse motion of the particles can be described by classical methods. A particle moving in a bent crystal channel will undergo not only the action of the lattice field, but also the actions of the damping force and the centrifugal force. According to Newton's second law, the motion equation of a particle can be expressed as

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2m\mu_0\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}x}V(x) = pv\kappa(z), \tag{2}$$

where $V(x)$ is the interaction potential between particle and crystal, p the particle momentum, v the particle velocity and $v \approx c$ in the ultra-relativistic case, μ_0 the damping coefficient, $m = m_0\gamma$, m_0 being the particle rest mass, and γ the relativistic factor,

$$\kappa(z) = \kappa_0 \sin\frac{2\pi}{\lambda_u}z \tag{3}$$

is curvature of the bent crystal and given by the 2nd order derivative of Eq. (1), with $\kappa_0 = \Omega_u^2 a$, and $z \approx ct$.

In general, the interaction potential can take different forms, such as, Lindhard potential, Moliere potential, sine-squared potential, etc. In this paper we adopt the sine-squared potential[9,10]

$$V(x) = V_0\sin^2(\pi x/d), \tag{4}$$

where d is the distance between the crystal face, $V_0 = K\beta$ is the depth of the potential well, with β being the potential parameter, and

$$K = \pi Z_1 Z_2 e^2 N d^2, \tag{5}$$

with the Nd^2 being the area density of the crystal atom, Z_1 the atomic number of the incident particles, Z_2 the target atomic number.

Let

$$\xi = 2\pi x/d, \quad \delta = \frac{2\pi^2 K\beta}{md^2}, \quad \tau = \delta^{1/2}t, \tag{6}$$

$$\mu = \mu_0/\delta^{1/2}, \quad \omega = \Omega_u/\delta^{1/2}, \quad f = \frac{2\pi pv\kappa_0}{m\delta d}, \tag{7}$$

equation (2) can be reduced to

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}\tau^2} + 2\mu\frac{\mathrm{d}\xi}{\mathrm{d}\tau} + \sin\xi = f\cos\omega\tau. \tag{8}$$

If the damping term and forcing term are relatively small, the equivalent system of Eq. (8) can be represented as

$$\begin{aligned}\xi' &= \varsigma,\\ \varsigma' &= -\sin\xi + \varepsilon(-2\mu\varsigma + f\sin\omega\tau),\end{aligned} \tag{9}$$

where ε is a small parameter, it indicates that its associated term is small. In the following, we further analyze the stability of the system by the Melnikov method.

3. Stability of the system

3.1. Periodic orbit

If $\varepsilon = 0$, equation (9) can be reduced to the canonical equation as follows:

$$\begin{aligned}\dot{\xi} &= \varsigma,\\ \dot{\varsigma} &= -\sin\xi.\end{aligned} \tag{10}$$

$\xi_s = 0$ and $\xi_u = \pm\pi$ are the equilibrium point and non-equilibrium point of the system. By integrating once, we can obtain the Hamiltonian of the system

$$h = \frac{\varsigma^2}{2} + (1-\cos\xi). \tag{11}$$

According to the value of h, orbits on the phase plane can be divided into three types: the heteroclinic orbit ($h = 2$), the periodic orbits ($0 < h < 2$), and the rotation orbit ($h > 2$). We are concerned only about the periodic orbits because this family orbits describe channelling effects and channeling radiation of charged particles in a bent crystal. When $0 < h < 2$, the periodic orbit of Eq. (10) can be expressed as

$$\left.\begin{aligned}\xi &= 2\arcsin(\kappa\,\mathrm{sn}\,\tau)\\ \varsigma &= 2\kappa\,\mathrm{cn}(\tau)\end{aligned}\right\}, \tag{12}$$

where $\kappa = h/2$, $\kappa \in (0,1)$, $0 < h < 2$, $\mathrm{sn}\,\tau$ and $\mathrm{cn}\,\tau$ are the Jacobian elliptic function. The motion period of particles along this family orbits is

$$T = 4\mathrm{K}(\kappa), \tag{13}$$

where $\mathrm{K}(\kappa)$ is the first kind of complete elliptic integral. When h increases monotonically, the motion period will increase from zero to infinity. The periodic orbits describe the motion behavior of particles around the equilibrium orbit, such particles are called the channelling particles. However, since the system fluctuations or electron multiple scattering, channelling particles can obtain additional energy, so that the energy constant $h > 2$ of the system. If this case happens, the system becomes unstable, and the channelling particle becomes dechannelling particle, the intensity of the CUR will be weakened. In what follows, we further analyze the bifurcation and stability of the system by using the Melnikov method.

3.2. Melnikov method

There exists only $k_{m/n} \in (0,1)$ for arbitrary coprime positive integers m and n, such that

$$T(k_{m/n}) = 4\mathrm{K}(k_{m/n}) = \frac{m}{n}T = \frac{2\pi m}{n\omega},$$

by using the Melnikov method, the Melnikov function of periodic orbit (12) can be constructed as follows:[9–11]

$$\begin{aligned} M^{m/n}(\tau_0) &= \int_0^{mT} \varsigma^o(\tau,\kappa)[Q - 2\mu\varsigma(\tau,\kappa) + f\sin\,\omega(\tau+\tau_0)]\,\mathrm{d}\tau \\ &= \begin{cases} \text{constant unrelated with } \tau_0, & \text{if } n \neq 1 \text{ or } m \text{ is even}, \\ -\dfrac{4f\pi}{\mathrm{ch}\,(\omega\mathrm{K}'(\kappa))}\left[\dfrac{2\mu}{\delta_m} - \sin\,\omega\tau_0\right], & \text{if } n = 1 \text{ and } m \text{ is odd}, \end{cases} \end{aligned} \tag{14}$$

where ς is given by Eq. (12), and $\mathrm{K}'(\kappa) = \mathrm{K}(\kappa')$, $\kappa' = \sqrt{1-\kappa^2}$ is the complement modulus of the elliptic function, $\mathrm{E}(\kappa)$ is the second kind of complete elliptic integral, and

$$\delta_m = \frac{f}{\mathrm{ch}\,(\omega\,\mathrm{K}'(\kappa))\left\{\dfrac{4}{\pi}(\mathrm{E}(\kappa) - (1-\kappa^2)\mathrm{K}(\kappa))\right\}}. \tag{15}$$

In the parameter plane $(\mu,\ f)$, if the condition

$$\mu \leq |\delta_m/2| \tag{16}$$

is satisfied, likewise there exists the periodic orbit in the vicinity of periodic solution (12), and the system becomes unstable.

3.3. System stability

Substituting Eq. (15) into Eq. (16) we have

$$\frac{2\mu}{f} \leq \frac{\pi\,\mathrm{sech}(\omega\mathrm{K}'(\kappa)}{4[\mathrm{E}(\kappa) - \dfrac{\pi}{4}(1-\kappa^2)\mathrm{K}(\kappa)]}. \tag{17}$$

When the condition (17) is satisfied, in the system subharmonic bifurcation appears, but bifurcation means instability. Suitably selecting parameters of the system, can avoid such a situation. From inequality (17) it can be seen that the condition (17) is not always satisfied if the external force $f = 0$, thus the system is stable. If it reaches

$$(f)_{\mathrm{c}} = \frac{8\mu\left[\mathrm{E}(\kappa) - \dfrac{\pi}{4}(1-\kappa^2)\right]\mathrm{K}(\kappa)}{\pi\,\mathrm{sech}(\omega\mathrm{K}'(\kappa))}, \tag{18}$$

the system is in a critical state. If $f < (f)_{\mathrm{c}}$, the system is stable, otherwise the system is unstable.

3.4. Numerical analysis

The system described by Eq. (8) or Eq. (9) is a typical dynamical system. This system has the typical global bifurcation and chaotic behavior, and it also has the dimensionless adjustable parameters μ, ω, and f. When these parameters vary, the system lies in different states. Note that equation (8) can be reduced to a generalized Duffing equation in the small amplitude approximation, and the resonance analysis of the system shows that in the system chaotic phenomena may appear if the excitation frequency is less than the inherent frequency (i.e., $\omega < 1$). Hence we may select parameter $\omega = 2/3$; in addition, we assume that $\mu = 0.25$. The results of numerical analysis are given in Fig. 2. Figure 2 shows the bifurcation and chaotic behavior of the system in $1.05 \leq f \leq 1.15$. From Fig. 2 we can see that the system is only one state if $f < 1.04815$; if $f = 1.04815$, time bifurcation appears in the system, the phase trajectories are closed, the symmetry is changed, and the phase diagram is antisymmetric; with f increasing the cascade bifurcation appears in the system, and the system successively enters into two-period, 4-period, and 8-period time bifurcation states, the phase trajectory is closed after rotating two laps, 4 laps, and 8 laps respectively around the stable point. If $f > 1.0942$ the particles may move randomly in the adjacent potential well, the system enters into the mode-locked state, the system is still stable, what state the system enters into depends on what the initial conditions of the system are. If $f > 1.0973$, the particles may move randomly between different potential wells, and the system enters into a chaotic state. If f continues to increase, for example $f > 1.149$, the system

will re-enter into the chaotic state.

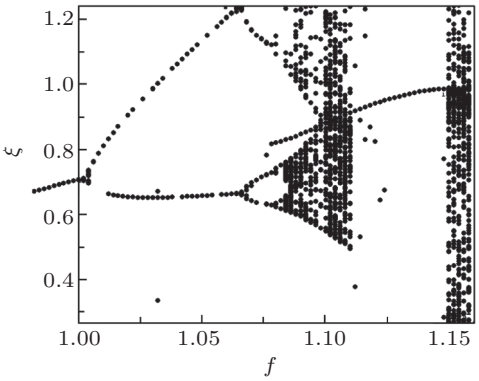


Fig. 2. Bifurcation and chaotic behavior of the system (8) if $1.05 \leq f \leq 1.15$.

4. Physical significance of subharmonic bifurcation

In the parameter plane $(f,\ \mu)$ the conditions of subharmonic bifurcation is given by Eq. (16). In the following, we further discuss the physical significance of subharmonic bifurcation. Substituting Eqs. (6) and (7) into Eq. (16), by using the original parameters, the subharmonic bifurcation conditions of the system can be expressed as

$$\frac{\mu_0}{a\Omega_u^2} < \left| \frac{pv\,\mathrm{sech}\,[\Omega_u d\sqrt{m/2V_0}v]}{\sqrt{2mV_0}\left[\dfrac{4}{\pi}\mathrm{E}(\kappa)-(1-\kappa^2)\mathrm{K}(\kappa)\right]} \right|. \tag{19}$$

If the condition (19) is satisfied, subharmonic bifurcation exists in the system. From Eq. (19) the following points can be seen.

(i) If the crystal is not bent ($a = 0$), the condition (19) is always not satisfied, the system is stable.

(ii) For the greater curvature $\kappa_0 = \Omega_u^2 a$, the condition (19) is more easily satisfied, the system is more unstable. When it reaches

$$(a\Omega_u^2)_{\mathrm{c}} = \frac{\mu_0\sqrt{2mV_0}\left[\dfrac{4}{\pi}\mathrm{E}(\kappa)-(1-\kappa^2)\mathrm{K}(\kappa)\right]}{pv\,\mathrm{sech}[\Omega_u d\sqrt{m/(2V_0)}v]}, \tag{20}$$

the system is in a critical state, and if $a\Omega_u^2 < (a\Omega_u^2)_{\mathrm{c}}$ the system is stable, otherwise the system is unstable. It is noteworthy that Ref. [5] pointed out that intensity of CUR is proportional to a/d, and emphasized that the greater the ratio a/d, the greater the radiation intensity is, if $a/d > 10$ the radiation intensity has significant extremum. However, the above analysis shows that amplitude a and frequency Ω_u may not be selected arbitrarily, their selection must satisfy the stability conditions (19), otherwise the system is unstable. Perhaps this is one of the reasons why the CUR has not yet been found.

(iii) The smaller the damping coefficient μ_0, the more easily the conditions (19) are satisfied and the more unstable the system becomes. In other words, only if the damping is strong enough, can the occurrence of bifurcation be avoided or suppressed. The critical value can be expressed as

$$(\mu_0)_{\mathrm{c}} = \left| \frac{a\Omega_u^2 pv\,\mathrm{sech}\,[\Omega_u d\sqrt{m/2V_0}v]}{\sqrt{2mV_0}\left[\dfrac{4}{\pi}\mathrm{E}(\kappa)-(1-\kappa^2)\mathrm{K}(\kappa)\right]} \right|. \tag{21}$$

If $\mu > (\mu_0)_{\mathrm{c}}$, the system is stable, otherwise the system is unstable. Numerical analysis shows that if the damping coefficient $2\mu = 0.5$, $f < 0.99$, the system is stable. This means that if the selected $2\mu > 0.5$, $f < 0.99$, the system is still stable. The rheoretical analysis is basically consistent with the numerical simulation.

5. Conclusion

The CUR has all the characteristics of the FEL. Because the lattice field is very strong, the energy of the CUR may enter into the X-energy region. In the classical mechanics, the motion equation of the particle is reduced to a generalized pendulum equation with a damping term and forcing term. There are three kinds of orbits in the phase plane, among which the periodic orbit describes the motion behavior of the channelling particle, and its bifurcation determines the stability of the system. The bifurcation behavior of the family orbits is analyzed by Melnikov method and also simulated by the numerical method. The results show that amplitude a and frequency Ω_u of a bent crystal may not be arbitrarily chosen, their choice must satisfy the stability conditions. It should be pointed out that the stability of the system can be one of the reasons why the desired CUR has not been found yet so far; it should also be emphasized that the stability of the system relates to its parameters, and only by adjusting these parameters appropriately, in principle can the occurrence of bifurcation be avoided or suppressed.

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