Unified analytical treatments of qubit–oscillator systems*

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An effective scheme within two displaced bosonic operators with equal positive and negative displacements is extended to study qubit–oscillator systems analytically in a unified way. Many previous analytical treatments, such as generalized rotating-wave approximation (GRWA) [*Phys. Rev. Lett.* **99**, 173601 (2007)] and an expansion in the qubit tunneling matrix element in the deep strong coupling regime [*Phys. Rev. Lett.* **105**, 263603 (2010)] can be recovered straightforwardly within the present scheme. Moreover, further improving GRWA and the extension to the finite-bias case are implemented easily. The algebraic formulae for the eigensolutions are then derived explicitly and uniquely, which work well in a wide range of the coupling strengths, detunings, and static bias including the recent experimentally accessible parameters. The dynamics of the qubit for an oscillator in the ground state is also studied. At the experimentally accessible coupling regime, GRWA can always work well. When the coupling is enhanced to the intermediate regime, only the improving GRWA can give the correct description, while the result of GRWA shows strong deviations. The previous Van Vleck perturbation theory is not valid to describe the dynamics in the present-day experimentally accessible regime, except for the strongly biased cases.

Keywords: numerical exact solution, qubit–oscillator system, dynamics

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1. Introduction

Matter–light interaction is fundamental and ubiquitous in modern physics ranging from quantum optics, quantum information science to condensed matter physics. The simplest paradigm is a two-level atom (qubit) coupled to the electromagnetic mode of a cavity (oscillator).[1] In the strong coupling regime where the coupling strength g/ω (ω is the cavity frequency) between the atom and the cavity mode exceeds the loss rates, the atom and the cavity can repeatedly exchange excitations before the coherence is lost. The Rabi oscillations can be observed in this strong coupling atom–cavity system, which is usually called cavity quantum electrodynamics (QED).[2] Typically, the coupling strength in cavity QED reaches $g/\omega \sim 10^{-6}$. It can be described by the well-known Jaynes–Cummings (JC) model[3] without the rotating-wave approximation (RWA).

Recently, for superconducting qubits,[4–9] a one-dimensional transmission line resonator[10] or an LC circuit[9–16] can play a role of the cavity, which is known today as circuit QED. More recently, an LC resonator inductively coupled to a superconducting qubit[17–19] has been realized experimentally. The qubit–resonator coupling has been strengthened by a few percentages[15,16] and even ten percentages most recently[17–19] from $g/\omega \sim 10^{-3}$ in the earlier realization.[10] Due to the ultra-strong coupling strength $g/\omega \sim 0.1$, the evidence for the breakdown of the RWA has been provided.[17] Recently, some works have been devoted to this qubit–oscillator system in the ultra-strong coupling regime.[20–24]

Actually, the JC model without the RWA in a wide coupling regime has been studied extensively for more than 40 years. By polaronic-like transformations or displaced operators, various analytical and numerical approaches have been developed in recent years, an incomplete list of which is given by Refs. [25]– [36]. Various forms of RWA energies have been developed[37–39] and extensions to the N-level case have been performed recently.[40] Many explicit and implied analytical approximated solutions have been obtained.

For the zero static bias, a generalized RWA (GRWA)[25,26] was proposed by Irish, which can give good eigensolutions in a wide parameter regime. Although the same formula was derived within substantially different approaches earlier,[41,42] it is fundamentally interesting[25,26] to link the original RWA to

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the subsequent extension. It was however found later[23] that the GRWA result deviates strongly with the increase of detuning and the coupling strength when exploring wider detuning regime. So the further improvement to the GRWA is also necessary. The finite biased qubit–oscillator model is closely relevant to the recent circuit QED system where the bias can be tuned by the external flux. Regarding the qubit tunneling matrix element as a perturbation, Hausinger and Grifoni[23] have performed a Van Vleck perturbatve (VVP) study, and systematically compared it with the numerical exact results for large static bias. They found that the VVP theory is especially suited to the deep strong coupling (DSC) regime or large static bias. However, the small and intermediate bias is not discussed, which is certainly also interesting. In Ref. [33], to obtain the exact energy levels, the numerical task to locate the zeros of complicate transcendental function defining through power series cannot be avoided. To the best of our knowledge, the algebraic formulae for the exact energy levels $E_{\rm exact}^{(n)}$ (model parameters) are lacking to date, which may not exist in principle. Therefore, the analytical explicit expression with higher accuracy for eigensolutions should be of practical interest.

In this paper, by using two displaced bosonic operators with equal positive and negative displacements, we can recover many previous analytical approximated treatments for both zero and finite static bias within the same scheme. What is more, we can extend analytically the previous GRWA to the finite static bias. Beyond the GRWA for the zero bias is also performed. The explicit expression is uniquely given and the results are very close to numerical exact ones for a wide range of the model parameters which cover the present-day experimentally accessible parameters. The dynamics for the qubit is performed as a preliminary application.

2. Model and numerical exact solution

The Hamiltonian for a superconducting qubit coupled to a harmonic oscillator in the circuit QED consists of three parts.[22,23] The first one is the interaction between the qubit and the LC resonator, which is described by

$$H_{\rm int} = \hbar g(a^\dagger + a)\sigma_z, \tag{1}$$

where $a^\dagger$, a are the photon creation and annihilation operators in the basis of Fock states of the LC resonator, g is the qubit–cavity coupling constant. The RWA has not been employed here. The effective Hamiltonian for the qubit can be written as the standard one for a two-level system

$$H = -\frac{1}{2}\left(\varepsilon\sigma_z + \Delta\sigma_x\right), \tag{2}$$

where ε and Δ are qubit static bias and tunneling matrix element. In the recent circuit QED[17,18] operating in the ultrastrong coupling regime, they describe the transition frequency of the flux qubit and the tunneling coupling between the two persistent current states. So we have $\varepsilon = I_{\rm p}(\Phi - \Phi_0/2)$ with $I_{\rm p}$ the persistent current in the qubit loop, Φ an externally applied magnetic flux, and Φ_0 the flux quantum. In contrast to the atomic cavity QED systems, ε is easily tunable in circuit QED systems using superconducting qubit. In the above two equations, the Pauli matrix notations σ_k $(k = x, y, z)$ are used in the basis of the two persistent current states. The third one is LC resonator $\omega a^\dagger a$ with single mode frequency ω. Then the Hamiltonian for the whole system reads ($\hbar = \omega = 1$)

$$H = -\frac{1}{2}\left(\varepsilon\sigma_z + \Delta\sigma_x\right) + a^\dagger a + g(a^\dagger + a)\sigma_z. \tag{3}$$

Motivated by the work in the Dicke model,[27] we have introduced two displaced bosonic operators with equal positive and negative displacements in this system[24]

$$A = a + g, \quad B = a - g, \tag{4}$$

then the Hamiltonian can be written in the following matrix form

$$H = \begin{pmatrix} A^\dagger A - g^2 - \varepsilon/2 & -\Delta/2 \\ -\Delta/2 & B^\dagger B - g^2 + \varepsilon/2 \end{pmatrix}. \tag{5}$$

Note that the linear term for the original bosonic operator $a^\dagger$ (a) is removed, and only the number operators $A^\dagger A$ and $B^\dagger B$ are left. Therefore the wavefunction can be expanded in terms of these new operators as

$$|\,\rangle = \begin{pmatrix} \sum_{n=0}^{N_{\rm tr}} c_n |n\rangle_A \\ \sum_{n=0}^{N_{\rm tr}} (-1)^n d_n |n\rangle_B \end{pmatrix}, \tag{6}$$

where $N_{\rm tr}$ is the truanted number. For operator A, we have

$$|n\rangle_A = \frac{\left(A^\dagger\right)^n}{\sqrt{n!}}|0\rangle_A = \frac{\left(a^\dagger + g\right)^n}{\sqrt{n!}}|0\rangle_A, \tag{7}$$

$$|0\rangle_A = {\rm e}^{-g^2/2 - g a^\dagger}|0\rangle_a, \tag{8}$$

operator B has the same properties. Inserting Eqs. (6) and (7) into the Schrödinger equation, we have

$$\left(m - g^2 - \frac{\varepsilon}{2}\right)c_m - \sum_{n=0} D_{mn} d_n = E c_m, \tag{9}$$

$$\left(m - g^2 + \frac{\varepsilon}{2}\right)d_m - \sum_{n=0} D_{mn} c_n = E d_m, \tag{10}$$

where

$$D_{mn} = \frac{\Delta}{2}(-1)^m {}_B\langle m|\,|n\rangle_A, \tag{11}$$

$${}_B\langle m|\,|n\rangle_A = (2g)^{n-m}\exp(-2g^2)\sqrt{\frac{m!}{n!}}\mathrm{L}_m^{n-m}\left(4g^2\right), \tag{12}$$

for $n \geq m$, $\mathrm{L}_m^{n-m}(x)$ is Laguerre polynomial, $D_{mn} = D_{nm}$.

Based on Eqs. (9) and (10), we have given a numerically exact spectrum to the qubit–oscillator system with any finite

static bias ε.[24] In this paper, we alternatively present some analytical approximated results in the framework of the above formalism. One can see that some recent analytical results by other authors are explicitly recovered in the present framework. Moreover, the present scheme is more convenient for performing further analytical studies.

3. Analytical approximated treatments

3.1. Variational study for $\varepsilon = 0$

To have a sense of two displaced bosonic operators in Eq. (4), we relax the displacement to be a variational parameter α,

$$A = a + \alpha, \quad B = a - \alpha, \tag{13}$$

then we study the unbiased Hamiltonian ($\varepsilon = 0$) variationally. Suppose that the trial state is the vacuum state in these displaced operators as follows:

$$|\Phi_0\rangle = \begin{pmatrix} |0\rangle_A \\ |0\rangle_B \end{pmatrix}.$$

The energy expectation is derived as

$$E_0 = -\frac{\Delta}{2} e^{-2\alpha^2} - 2g\alpha + \alpha^2. \tag{14}$$

Minimizing the energy gives

$$\Delta\alpha e^{-2\alpha^2} - g + \alpha = 0. \tag{15}$$

In the weak coupling limit, we can obtain the variational parameter and the ground state (GS) energy respectively

$$\alpha = \frac{g}{1+\Delta}, \tag{16}$$

$$E_0 = -\frac{\Delta}{2} \exp\left[-2\left(\frac{g}{1+\Delta}\right)^2\right] - g^2 \frac{1+2\Delta}{(1+\Delta)^2}, \tag{17}$$

which are exactly the same as Eqs. (7) and (8) obtained in Ref. [43].

In the strong-coupling limit, the first term in Eq. (14), which is originated from the qubit tunneling, is too small and can be neglected, then we simply have

$$\alpha = g, \tag{18}$$

and the GS energy

$$E^{\mathrm{SCL}} = -g^2. \tag{19}$$

For the arbitrary coupling, one can solve Eq. (15) consistently and the reasonable GS energy will be derived, which is not shown here.

3.2. Perturbation theory based on the exact solution in the strong coupling limit

Note above that in the strong coupling limit, the variational parameter is just exactly $\alpha = g$. It can also be readily obtained by neglecting the qubit tunneling term $-\frac{1}{2}\Delta\sigma_x$ in Hamiltonian Eq. (5) with zero static-bias. In this case, based on the displaced operators A and B, the eigenstates are easily obtained as

$$|m\rangle^{(0)} = \begin{pmatrix} |m\rangle_A \\ \pm(-1)^m |m\rangle_B \end{pmatrix}, \tag{20}$$

and the eigenvalues are $E_m^{\pm 0} = m - g^2$ for the m state. Note that the eigenstates are two-fold degenerate.

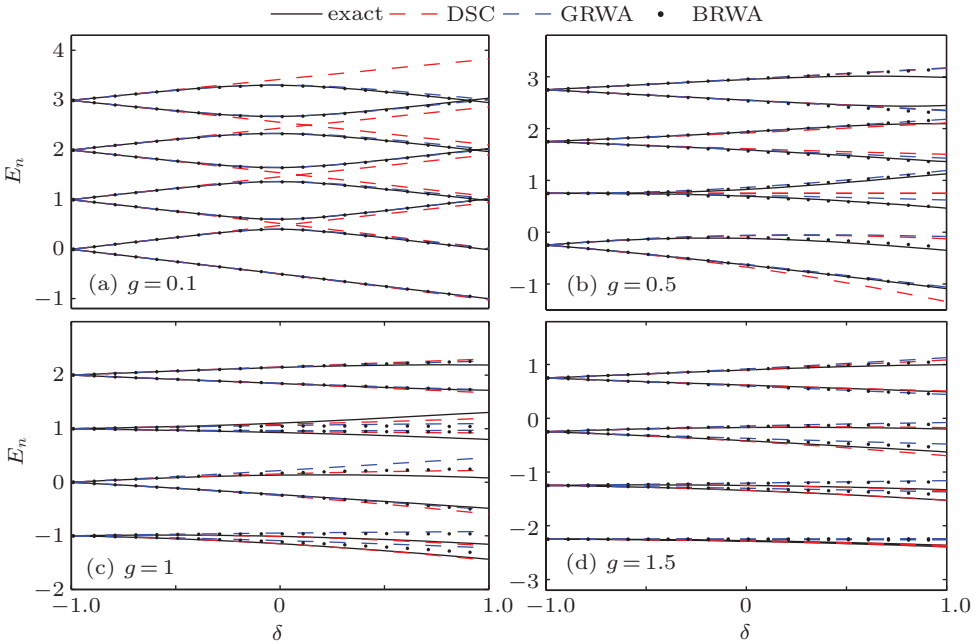


Fig. 1. (color online) The energy levels for zero bias $\varepsilon = 0$ at (a) $g = 0.1$, (b) $g = 0.5$, (c) $g = 1.0$, and (d) $g = 1.5$. The present BRWA results (black filled circles) are compared to the numerical exact (solid lines), DSC (red dashed lines), and GRWA (blue dashed lines) ones.

Next, considering $H' = -\frac{1}{2}\Delta\sigma_x$ as a perturbation, within the second-order perturbation theory, we can readily derive the eigenenergy with even (odd) parity for zero qubit static bias as

$$E_m^{\pm} = m - g^2 \mp (-1)^m D_{mm} + \sum_{m\neq n} \frac{|D_{mn}|^2}{m-n}. \tag{21}$$

It is interesting to note that it is just the same as Eq. (5) in Ref. [31] obtained by Casanova *et al.* in the DSC regime of the JC model.

The energy levels by Eq. (21) against the qubit–oscillator detuning $\delta = \Delta - 1$ for $g = 0.1, 0.5, 1.0$, and 1.5, ranging from weak to deep strong coupling are displayed in Fig. 1. The exact results by using the numerically exact diagonalization are also collected. Actually, this comparison has been already done in Ref. [23]. We will add our further results to this figure later for complete comparison. The DSC results are especially suited to the DSC regime or small detunings. =Casanova *et al.*[31] focused on the physical features, such as parity chains and photon number wave packets in the DSC regime ($g/\omega = 2$) or small detunings ($\Delta \leq 0.5$). At weak coupling $g = 0.1$, it is shown in Fig. 1(a) that, even for the negative detuning δ, the DSC deviates from the numerical exact ones. Note that in the present experimentally accessible systems the maximum value for the coupling strength is generally realized in the superconducting flux qubit coupled to a circuit resonant,[17] which is only around $g = 0.1$, to our knowledge. So it should be practically interesting to find a good solution in this coupling regime.

For any value of the qubit bias ε, the Hamiltonian Eq. (5) with a vanishing tunneling element $\Delta = 0$ can be diagonalized in terms of two eigenstates $|\uparrow, m\rangle_A$ and $|\downarrow, m\rangle_B$ with $|\uparrow\rangle$ ($|\downarrow\rangle$) the eigenstate of σ_z, the corresponding eigenvalues are

$$E^{(0)}_{\uparrow/\downarrow,m} = \pm\frac{1}{2}\varepsilon + m - g^2. \tag{22}$$

For the finite Δ, the perturbative matrix elements become

$$-\frac{1}{2}\,{}_B\langle\downarrow, m|\,\Delta\sigma_x\,|\uparrow, n\rangle_A = -(-1)^m D_{mn}. \tag{23}$$

Note that these two equations are exactly the same as Eqs. (7) and (8) in Ref. [23]. Then the full Hamiltonian can be diagonalized perturbatively to second-order in Δ by using VVP theory as outlined in Ref. [23]. The eigenvalue is given by

$$\begin{aligned} E_m^{\pm} &= m + \frac{l}{2} - g^2 + \frac{1}{2}\sum_{k=0,k\neq m\pm l}^{\infty}\left(\frac{D_{mk}^2}{\varepsilon + m - k} - \frac{D_{nk}^2}{\varepsilon + k - n}\right) \mp \frac{1}{2}\sqrt{\left[\varepsilon - l + \sum_{k=0,k\neq m\pm l}^{\infty}\left(\frac{D_{mk}^2}{\varepsilon + m - k} + \frac{D_{nk}^2}{\varepsilon + k - n}\right)\right]^2 + 4D_{mn}^2}, \\ n &= m + l (l \geq 0), \quad m = 0, 1, 2, \ldots, \end{aligned} \tag{24}$$

which is the same as Eqs. (12) for VVP in Ref. [23]. So the VVP for finite bias can also be recovered easily in the present scheme. Note that in the derivation, the qubit bias is assumed as a multiple of the oscillator frequency $\varepsilon = l$, so if it is not exactly a multiple, l cannot be identified unambiguously. This ambiguity is quite special in this approach, and does not appear in other approaches in this field to our knowledge. It can be reduced to the zero-bias case Eq. (21) by setting $\varepsilon = 0$ $(m = n)$, where the above ambiguity disappears.

It has been shown[23] that VVP works very well in the deep strong coupling or large static bias. It is consistent with the fact that the unperturbative Hamiltonian includes the qubit–oscillator interaction and qubit bias. What happen for the accessible parameters of the present-day experiments? In addition, VVP at small static bias $\varepsilon \leq 1$ has not been discussed either so far, which might however be more important.

Here, we calculate energy levels in the VVP for different static bias $\varepsilon \leq 1$, which are exhibited in Fig. 2. Comparing with the numerical exact results, one can find that VVP deviates strongly with the increase of the tunneling parameters Δ in a wide coupling regime $g < 0.5$, and becomes more pronounced at small static bias. Especially, around the experimentally accessible coupling strength around $g = 0.1$, VVP becomes considerably worse. Therefore a new analytical approximated treatment is highly desirable.

3.3. Analytical approximations at different levels

In the framework of Eqs. (9) and (10), analytical approximations can be performed systematically. First, as a zero-order approximation (ZOA), we omit the off-diagonal terms and have

$$\begin{aligned} \left(m - g^2 - \frac{\varepsilon}{2} - E\right) c_m - D_{m,m} d_m = 0, \\ -D_{m,m} c_m + \left(m - g^2 + \frac{\varepsilon}{2} - E\right) d_m = 0. \end{aligned}$$

Nonzero coefficients will give the following equation

$$\left(m - g^2 - \frac{\varepsilon}{2} - E\right)\left(m - g^2 + \frac{\varepsilon}{2} - E\right) - D_{m,m}^2 = 0.$$

The eigenvalues are then given by

$$E_{\pm} = m - g^2 \mp \frac{1}{2}\sqrt{\varepsilon^2 + 4D_{m,m}^2}. \tag{25}$$

The corresponding eigenstate is

$$|m\rangle_{\pm} \propto \begin{pmatrix} (-1)^m D_{mm}\,|m\rangle_A \\ \left(m - g^2 - \frac{\varepsilon}{2} - E_{\pm}\right)|m\rangle_B \end{pmatrix}. \tag{26}$$

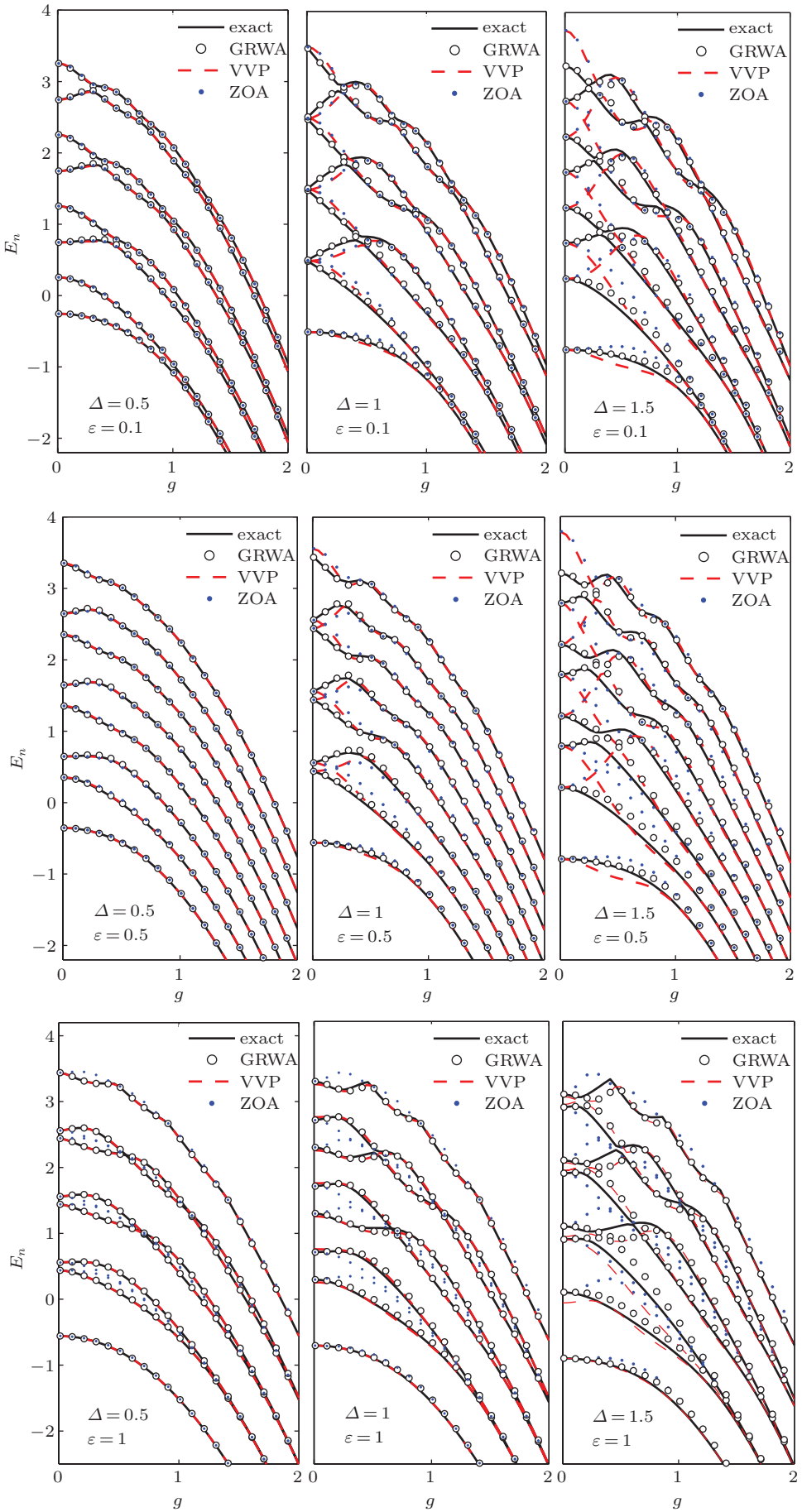


Fig. 2. (color online) The energy levels as a function of the coupling constant for different qubit bias $\varepsilon = 0.1$ (upper panel), 0.5 (middle panel), and 1.0 (bottom panel). The values of Δ are 0.5, 1.0, and 1.5 from left to right column. The present GRWA results (black open circles) are compared with the numerical exact (black solid lines), VVP (red dashed lines), and ZOA (blue filled circles) ones.

The ZOA energies with zero static bias are just the three terms obtained in Eq. (21). In Fig. 2, we also plot the ZOA energy levels against the coupling constant g for several static bias. It is demonstrated from the upper and middle panels that for small static bias ($\varepsilon \leq 0.5$), ZOA is almost equivalent to the VVP in all parameters. If the high accuracy is not required, the simple expression of the eigensolutions in the ZOA should be practically very useful, at least as preliminary estimations for some physical quantities.

The approximation can be easily improved step by step with the consideration of more off-diagonal elements in the present formalism. The first-order approximation (FOA) is performed by selecting two coefficients $c_m\ d_m\ c_{m+1}\ d_{m+1}$. The determinants for any m is given by

$$\begin{vmatrix} \Omega_m^-(E) & -D_{mm} & 0 & -D_{m,m+1} \\ -D_{mm} & \Omega_m^+(E) & -D_{m,m+1} & 0 \\ 0 & -D_{m+1,m} & \Omega_{m+1}^-(E) & -D_{m+1,m+1} \\ -D_{m+1,m} & 0 & -D_{m+1,m+1} & \Omega_{m+1}^+(E) \end{vmatrix} = 0, \quad (27)$$

where

$$\begin{aligned} \Omega_m^-(E) &= m - g^2 - E - \frac{\varepsilon}{2}, \\ \Omega_m^+(E) &= m - g^2 - E + \frac{\varepsilon}{2}. \end{aligned} \quad (28)$$

Some roots of this quartic equation will give the energy levels. The analytical approximated expression might be a little bit complicate but should be given unambiguously.

We first revisit the zero-bias case $\varepsilon = 0$. In this case, due to the parity symmetry, we can set $d_n = \pm c_n$, then both equations give $\left(m - g^2\right) c_m \mp \sum_{n=0} D_{mn} c_n = E c_m$. In the FOA, the determinant takes the following 2×2 block form

$$\begin{vmatrix} \left(m - g^2 - E \mp D_{m,m}\right) & \mp D_{m,m+1} \\ \mp D_{m+1,m} & \left(m + 1 - g^2 - E \mp D_{m+1,m+1}\right) \end{vmatrix} = 0,$$

where the sign $-(+)$ is for even (odd) parity. We can readily have two roots for even parity

$$\begin{aligned} E_m^{(1,2)} = {} & m - g^2 + \frac{1}{2} - \frac{1}{2}\left(D_{m+1,m+1} + D_{m,m}\right) \\ & \pm \frac{1}{2}\sqrt{\left[1 + \left(D_{m,m} - D_{m+1,m+1}\right)\right]^2 + 4D_{m,m+1}^2}, \end{aligned} \quad (29)$$

and other two roots for odd parity

$$\begin{aligned} E_m^{(3,4)} = {} & m - g^2 + \frac{1}{2} + \frac{1}{2}\left(D_{m+1,m+1} + D_{m,m}\right) \\ & \pm \frac{1}{2}\sqrt{\left[1 - \left(D_{m,m} - D_{m+1,m+1}\right)\right]^2 + 4D_{m,m+1}^2}. \end{aligned} \quad (30)$$

In the ansatz of the wavefunction Eq. (6), the dimensions of the Hilbert space is only $2(N_{\rm tr}+1)$. So for each m, we only have two eigenvalues for excited states. The other two roots for each m should be omitted. Note that at weak coupling, the parity for each eigenstate is fixed and arranged from bottom to top with the order as the first even state, then followed by two odd states, two even states, two odd states, and so on. Therefore, the excited states 1 and 2 are of odd parity, which should be given by $E_0^{(3,4)}$, the excited states 3 and 4 are of even parity, then given by $E_1^{(1,2)}$, the excited states 5 and 6 are of odd parity, then given by $E_2^{(3,4)}$, and so on. In this way, two eigenvalues for excited states for any m can be summarized as

$$\begin{aligned} E_m = {} & m - g^2 + \frac{1}{2} + \frac{(-1)^m}{2}\left(D_{m+1,m+1} + D_{m,m}\right) \\ & \pm \frac{1}{2}\sqrt{\left[1 - (-1)^m\left(D_{m,m} - D_{m+1,m+1}\right)\right]^2 + 4D_{m,m+1}^2}. \end{aligned} \quad (31)$$

Besides, the GS energy is given by $E_0^{(1)}$

$$\begin{aligned} E_{\rm GS} = {} & \frac{1}{2} - g^2 - \frac{1}{2}\left(D_{1,1} + D_{0,0}\right) \\ & - \frac{1}{2}\sqrt{\left[1 + \left(D_{0,0} - D_{1,1}\right)\right]^2 + 4D_{0,1}^2}. \end{aligned} \quad (32)$$

The FOA results in Eqs. (31) and (32) have been given directly by the determinant with 2×2 block form in Ref. [28] by two present authors and one collaborator previously. We here display the derivation in detail. Especially we rule out two pseudosolutions for each m by taking the fixed parity of the eigenstates into account.

Surprisingly, Eq. (31) is exactly the same as the previous GRWA result Eq. (20) in Refs. [25] and [26]. What is more, we can straightforwardly perform the second-order approximation for the further improvement, and extension to the biased case in the present framework, which is however not so easy to operate within Irish's approach. To the best of our knowledge, GRWA with finite static bias does not exist in the literature.

For the finite bias $\varepsilon \neq 0$, the parity symmetry is broken with ε, we have to solve the original quartic equation Eq. (27) without any simplification. The detailed expression for this quartic equation is presented in Appendix B and its solution is given in Appendix A. Comparing to the numerical exact solutions, we find that the second and third roots x_2 and x_3 in Eqs. (A9) and (A10) are generally the true solutions. The GS energy is given by the first root x_1 in Eq. (A8) for $m = 0$. We also call the FOA with finite static bias as GRWA. In this way, we can calculate the eigenenergies uniquely and straightforwardly, which are shown in Fig. 2 with black circles. It is very interesting to find that the present GRWA results are very close to the numerical exact ones in the whole coupling regime for a wide range of the static bias. Compared to the VVP at static bias $\varepsilon \leq 1$, the present GRWA is obviously much better.

As stated above, for zero-static bias case $\varepsilon = 0$, there is still room to improvement by performing the higher order approximation. In Refs. [25] and [26], after a unitary transformation, only the energy-conserved excitation terms like Eq. (18) in Refs. [25] and [26], a generalization of the energy-conserved term in the usual RWA, is kept in Eqs. (13) and

(14) in Refs. [25] and [26], so it is called as GRWA. Because the present FOA is equivalent to GRWA, so in the second-order approximation, the terms beyond Eq. (18) in Ref. [25] and [26] must be included, so we term this improvement to GRWA as beyond the RWA (BRWA). In other words, BRWA cannot be implemented within any renormalized RWA form like in Refs. [25] and [26].

In the BRWA, the analytical approximated expression can be uniquely and clearly derived within the following procedure. The determinant is

$$\begin{vmatrix} \left(m-g^2-E\mp D_{m,m}\right) & \mp D_{m,m+1} & \mp D_{m,m+2} \\ \mp D_{m,m+1} & \left(m+1-g^2-E\mp D_{m+1,m+1}\right) & \mp D_{m+1,m+2} \\ \mp D_{m,m+2} & \mp D_{m+1,m+2} & \left(m+2-g^2-E\mp D_{m+2,m+2}\right) \end{vmatrix} = 0, \tag{33}$$

where $-(+)$ for even (odd) parity, which results in a cubic equation described in Appendix C in detail.

The three different roots to the cubic equation can be found in the Appendix A. In this approximation, we have more than one eigenvalues for each eigenstate with fixed parity, which are all true solutions physically, but only some of them would be selected. The criterion for the unique formulae for the BRWA is that the solutions are the most close to the numerical exact results in the whole parameter regime. In this way, we find that, for even (odd) number m, two roots y_1 and y_2 in Eqs. (A2) and (A3) of the determinant with odd (even) parity would generally give the best eigenvalues for the excited states. The GS state is given by the first root y_1 of the $m=0$ determinant with even parity. In Ref. [28], the detailed explicit expression for the eigenvalues in BRWA was not presented. In addition, the direct comparisons between the different order approximations and the GRWA[25,26,41,42] have not been performed, which however could provide direct judgement of the two approaches.

We examine the BRWA energy levels against the qubit–oscillator detuning δ for fixed couplings $g=0.1$, 0.5, 1.0, and 1.5, respectively in Fig. 1, where the GRWA results have been also collected. It is interesting to note that BRWA result is always closer to the numerical exact one than GRWA one in all values of the coupling strength, which becomes more pronounced with the increase of δ.

Due to the counter-rotating wave terms, the eigenfunctions and eigenvalues of the JC model without the RWA present an open problem because they are not known in anything like a closed form, even given the work reported recently.[32,33] The accuracy eigenvalues can be given by zeros of two well-defined functions numerically in the practical evaluation. No analytical explicit expressions for the exact eigenvalues are available in the literature, to the best of our knowledge. The analytical approximated explicit expressions presented in this paper might be more practically useful.

For the biased case in the second approximation, which is also called BRWA, we will derive a polynomial equations with order six. However, explicit analytical expression is not available, due to the Abel–Ruffini theorem that the general solution in radicals is impossible to polynomial equations of degree five or higher.[44] So the numerical solutions to locate the roots of the above polynomial equations with order six are needed. For each m, we have six roots. Compared with the numerical exact solutions, we find that the fist four small roots for $m=0$ and the intermediate two roots for other m would give the best eigenstates, which will be used in the next section.

4. Dynamics

Due to the inevitable interaction with the environment, the decoherence must occur in the long time regime. So the dynamical evolution before the coherence is lost is crucial to understand the various physical properties. We will study the earlier dynamics of the qubit–oscillator system in this section. The accuracies of different approaches determine crucially the dynamics, because the transition frequency defined by the energy difference determines the peak position of the Fourier transformation of the real time dynamics and the amplitude of the dynamical evolution is determined by the eigenfunction.

The dynamics at $g=1.0$ and 2.0 have been studied in Ref. [23] within VVP, which is actually in the deep strong coupling regime. In this paper, we will consider the present-day experimentally accessible coupling regime and beyond not too much, such as $g\leq 0.5$. The time-dependent population difference can be given by

$$\langle\sigma_z(t)\rangle = 2\langle\uparrow|\rho_{\rm red}(t)|\uparrow\rangle - 1,$$

where the reduced density operator $\rho_{\rm red}(t)$ is obtained after tracing out the oscillator degrees of freedom of the total one ρ, which matrix elements are

$$\rho_{ij}(t) = \langle i|\rho(t)|j\rangle = \rho_{ij}(0)\,{\rm e}^{-{\rm i}(E_i-E_j)t},$$

where $|i\rangle$ is the i-th eigenstate with eigenenergy E_i of the present qubit–oscillator system. Initially, the qubit is pumped into the upper state with the oscillator being at the ground-state. It is very close to that used in Ref. [23] where the ratio of the components of the ground-state and the first excited state of oscillator in the density matrix element are around 5×10^{-5}. In what follows, we investigate the dynamics for both the unbiased and biased cases respectively.

4.1. Dynamics for zero static bias $\varepsilon = 0$

First, we examine the dynamics of the population difference $\langle\sigma_z(t)\rangle$ for zero static bias $\varepsilon = 0$ at resonance for the coupling strengths $g = 0.1$, 0.2, and 0.5 with different methods, such as DSC (VVP), GRWA, and BRWA. The results for $\langle\sigma_z(t)\rangle$ are shown in Fig. 3(a). At small coupling $g = 0.1$, both GRWA and BRWA can describe the correct dynamics in longer time scale excellently; the GRWA result begins to deviate after a long time. For $g = 0.2$, both can give basically right results, but GRWA result becomes deviated strongly for longer time scales. At intermediate coupling $g = 0.5$, BRWA can give basically a right description, and a slight deviation from the numerical exact ones for long time scale can be observed. But the GRWA result totally fails to match the oscillation.

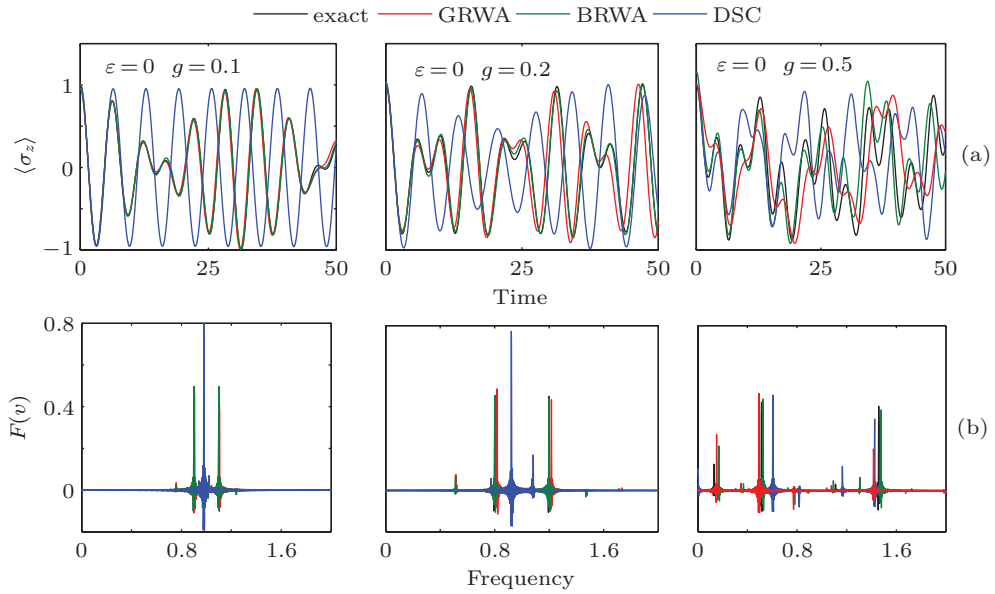


Fig. 3. (color online) (a) Population differences for the zero static bias at resonance. $g = 0.1, 0.2$, and 0.5 from left to right. The GRWA, BRWA, and DSC results are compared to the numerical results. (b) The Fourier transform of the population difference.

For better understanding, we turn to the analysis of Fourier transform for the long time window $t[0:1000]$ in Fig. 3(b). It is observed that BRWA is always better than GRWA, and its result becomes pronounced with the increase of the coupling constant. For $g = 0.1$ and 0.2, two main peaks appear, BRWA ones are almost the same as the numerical exact ones, and GRWA ones deviate obviously but are still acceptable. For $g = 0.5$, three frequency peaks in the Fourier transform in the numerical exact solutions are roughly recovered by both GRWA and BRWA, but BRWA can still describe the population difference well, and the GRWA results deviate strongly.

Obviously, for $g \leq 0.5$, DSC fails completely to describe the dynamics, although it works well in the deep strong coupling regime.[23] From the DSC eigenenergies in Eq. (21), the frequency from the transitions between energy levels with the same index m is approximately given by $2D_{mm}$. The $m = 0$ frequency is given by $\Delta \exp\left(-2g^2\right)$, which is dominate in the weak coupling regime, which accounts for only one dominating oscillation with one frequency $g = 0.1$. For $g = 0.2$, it can be seen obviously that the evolution is the superposition from a few oscillations with different frequencies. The dominating frequency peak still originates from the transitions between energy levels with $m = 0$. However, the second dominating frequency peak originates from the transition of the energy levels E_1^+ and E_0^+, in contrast with the previous findings in the deep strong coupling regime[23] that the dynamics is dominated by the peaks belonging to the transitions with the same index. With the increase of the coupling strength, more frequency oscillations appear, and the characteristic in the DSC regime emerges. Although DSC yields some frequency peaks close to the numerical exact ones in the Fourier space, it still deviates far from the numerical exact dynamics for the population difference.

4.2. Dynamics for finite static bias $\varepsilon \neq 0$

We now turn to the dynamics of the population differences for a weakly static bias $\varepsilon = \sqrt{0.5}$ and a large one 1.5 at resonance for different coupling strengths $g = 0.1$, 0.2, and 0.5 with different methods, such as VVP, GRWA, and BRWA. Here we chose $l = 0$ for $\varepsilon = \sqrt{0.5}$ and $l = 1$ for $\varepsilon = 1.5$ to obtain the best results. Any other choice of l gives a worse result. As shown in Fig. 4, both GRWA and BRWA can give correct dynamics in an experimentally realized coupling regime $g = 0.1$, 0.2. In the intermediate coupling regime, that is $g = 0.5$, BRWA still yields accepted results, but the GRWA result gets out of phase. At $\varepsilon = \sqrt{0.5}$, VVP cannot work at all in the weak and intermediate coupling regime $0 < g \leq 0.5$. This situation can be improved with the increase of the static bias, as shown in the bottom panel for $\varepsilon = 1.5$, consistent with the perturbative nature, because the unperturbative part includes $-\varepsilon\sigma_z/2$ in VVP.

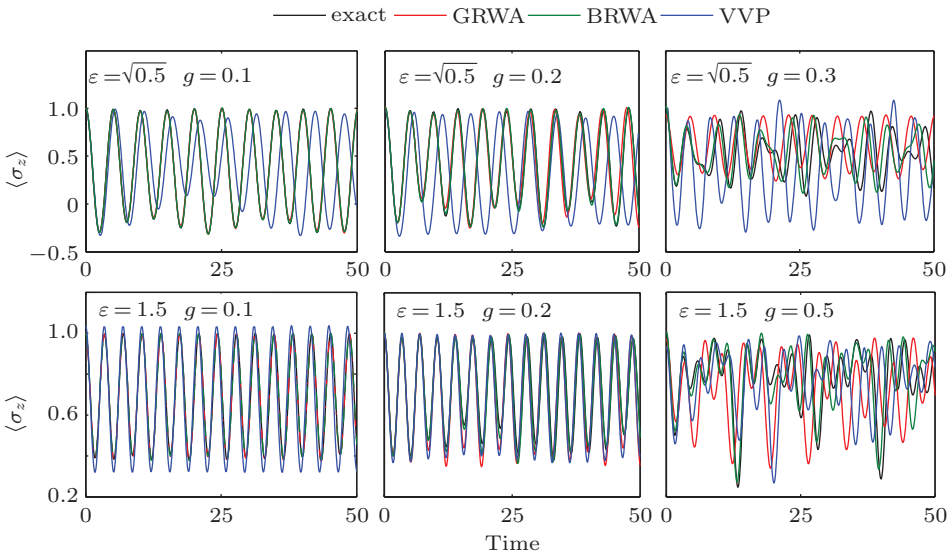


Fig. 4. (color online) Population differences at resonance for the finite static bias $\varepsilon = \sqrt{0.5}$ (a) and $\varepsilon = 1.5$ (b), $g = 0.1, 0.2$, and 0.5 from left to right. The GRWA and VVP results are compared with the numerical exact results.

5. Summary

In this paper, by an effective scheme within two displaced bosonic operators with equal positive and negative displacements, we study the qubit–oscillator systems analytically in a unified way. Many previous analytical treatments, such as GRWA and an expansion in the qubit tunneling matrix element in the DSC regime, can be recovered in the present scheme. It should be pointed out here that all mentioned previous analytical results are original. We show here that they can also be re-derived with the present scheme in a unified way. Moreover, we extend the GRWA to the finite-bias case. The results are much better than that of VVP in the weak and intermediate coupling regime, which is more experimentally interesting. For the zero static bias, the GRWA is further improved to BRWA, the result of which is more close to the numerical exact ones at large detuning regime while the result of GRWA deviates strongly. The analytical approximated expression is explicitly given for future applications. The preliminary applications to the dynamics of the qubit have also been performed. At the experimentally accessible coupling regime, GRWA can always work well, but its result deviates with the increase of the coupling strengths. BRWA can give the correct description in a wide coupling regime for any static bias. The VVP theory is not valid in the present-day experimentally interesting coupling regime, except for very large static bias cases.

Appendix A: Solutions to univariate cubic and quartic equations

The univariate cubic equation can be generally expressed as

$$x^3 + bx^2 + cx + d = 0. \tag{A1}$$

Its solutions can be found in any mathematics manual. If

$$\Gamma = B^2 - 4AC < 0,$$

with

$$A = b^2 - 3c, \quad B = bc - 9d, \quad C = c^2 - 3bd,$$

there are three different real roots

$$y_1 = \frac{-b - 2\sqrt{A}\cos\theta}{3}, \tag{A2}$$

$$y_2 = \frac{-b - 2\sqrt{A}\cos\left(\theta - \frac{2\pi}{3}\right)}{3}, \tag{A3}$$

$$y_3 = \frac{-b - 2\sqrt{A}\cos\left(\theta + \frac{2\pi}{3}\right)}{3}, \tag{A4}$$

where

$$\theta = \frac{1}{3}\arccos\left(\frac{2Ab - 3B}{2\sqrt{A^3}}\right). \tag{A5}$$

The univariate quartic equation can be generally expressed as

$$x^4 + bx^3 + cx^2 + dx + e = 0, \tag{A6}$$

Its four solutions are exactly the four solutions of the following two quadratic equations

$$x^2 + \frac{b+z}{2}x + \left(y + \frac{by - d}{z}\right) = 0, \tag{A7}$$

$$x^2 + \frac{b-z}{2}x + \left(y - \frac{by - d}{z}\right) = 0, \tag{A8}$$

where $z = \sqrt{8y + b^2 - 4c}$ and y is the third root y_3 in Eq. (A4) of the following cubic equation

$$y^3 - \frac{c}{2}y^2 + \left(\frac{bd}{4} - e\right)y + \frac{e\left(4c - b^2\right) - d^2}{8} = 0.$$

We have checked that $\Gamma < 0$ in all parameters in the present case. Therefore the four roots are

$$x_1 = -\frac{1}{4}(b+z) - \frac{1}{4}\sqrt{(b+z)^2 - \frac{16y(b+z)-16d}{z}}, \quad \text{(A9)}$$

$$x_2 = -\frac{1}{4}(b+z) + \frac{1}{4}\sqrt{(b+z)^2 - \frac{16y(b+z)-16d}{z}}, \quad \text{(A10)}$$

$$x_3 = -\frac{1}{4}(b-z) - \frac{1}{4}\sqrt{(b-z)^2 + \frac{16y(b-z)-16d}{z}}, \quad \text{(A11)}$$

$$x_4 = -\frac{1}{4}(b-z) + \frac{1}{4}\sqrt{(b-z)^2 + \frac{16y(b-z)-16d}{z}}. \quad \text{(A12)}$$

Appendix B: Cubic equations in BRWA for $\varepsilon = 0$

In the BRWA without static bias, the analytical expression can be uniquely and clearly derived within the following procedure. The determinant is

$$\begin{vmatrix} (m-g^2-E\mp D_{m,m}) & \mp D_{m,m+1} & \mp D_{m,m+2} \\ \mp D_{m,m+1} & (m+1-g^2-E\mp D_{m+1,m+1}) & \mp D_{m+1,m+2} \\ \mp D_{m,m+2} & \mp D_{m+1,m+2} & (m+2-g^2-E\mp D_{m+2,m+2}) \end{vmatrix} = 0, \quad \text{(B1)}$$

where $-(+)$ for even (odd) parity, which can be simplified as

$$\begin{vmatrix} -X\mp u & \mp x & \mp y \\ \mp x & (1-X\mp v) & \mp z \\ \mp y & \mp z & (2-X\mp w) \end{vmatrix} = 0,$$

where

$$\begin{aligned} E &= X+m-g^2, \\ u &= D_{m,m}, \quad v = D_{m+1,m+1}, \quad w = D_{m+2,m+2}, \\ x &= D_{m,m+1}, \quad y = D_{m,m+2}, \quad z = D_{m+1,m+2}, \end{aligned}$$

which gives the following cubic equation

$$X^3 + bX^2 + cX + d = 0, \quad \text{(B2)}$$

where

$$\begin{aligned} b &= (u+v+w)-3, \\ c &= -\left(x^2+y^2+z^2\right) + u(v-1) + (u+v-1)(w-2), \\ d &= \left(2u-uw+y^2\right)(1-v) - z^2u + x^2(2-w) + 2xyz, \end{aligned}$$

for even parity, and

$$\begin{aligned} b &= -(u+v+w)-3, \\ c &= -\left(x^2+y^2+z^2\right) + u(1+v) + (u+v+1)(2+w), \\ d &= \left(y^2-2u-uw\right)(1+v) + z^2u + x^2(2+w) - 2xyz, \end{aligned}$$

for odd parity.

Appendix C: Quartic equations in GRWA for $\varepsilon \neq 0$

For the finite bias $\varepsilon \neq 0$, the parity symmetry is broken with ε, with the following notation

$$\begin{aligned} E &= x+m-g^2-\frac{\varepsilon}{2}, \quad \text{(C1)} \\ u &= -D_{mm}, \quad v = -D_{m,m+1}, \quad w = -D_{m+1,m+1}. \end{aligned}$$

The determinant can be reduced to

$$\begin{vmatrix} -x & u & 0 & v \\ u & -x+\varepsilon & v & 0 \\ 0 & v & 1-x & w \\ v & 0 & w & 1-x+\varepsilon \end{vmatrix} = 0. \quad \text{(C2)}$$

The corresponding quartic equation is

$$x^4 + bx^3 + cx^2 + dx + e = 0,$$

where

$$\begin{aligned} b &= -2-2\varepsilon, \\ c &= 1+3\varepsilon+\varepsilon^2 - \left(2v^2+u^2+w^2\right), \\ d &= \left(2v^2+u^2+w^2-\varepsilon-1\right)\varepsilon + 2\left(u^2+v^2\right), \\ e &= \left(uw-v^2\right)^2 - u^2(1+\varepsilon) - v^2\varepsilon. \end{aligned}$$

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