# Brownian localization: A generalized coupling model yielding a nonergodic Langevin equation description*

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(Received 1 December 2012; revised manuscript received 20 February 2013)

A minimal system-plus-reservoir model yielding a nonergodic Langevin equation is proposed, which originates from the cubic-spectral density of environmental oscillators and momentum-dependent coupling. This model allows ballistic diffusion and classical localization simultaneously, in which the fluctuation–dissipation relation is still satisfied but the Khinchin theorem is broken. The asymptotical equilibrium for a nonergodic system requires the initial thermal equilibrium, however, when the system starts from nonthermal conditions, it does not approach the equilibration even though a nonlinear potential is used to bound the particle, this can be confirmed by the zeroth law of thermodynamics. In the dynamics of Brownian localization, due to the memory damping function inducing a constant term, our results show that the stationary distribution of the system depends on its initial preparation of coordinate rather than momentum. The coupled oscillator chain with a fixed end boundary acts as a heat bath, which has long been used in studies of collinear atom/solid-surface scattering and lattice vibration, we investigate this problem from the viewpoint of nonergodicity.

**Keywords:** ocalization, nonergodicity, generalized coupling model, coupled oscillator chain

**PACS:** 05.70.Ln, 05.40.Jc, 05.40.Ca **DOI:** 10.1088/1674-1056/22/6/060513

## 1. Introduction

Localization and coherence at the nanometer scale are currently subjects of intense debate in both theoretical and experimental physics. Leggett first drew attention to quantum coherence in connection with a macroscopic degree of freedom coupled to an environment.[1] Coherence dictates that a particle can be localized in one well of a double-well potential[2] or a periodic potential.[3] Grabert *et al.*[4] proved that an initially localized particle that is subject to sub-Ohmic damping remains permanently localized. Localization can occur by pure coherence via external driving,[5,6] besides, dynamical localization of a particle coupled to a two-level thermal reservoir was found in the sub-Ohmic regime at low temperature.[7] The above results are induced by the quantum effect, nevertheless, the localization in the classical sense has much greater potential importance and application. Some examples of classical localization include the Bethe lattice, which is not subject to diffusion and thus behaves as a frozen system;[8,9] a heavy particle coupled to a semi-infinite coupled oscillator chain;[10] intrinsic localized modes in anharmonic crystals;[11] and localization in discrete breathers.[12] Until now, understanding the phenomenon and behavior of the non-diffusion of unbounded particles under the influence of fluctuations from the viewpoint of nonergodicity has not been discussed.

To study the dissipative dynamics, the system-plus-reservoir (heat bath) model has been extensively used in statistical mechanisms. The properties of any macroscopic system are determined by its initial condition and relation with the environment; an isothermal system without external time-dependent influences will approach the Gibbs–Boltzmann distribution (GBD); however, the reservoir structure and coupling form control the dynamics and stationary state of the system at all events. Usually, the heat bath is modeled as an infinite number of independent harmonic oscillators[13–15] for which the fluctuation–dissipation theorem (FDT) is valid. A nonlinear heat bath was used in the quantum case.[16,17] Recently, the model has received increasing attention in small systems and finite-size heat baths.[18–20] Prefiguring and classifying nonequilibrium systems in terms of their dissipative environments[21–25] may reveal some surprising findings, because the long-time behavior of a stochastic variable depends on its Laplace transform at low frequency, so that the frequency-dependent memory damping plays a key role.

The nonergodic behavior of non-Markovian Brownian motion in free and biased Brownian motion was also introduced recently.[26–28] In this work, a nonergodic generalized Langevin equation (GLE) is derived based on the momentum-dependent system and reservoir coupling model, the two limits of thermal diffusion, ballistic diffusion, and localization can be considered successionally. A phase transition for diffusive coefficient is found and the condition of localization for the spectral density of the bath is obtained. It is demonstrated that a nonergodic coordinate variable cannot be equilibrated yet if a nonlinear potential is used to bound it. Also, it is then combined into a single framework and ample numerical evidence

*Project supported by the National Natural Science Foundation of China (Grant No. 11175021).

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is provided.

## 2. Generalized coupling model

We consider a system $A$ coupled to a subsystem $B$, the latter is coupled in turn to a large reservoir $C$ of independent oscillators at a finite temperature $T$ (see Fig. 1). $A$ and $B$ are two coupled dynamical systems, alternatively, the subsystem $B$ plus bath $C$ can be regarded as an effective bath as seen by the system $A$. The whole Hamiltonian reads

$$\begin{aligned} H &= H_A(P,Q)+H_B(p,q)+H_C(p_j,y_j) \\ &= \frac{P^2}{2M}+U(Q)+\frac{(p-\lambda Q)^2}{2m}+u(q) \\ &\quad +\sum_{j=1}^{N}\frac{1}{2}\left[\frac{p_j^2}{\mu_j}+\mu_j\omega_j^2\left(y_j-\frac{c_j q}{\mu_j\omega_j^2}\right)^2\right], \end{aligned} \tag{1}$$

where $\lambda$ and $c_j$ are two sets of coupling constants, $u(q)$ and $U(Q)$ are two potentials. Note that we use the coordinate–momentum coupling between $A$ and $B$ while the coupling between $B$ and $C$ is chosen as the usual form. The purpose is to find two classes of nonergodic non-Markovian Brownian motions simultaneously within a set of coupling equations. As a matter of fact, the momentum-dependent coupling has been found in actual electromagnetic systems including a superconducting quantum interference device (SQUID),[13,14] a blackbody radiation field,[15] the vortex tunneling in superconductors,[29] and other systems.[30]

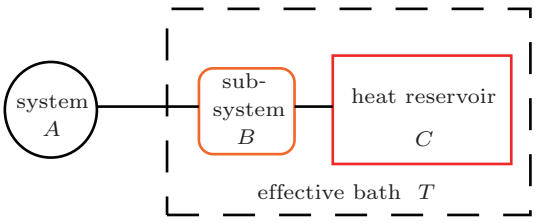



**Fig. 1.** (color online) Schematic representation of a nonergodic heat bath coupled to system $A$.

Solving $(2N+4)$-dimensional Hamiltonian canonical equations for $P$, $Q$; $p$, $q$; $y_j$, $p_j$ corresponding to Eq. (1), we have firstly the solution of bath variables as

$$\begin{aligned} y_j(t) &= y_j(0)\cos\omega_j t+\frac{p_j(0)}{\mu_j\omega_j}\sin\omega_j t \\ &\quad +\int_0^t\frac{c_j}{\mu_j\omega_j}\sin[\omega_j(t-t')]q(t')\,\mathrm{d}t'. \end{aligned} \tag{2}$$

Substituting $y_j(t)$ into the equations of motion of two systems $A$ and $B$, we get a set of coupling equations:

$$\begin{aligned} &\dot q=\frac{p}{m}, \\ &\dot p+\int_0^t\gamma(t-t')p(t')\,\mathrm{d}t'+u'(q)=-\frac{\lambda}{M}P(t)+\varepsilon(t), \end{aligned} \tag{3}$$

$$\dot Q=\frac{P}{M},\quad \dot P+U'(Q)=\frac{\lambda}{m}p(t), \tag{4}$$

where the noise $\varepsilon(t)$ and the memory-damping function $\gamma(t)$ in Eq. (3) are given by

$$\begin{aligned} \varepsilon(t) &= \sum_{j=1}^{N}c_j\left\{\left[y_j(0)-\frac{c_j}{\mu_j\omega_j^2}q(0)\right]\cos\omega_j t\right. \\ &\quad \left.+\frac{p_j(0)}{\mu_j\omega_j}\sin\omega_j t\right\}, \end{aligned} \tag{5}$$

$$\gamma(t) = \frac{1}{m}\sum_{j=1}^{N}\frac{c_j^2}{\mu_j\omega_j^2}\cos(\omega_j t). \tag{6}$$

As $y_j(0)$ and $p_j(0)$ are assumed to be distributed according to thermal equilibrium,[13–15] the FDT holds: $\langle\varepsilon(t)\varepsilon(t')\rangle = k_{\mathrm{B}}Tm\gamma(t-t')$. The influence between $A$ and $B$ is controlled through the coupling constant $\lambda$, and the coupled equations (3) and (4) need to be solved numerically in general.

It is known that the behavior of thermal noise and the damping function in GLE is determined by the coupling constants and frequencies of environmental oscillators, namely, the Laplace transform of the damping function $\gamma(t)$ is

$$\hat\gamma(s)=\frac{2s}{\pi m}\int_0^\infty\mathrm{d}\omega'\frac{J(\omega')}{\omega'}\frac{1}{\omega'^2+s^2}, \tag{7}$$

where the spectral density is defined by

$$J(\omega)=\frac{\pi}{2}\sum_j c_j^2\omega_j\mu_j^{-1}\delta(\omega-\omega_j).$$

In the absence of coupling to $A$ and $u(q)=0$ in Eq. (3), we have the two conditions:[23]

$$\hat\gamma(0)=\int_0^\infty\gamma(t)\,\mathrm{d}t=0\ \text{ and }\ \hat\gamma'(0)=\lim_{s\to0}\frac{\hat\gamma(s)}{s}=\text{finite}, \tag{8}$$

which leads to the ballistic diffusion of the system $B$ in the coordinate space and thus $p(t)$ is a nonergodic variable. This corresponds to a spectral density $J(\omega)$ which is proportional to cubic frequency at low frequency.

A simple model which satisfies these two conditions [Eq. (8)], $\varepsilon(t)$ is now modeled to be the time derivative of the well-known harmonic colored noise,[31]

$$\dot\varepsilon(t)+\Gamma\varepsilon(t)+\Omega^2\int_0^t\varepsilon(t')\,\mathrm{d}t'=\xi(t), \tag{9}$$

where $m=1.0$, $\Gamma$ and $\Omega$ are the damping and frequency parameters, $\xi(t)$ is the Gaussian white noise with $\langle\xi(t)\rangle=0$ and $\langle\xi(t)\xi(t')\rangle=\gamma\Omega^4k_{\mathrm{B}}T\delta(t-t')$. $\varepsilon(t)$ is called the harmonic velocity noise (HVN)[31] and its correlation function is given by

$$\begin{aligned} \langle\varepsilon(t)\varepsilon(t')\rangle &= \frac{\gamma\Omega^4k_{\mathrm{B}}T}{\mu_1^2-\mu_2^2}[-\mu_1\exp(\mu_1|t-t'|) \\ &\quad +\mu_2\exp(\mu_2|t-t'|)], \end{aligned} \tag{10}$$

where $\mu_1$ and $\mu_2$ are the two roots of $\mu^2+\Gamma\mu+\Omega^2=0$ and $\gamma$ is the damping strength of the subsystem $B$. Observably, $\int_0^\infty\langle\varepsilon(t)\varepsilon(0)\rangle\exp(st)\,\mathrm{d}t\propto s$, equation (8) is obeyed.

The GLE (3) with Eq. (9) can then be transferred into a set of embedding higher-dimensional Markovian Langevin equations as follows:

$$\dot{q}(t) = p, \quad \dot{p}(t) = -u'(q) - \lambda \dot{Q}(t) + w, \tag{11}$$

$$\dot{w}(t) = -\Gamma w - \gamma\Gamma p - \Omega^2 \varepsilon - h + \xi(t), \tag{12}$$

$$\dot{h}(t) = \Omega^2 (w - \eta), \tag{13}$$

$$\dot{\varepsilon}(t) = \eta(t), \quad \dot{\eta}(t) = -\Gamma\eta - \Omega^2\varepsilon + \xi(t). \tag{14}$$

To produce a stationary noise process, the distributions of noise $\varepsilon(t)$ and driven variables at the initial time are chosen to be independent Gaussian ones with zero mean and variances: $\{\varepsilon^2(0)\} = \gamma\Gamma k_{\rm B}T/\Omega^2$, $\{\eta^2(0)\} = \gamma\Gamma k_{\rm B}T$, as well as $\{w(0)\} = \eta(0)$ and a $\delta$ distribution for $h(0)$. Here and blow, $\{\cdot\}$ denotes the initial average.

In this case, the system $A$ is driven by the momentum variable $p(t)$ of the system $B$. The diffusive behavior of coordinate of the system $A$ in the absence of potential $[U(Q) = 0]$ is of interest to investigate, here we consider the subsystem $B$ to be a periodic structure determined by

$$u(q) = -\frac{u_0}{2\pi}\cos(2\pi q). \tag{15}$$

This implies that system $A$ is subjected to an intrinsic action through nonlinear noise dynamics.

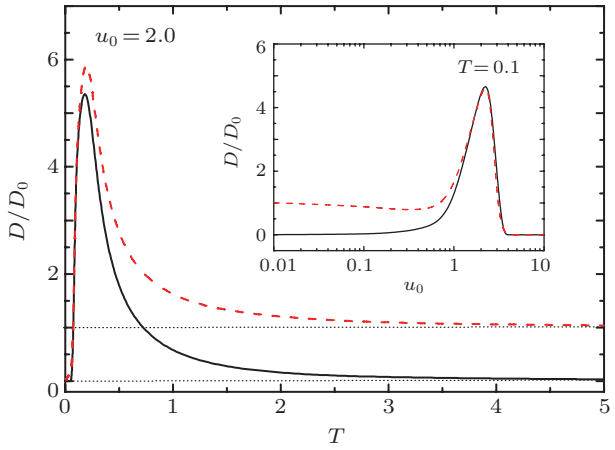



**Fig. 2.** (color online) Temperature dependence of the diffusion coefficient of a force-free particle (system $A$) for the normal case $D_0 = k_{\rm B}T/(m\gamma)$ at fixed $u_0 = 2.0$. The inset shows the barrier height at fixed $T = 0.1$. The solid and dashed lines are the results for HVN and white noise, respectively. The parameters used are: $\gamma = 1.0$, $\Gamma = 2.0$, and $\Omega = 1.0$.

Figure 2 shows the diffusion coefficient of system $A$ as a function of the temperature, which is calculated using

$$D = \frac{1}{2}\lim_{t\to\infty}\frac{\langle Q^2(t)\rangle - \langle Q^2(t_{\rm a})\rangle}{t - t_{\rm a}} \tag{16}$$

for $t \gg t_{\rm a}$, where $t_{\rm a}$ is the time of system after entering the stationary state; in other words, the mean square displacement (MSD) of particle becomes a linear function of time after $t = t_{\rm a}$. Similar phenomena are reported in Refs. [32]–[35]. Here and below dimensionless units and $k_{\rm B} = M = \lambda = 1$ are used. It is seen that the diffusion coefficient $D$ decreases and vanishes with increasing temperature or decreasing barrier height of periodic potential $u(q)$. This is opposite to the case of external periodic potential and represents a phase transition from diffusion to localization. Moreover, we observe a resonance-like phenomenon, namely, large enhancement of the effective diffusion coefficient relative to the normal free diffusion appears at a moderate temperature. This is due to nonlinear noise, which makes the shape of the effective bath spectrum depend on the temperature, as discussed below.

The spectrum of stochastic variable $p(t)$ driving the system $A$ is numerically calculated by

$$S_p(\omega) = \frac{1}{\langle p^2(\infty)\rangle}\lim_{t\to\infty}\int_0^\infty \langle p(t+\tau)p(t)\rangle\cos(\omega\tau)\,{\rm d}\tau. \tag{17}$$

In Fig. 3, we plot the spectrum of stochastic variable $p(t)$. It is found that the low-frequency behavior of the spectrum of $p(t)$ depends strongly on the temperature. A similar situation is used to study quantum decoherence reduction.[17] As one knows that the MSD of a force-free particle is inversely proportional to the zero-frequency friction: $\hat{\gamma}(0) = \int_0^\infty \gamma(t)\,{\rm d}t$, i.e., the value of $S_{\rm p}(0)$ varies according to the FDT, hence the diffusion coefficient of the system $A$ decreases and eventually vanishes when $S_{\rm p}(0)$ increases and becomes divergent.

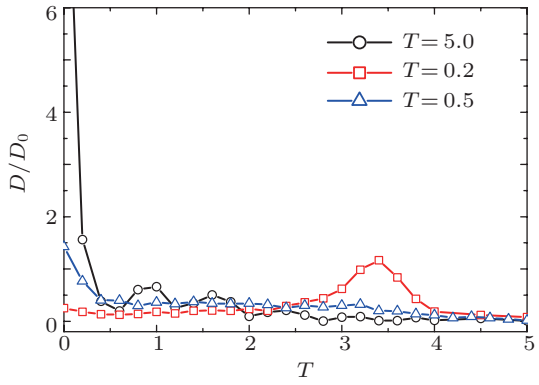



**Fig. 3.** (color online) The spectrum of $p(t)$ for various values of $T$ at fixed $u_0 = 0.2$. The parameters used are: $\gamma = 1.0$, $\Gamma = 2.0$, and $\Omega = 1.0$.

## 3. Brownian localization

To get a generic conclusion for the localization phenomenon in non-Markovian dynamics, i.e., what is the condition for a force-free particle to be localized? We consider $u(q) = 0$ case but do not restrict special noise. The analytical solution of subsystem $B$ is

$$p(t) = R(t)p(0) + \int_0^t R(t-t')[\varepsilon(t') - \lambda\dot{Q}(t')]\,{\rm d}t', \tag{18}$$

where the response function $R(t)$ is given by $R(t) = \mathscr{L}^{-1}[s + \hat{\gamma}(s)]^{-1}$ with $R(0) = 1$, $\hat{\gamma}(s) = \mathscr{L}\gamma(t)$, $\mathscr{L}$ and $\mathscr{L}^{-1}$ denote the Laplace and its inverse transforms. Further, in the absence

of coupled to the system $A$, i.e., $\lambda = 0$, we get the momentum correlation of the subsystem $B$ as

$$\begin{aligned}&\{\langle p(t+\tau)p(t)\rangle\}\\ &= mk_{\rm B}TR(|\tau|) + (\{p^2(0)\} - mk_{\rm B}T)R(t+\tau)R(t). \quad (19)\end{aligned}$$

If $R(\infty) = [1+\hat{\gamma}'(0)]^{-1} \neq 0$, the correlation function of momentum in the long-time limit is $\lim_{t,\tau\to\infty}\{\langle p(t+\tau)p(t)\rangle\} \neq 0$ which is apparently breakdown of the Khinchin theorem[36–38] which is judgment for ergodicity, so $p(t)$ is a nonergodic variable, simultaneously, the particle shows the ballistic diffusion in the coordinate space, i.e., $\langle q^2(t)\rangle \propto t^2$ in the long-time limit.[39,40] In such a case, the subsystem $B$ plus the heat bath $C$ can be regarded as a nonergodic heat bath for the system $A$.

The GLE for system $A$ is written as

$$\dot{P} + \frac{\lambda^2}{m}\int_0^t R(t-t')P(t')\,{\rm d}t' + U'(Q) = \varepsilon(t) \quad (20)$$

with

$$\varepsilon(t) = \frac{\lambda}{m}\left[R(t)p(0) + \int_0^t R(t-t')\eta(t')\,{\rm d}t'\right].$$

The correlation of noise $\varepsilon(t)$ is typically related to $R(t)$ through the FDT: $\langle\varepsilon(t)\varepsilon(t')\rangle = k_{\rm B}T\lambda m^{-1}R(t-t')$ as $\{p^2(0)\} = mk_{\rm B}T$.

Further, when $U(Q)$ is absent, we obtain the correction function of the coordinate of system $A$,

$$\begin{aligned}&\{\langle Q(t+\tau)Q(t)\rangle\}\\ &= \{Q^2(0)\} + \{Q(0)P(0)\}[\Psi(t+\tau)+\Psi(t)]\\ &\quad + \left(\{P^2(0)\} - Mk_{\rm B}T\right)\Psi(t+\tau)\Psi(t)\\ &\quad + Mk_{\rm B}T\Big(2\int_0^t \Psi(t')\,{\rm d}t'\\ &\quad + \int_0^\tau {\rm d}t'[\Psi(t+t') - \Psi(t')]\Big), \quad (21)\end{aligned}$$

where $\Psi(t) = \mathscr{L}^{-1}\hat{\Psi}(s)$ with $\hat{\Psi}(s) = [s^2 + s\hat{R}(s)]^{-1}$.

By using $\Psi(\infty) = \lim_{s\to 0}[s\hat{\Psi}(s)] = 0$ and $\int_0^\infty \Psi(t')\,{\rm d}t' = \hat{\Psi}(0)$ if $\hat{R}(0) = \infty$ but

$$\lim_{s\to 0}[s\hat{R}(s)] = \varpi^2 \neq 0, \quad (22)$$

we have

$$\lim_{t,\tau\to\infty}\{\langle Q(t+\tau)Q(t)\rangle\} = \{Q^2(0)\} + \frac{k_{\rm B}T}{M\varpi^2},$$

resulting in a breakdown of the Khinchin theorem. In particular,

$$\{\langle Q^2(\infty)\rangle\} = \{Q^2(0)\} + \frac{2k_{\rm B}T}{M\varpi^2}, \quad (23)$$

thus a force-free particle is located at a finite temperature.

In the case of harmonic potential $U(Q) = \frac{1}{2}c_2Q^2$, the exact second moment of coordinate and momentum of the system $A$ in the stationary state are obtained:

$$\{\langle Q^2(\infty)\rangle\} = \frac{k_{\rm B}T}{c_2} + \frac{\varpi^4}{(c_2+\varpi^2)^2}\left(\{Q^2(0)\} - \frac{k_{\rm B}T}{c_2}\right),$$

$$\langle P^2(\infty)\rangle = Mk_{\rm B}T. \quad (24)$$

This apparent violation of the energy equalization theorem, also, the dependence of the stationary result on the initial coordinate is a consequence of nonergodicity. This is due to the fact that a damped well potential is introduced around the initial position of particle. From GLE (20), we find that a linear force term separating from the memory damping integration $(\lambda^2/mM)R(\infty)[Q(t) - Q(0)]$ is added into the force term if a constant term exists in the damping kernel function. Considering that the noise driven the force-free system $B$ is the HVN determined by Eq. (9), we have $\varpi^2 = \lambda/(1+\gamma\Gamma/\Omega^2)$.

## 4. Applications

The role of the initial preparation of the system on its thermal equilibrium can be demonstrated in the zeroth law of thermodynamics. First, $A$ and $B$ relax towards a unique stationary state that equilibrates thermally with the reservoir $C$, thus $T_A = T_C$ and $T_{\rm B} = T_C$. Next, systems $A$ and $B$ are separated from the reservoir $C$, then thermally contact one another. In this case, the temperature of $B$ as the heat bath of $A$ is equal to $T_C$. Since the system $A$ is in initially thermal equilibrium, i.e., $T_A(t=0) = T_C$, after evolution and exchanging energy between $A$ and $B$, the system $A$ can arrive at thermal equilibration with the heat bath $B$ in the long-time limit, even when $A$ is a nonergodic system. Therefore, $T_A(t\to\infty) = T_C = T_{\rm B}$.

As another practical physical situation, the coupled oscillator chain acts such a heat bath, such that the coordinate of the particle is a nonergodic variable when it is coupled to this bath. The model constrained to periodic or fixed-end boundary conditions has long been used in the studies of collinear atom/solid-surface scattering, lattice vibration[41,42] as well as equilibration.[43] Nevertheless, the nonergodic behavior of lattice dynamics has not been discussed. The whole Hamiltonian ($H = H_{\rm s} + H_{\rm b} + H_{\rm int}$) of the system and the coupled-oscillator-chain reads[10,44]

$$\begin{aligned}H &= \frac{p_x^2}{2} + U(x) + \frac{\lambda}{2}(x-q_1)^2\\ &\quad + \sum_{j=1}^{N}\left[\frac{p_j^2}{2} + \frac{\omega_0^2}{2}(q_{j+1}-q_j)^2\right], \quad (25)\end{aligned}$$

where $x$ refers to the coordinate of atom, i.e., the Brownian particle, $p_j$ and $q_j$ are the momenta and coordinates of the $j$-th oscillator with $q_{N+1} = 0$, the coupled-oscillator-chain which is one-end free and other-end fixed represents the heat bath, the tagged particle interacts with the free-end oscillator. The statistical properties of the heat bath arise from the initial distribution of oscillators.

In order to produce the initial thermal distribution of the coupled-oscillator-chain, $P(q_1(0),\ q_2(0),\ldots,q_N(0)) =$

$C^{-1}\exp(-\beta H_{\rm b})$, we propose a recursion relation in terms of the normal Gaussian variables $\{\omega_i\}$ through a lower-triangle transformation of variables: $q_i = \sum_{k=1}^{j} a_{jk}\omega_k$, i.e.,

$$\begin{aligned} q_1(0) &= \sqrt{N\frac{k_{\rm B}T}{\omega_0^2}}\omega_1; \\ q_{j+1}(0) &= \frac{N-j}{N+1-j}q_j(0) + \sqrt{\frac{k_{\rm B}T}{\omega_0^2}\frac{N-j}{N+1-j}}\omega_{j+1}, \end{aligned} \quad (26)$$

where $j = 1,2,\ldots,N$, as well as $\langle\omega_j\rangle = 0$ and $\langle\omega_i\omega_j\rangle = \delta_{ij}$. Thus we get the variance:

$$\sigma_j^2(0) = \langle q_j^2(0)\rangle - \langle q_j(0)\rangle^2 = \frac{k_{\rm B}T}{\omega_0^2}(N+1-j).$$

Then, a set of equations of the motion for the tagged particle and the coupled-oscillator-chain from Eq. (25) are as follows:

$$\begin{aligned} \ddot{x} &= -\lambda(x-q_1) - U'(x), \\ \ddot{q}_1 &= -\lambda(q_1-x) + \omega_0^2(q_2-q_1), \\ \ddot{q}_j &= \omega_0^2(q_{j-1} - 2q_j + q_{j+1}) \end{aligned} \quad (27)$$

with $j = 1,2,\ldots,N$ and $q_{N+1} = 0$.

In Fig. 4, we plot the mean square displacement (MSD) of the $j$-th oscillator ($j = 1, 50, 100$) and the tagged particle subjected to a nonlinear potential:

$$U(x) = \frac{1}{2}c_2x^2 + \frac{1}{4}c_4x^4. \quad (28)$$

Here, we use the second-order stochastic Runge–Kutta algorithm with a small time step $\Delta t = 0.01$ and $10^4$ trajectories to simulate numerically whose equations, in which the initial condition of each oscillator is chosen as Eq. (26). It is seen that the MSD of the tagged particle depends on its initial coordinate preparation $\{x^2(0)\}$; this quantity increases with the increase of $\{x^2(0)\}$. The equilibration of the tagged particle can be reached only if it is initially thermal equilibrium. In addition, if the number of oscillators is enough, the ergodicity behavior of all the bath oscillators is ensured, namely, the MSD of oscillators do not depend on their different variances of the initial coordinate.

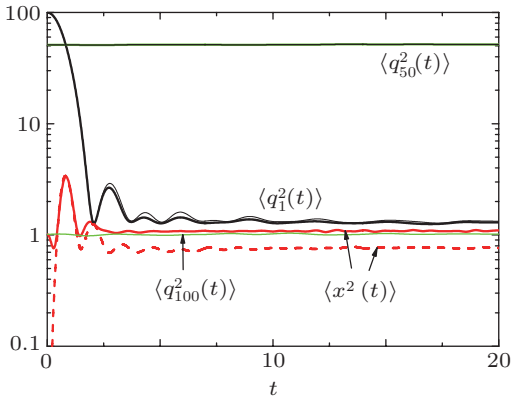



**Fig. 4.** (color online) The second moments of the Brownian particle and the bath oscillators for the initial distributions of two kinds: $\langle x^2(0)\rangle = k_{\rm B}T/c_2$ (solid lines) and $\langle x^2(0)\rangle = 0$ (dashed lines). The parameters used are: $\omega_0 = c_2 = 1.0$, $c_4 = 2.0$, and $T = 1.0$.

Let us analyze the origin of nonergodicity of the tagged particle. In fact, eliminating all the coupled oscillator chain variables, we obtain the dynamical equation of particle which has the same form as GLE (20) with the damping kernel function $\gamma(t) = \omega_0^2[J_0(2\omega_0 t) + J_2(2\omega_0 t)]$.[41,45] Note that the Laplace transformation of $\gamma(t)$ reads $\hat{\gamma}(z) = (\sqrt{z^2+4\omega_0^2} - z)/(2z)$. It is found that $\lim_{z\to 0} z\hat{\gamma}(z) = \omega_0$ and then Eq. (22) is obeyed. In the presence of a nonlinear potential $U(x)$, the dependence of distribution of tagged particle on its initial coordinate cannot be avoided yet, because there exists a damping trapping well being added into the initial position region of the particle. This is due to a fact that the memory damping function $\gamma(t)$ can separate out a constant, which contributes into the potential when the integration over the momentum term in the memory kernel of GLE is performed.

## 5. Summary

We have proposed a generalized coupling model including a nonlinear subsystem, a cubic spectrum, and momentum-dependent coupling. A phase transition has been demonstrated in the presence of an intrinsic periodic potential, namely, the diffusion coefficient of a force-free particle coupled to an effective heat bath vanishes (i.e., localization) when the temperature is larger than the height of the periodic potential. This anomalous phenomenon is due to the heat spectrum diverging at zero frequency. Hence the condition for localization of a force-free non-Markovian Brownian particle described by a generalized Langevin equation is proposed: the damping function does not vanish in the long-time limit, where the fluctuation–dissipation theorem holds. Indeed, the stationary distribution of a nonergodic particle of this kind depends on its initial coordinate preparation, because a damped trapping well around the initial position is introduced.

As applications of Brownian localization, we use the zeroth law of thermodynamics to illuminate the key role of initial thermal preparation in the asymptotical equilibrium of nonergodic system, i.e., the particle with initial thermalization can always arrive at the equilibrium state. Another example is the model of coupled-oscillator-chain, when a particle is coupled nonlinearly to an end of the chain, the Gibbs–Boltzmann statistics of the particle bounded by a potential no longer holds and its steady state depends on the initial coordinate preparation.

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