The KdV–Burgers equation in a modified speed gradient continuum model*

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Based on the full velocity difference model, Jiang *et al.* put forward the speed gradient model through the micro–macro linkage (Jiang R, Wu Q S and Zhu Z J 2001 *Chin. Sci. Bull.* **46** 345 and Jiang R, Wu Q S and Zhu Z J 2002 *Trans. Res. B* **36** 405). In this paper, the Taylor expansion is adopted to modify the model. The backward travel problem is overcome by our model, which exists in many higher-order continuum models. The neutral stability condition of the model is obtained through the linear stability analysis. Nonlinear analysis shows clearly that the density fluctuation in traffic flow leads to a variety of density waves. Moreover, the Korteweg-de Vries–Burgers (KdV–Burgers) equation is derived to describe the traffic flow near the neutral stability line and the corresponding solution for traffic density wave is derived. The numerical simulation is carried out to investigate the local cluster effects. The results are consistent with the realistic traffic flow and also further verify the results of nonlinear analysis.

Keywords: traffic flow, the speed gradient model, KdV–Burgers equation

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1. Introduction

In 1995, Daganzo,[1] a famous traffic flow theorist, questioned the hydrodynamic model of traffic flow. He thought gas-like behavior and vehicle back problems existed in these models, which caused a heated debate on the hydrodynamic model. In order to solve the problem, some anisotropic hydrodynamic models are put forward.

Chronologically, Aw and Rascle[2] who first proposed the anisotropic speed gradient model (Aw) thought that the reasonable dynamic equation should include not only the space derivative but also the density of convection derivative. The dynamic equation is

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\rho f'(\rho)\frac{\partial u}{\partial x}, \tag{1}$$

where f is a smooth monotone increasing function of density ρ. Compared with the density gradient model, the speed gradient model no longer has the density gradient term but the speed gradient term appears in the dynamic equation. So the Aw model solved the two problems proposed by Daganzo.

After that, Zhang[3] obtained the speed gradient model through the relation between the micro and the macro various. The governing equation reads

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-\rho u_{\rm e}\frac{\partial u}{\partial x}, \tag{2}$$

where $u_{\rm e}$ is the equilibrium speed.

Xue[4] established another anisotropic hydrodynamic model by considering two different delay time scales. The dynamic equation is

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{u_{\rm e}-u}{\tau}+\rho\frac{\hat{\tau}}{\tau}u'_{\rm e}\frac{\partial u}{\partial x}, \tag{3}$$

where τ is the relaxation time, and $\hat{\tau}$ is the driver reaction time.

Through the micro–macro linkages, the anisotropic macro speed gradient continuum model was also obtained by Jiang *et al.*[5,6] in 2002. Differently, the authors started with the full velocity difference model (for short, FVDM).[7] The governing equation is

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{u_{\rm e}-u}{T}+\frac{\Delta}{\tau}\frac{\partial u}{\partial x}, \tag{4}$$

where T is the relaxation time, Δ/τ is the disturbance propagation velocity. Jiang *et al.* made linear stability analysis to this model and got the neutral stability condition.

A lot of research has been done on car following models, and there have been a lot of studies relating to the traffic jams in congested traffic described by various density waves. The nonlinear wave equations, such as the Burgers equation, the Korteweg-de Vries (KdV) equation, and the modified Korteweg-de Vries (mKdV) equation have been obtained to show the corresponding density waves.[8–11] However, nonlinear research on the nonlinear density wave is relatively rare

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on hydrodynamic models. In 1995, Kurtze and Hong[12] derived the KdV equation by using the nonlinear analysis method from the hydrodynamic model. There are some works for KdV and mKdV equation in traffic continuum models.[13–18] Berg *et al.*[19] developed a systematic way to relate the car following model with the continuum model of road traffic in 2000. According to this relation, another aspect for car following models was analyzed for density waves.[20–22] Until now, the KdV–Burgers equation is hardly found in various traffic flow models. In this paper, we propose a modified speed gradient continuum model and conduct further research on it. It is worth pointing out that the fundamental principle that traffic flow is anisotropic and responds only to frontal stimuli is incorporated in our model, while most macroscopic models are isotropic.

The rest of this paper is organized as follows. In Section 2, the model is modified by Taylor expansion and the stability analysis is conducted in Section 3. The KdV–Burgers equation is derived through nonlinear analysis and the soliton solution is given in Section 4. Furthermore, a numerical simulation is described in Section 5. Finally, Section 6 makes a conclusion.

2. The modified continuum model

We regain the corresponding continuum traffic flow model on the basis of the FVDM, whose correlative equation is

$$\frac{\mathrm{d}v_n}{\mathrm{d}t} = k[V(\Delta x) - v_n(t)] + \lambda \Delta v_n, \tag{5}$$

where V is the optimal velocity.

Transferring the microscopic variables into the macroscopic ones as follows:

$$v_n(t) \to v(x,t), \quad v_{n+1}(t) \to v(x+\Delta,t),$$

$$V(\Delta x) \to V_{\mathrm{e}}(\rho), \quad k \to 1/T, \quad \lambda \to 1/\tau,$$

where T is the relaxation time, and τ is the time needed for the backward propagated disturbance to travel a distance Δ. Inserting the macro various into Eq. (5), the dynamic equation is converted into

$$\frac{\mathrm{d}v(x,t)}{\mathrm{d}t} = \frac{V_{\mathrm{e}}(\rho(x,t)) - v}{T} + \frac{v(x+\Delta,t) - v(x,t)}{\tau}. \tag{6}$$

Different from Jiang *et al.*, we make a Taylor expansion on the second term on the right-hand side of Eq. (6) around (x,t) to the second order, and neglect the higher-order terms. Then we get another continuum approximation of Eq. (6)

$$\frac{\mathrm{d}v(x,t)}{\mathrm{d}t} = \frac{V_{\mathrm{e}}(\rho(x,t)) - v}{T} + \frac{\Delta}{\tau}\frac{\partial v}{\partial x} + \frac{1}{2}\frac{\Delta^2}{\tau}\frac{\partial^2 v}{\partial x^2}. \tag{7}$$

Because $c_0 = \Delta/\tau \geq 0$ means the speed at which a small disturbance in traffic density (or speed) propagates back along a line of vehicles, so we call it traffic sound speed. We define

$$\mu_0 = \frac{1}{2}\frac{\Delta^2}{\tau} = \frac{1}{2}\tau c_0^2, \tag{8}$$

which depends on the relaxation time.

With the continuity equation, the modified speed gradient continuum model can be described by two partial differential equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \tag{9}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{V_{\mathrm{e}} - v}{T} + c_0\frac{\partial v}{\partial x} + \mu_0\frac{\partial^2 v}{\partial x^2}. \tag{10}$$

What is worth pointing out is that our model is concurrent with the fundamental principle that traffic flow is anisotropic and responds only to frontal stimuli, while most macroscopic models are isotropic. We will discuss the property in the following sections.

3. Stability analysis

Now we apply the method of characteristics to analyze the model. To aid this analysis, we rewrite Eqs. (9) and (10) in a vector form

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = E, \tag{11}$$

where

$$U = \begin{pmatrix} \rho \\ v \end{pmatrix}, \quad A = \begin{pmatrix} v & \rho \\ 0 & v - c_0 \end{pmatrix},$$

$$E = \begin{pmatrix} 0 \\ (V_{\mathrm{e}} - v)/T + \mu_0 \partial^2 v/\partial x^2 \end{pmatrix}. \tag{12}$$

The eigenvalues of the matrix A read

$$\lambda_1 = v, \quad \lambda_2 = v - c_0. \tag{13}$$

Because of $c_0 > 0, \lambda_2 < \lambda_1 = v$, the characteristic speed which is greater than the macroscopic traffic flow speed v does not exist, in other words, information can never reach the vehicles behind.

Then the linear stability method is used to analyze the qualitative properties of the model. We consider that the stability is a uniform flow. By assuming that traffic flow is initially in a state differing infinitesimally from the uniform flow, this flow is decomposed as

$$\begin{pmatrix} \rho(x,t) \\ v(x,t) \end{pmatrix} = \begin{pmatrix} \rho_0 \\ v_0 \end{pmatrix} + \sum_k \begin{pmatrix} \hat{\rho}_k \\ \hat{v}_k \end{pmatrix} \exp(\mathrm{i}kx + \sigma_k t). \tag{14}$$

By inserting the expression into Eq. (9) and Eq. (10), and neglecting the higher-order terms of the small perturbations, we have

$$(\sigma_k + \mathrm{i}kv_0)\hat{\rho}_k + \mathrm{i}k\rho_0\hat{v}_k = 0, \tag{15}$$

$$\sigma_k \hat{v}_k + \mathrm{i}k(v_0 - c_0)\hat{v}_k = \frac{1}{T}[V_e'(\rho_0)\hat{\rho}_k - \hat{v}_k] + \mu_0(\mathrm{i}k)^2\hat{v}_k. \quad (16)$$

For the linear algebraic equation in $\hat{\rho}_k$ and $\hat{v}_k$, we find that the linear growth rate σ_k satisfies the quadratic equation

$$(\sigma_k + \mathrm{i}kv_0)^2 + \left(\frac{1}{T} + \mu_0 k^2 - \mathrm{i}kc_0\right)(\sigma_k + \mathrm{i}kv_0) + \frac{1}{T}\mathrm{i}k\rho_0 V_e'(\rho_0) = 0. \quad (17)$$

According to the stability criterion of the control theories, the linear stability of the model depends on the roots of σ_k. The system is said to be stable if both roots of σ_k have negative real parts, because the uniform traffic flow is stable against all infinitesimal perturbations. However, if σ_k is a positive value, the uniform flow is unstable. So the neutral stability condition for this steady state is given by

$$T_s = \frac{-\rho_0 V_e'(\rho_0) - c_0}{\mu_0 c_0 k^2}. \quad (18)$$

Only when $-\rho_0 V_e'(\rho_0) > c_0$, does the above condition have physical sense

$$\mathrm{Im}(\sigma_k) = -kv_0 - \frac{k\rho_0 V_e'(\rho_0)}{1 + T\mu_0 k^2}. \quad (19)$$

Considering $(1 + T\mu_0 k^2)^{-1} \approx 1 - T\mu_0 k^2 + T^2\mu_0^2 k^4 - \cdots$, then we obtain

$$\mathrm{Im}(\sigma_k) \approx -k(v_0 + \rho_0 V_e'(\rho_0)) + O(k^3). \quad (20)$$

From Eq. (20), we deduce the critical speed of disturbance propagation

$$c(\rho_0) = v_0 + \rho_0 V_e'(\rho_0). \quad (21)$$

The above $c(\rho_0)$ is in accordance with the kinematic wave speed mentioned in Ref. [23].

4. Nonlinear analysis

From the stability analysis, one can find that in the unstable region, traffic flow will evolve into congestion after a small disturbance is added. We concentrate on the system behavior near the neutral stability condition given in Eq. (18), and transform the reference system of the wave by introducing the coordinate[19]

$$z = x - ct. \quad (22)$$

Then the system described by Eqs. (9) and (10) can be rewritten respectively as

$$-c\rho_z + q_z = 0, \quad (23)$$

$$-cv_z + (v - c_0)v_z = a[V_e(\rho) - v] + \mu_0 v_{zz}, \quad (24)$$

where q is the flow which equals the product of the density and speed. From Eq. (23), we get

$$v_z = \frac{c\rho_z}{\rho} - \frac{q\rho_z}{\rho^2}. \quad (25)$$

Inserting Eq. (25) into Eq. (24), we obtain the following equation

$$-\frac{\rho_z}{\rho^3}q^2 + \left[\frac{(2c + c_0)\rho_z}{\rho^2} + \frac{a}{\rho} + \frac{\mu_0\rho_{zz}}{\rho^2}\right]q = \frac{(c^2 + cc_0)\rho_z}{\rho} + aV_e(\rho) + \frac{c^2\rho_z}{\rho}. \quad (26)$$

We describe the flow q as

$$q = \rho V_e(\rho) + b_1\rho_z + b_2\rho_{zz}. \quad (27)$$

The first term on the right-hand side of the equation represents homogeneous, stabilty, and stable flow q, while b_1 and b_2 are parameters determined below. Substituting Eq. (27) into Eq. (26), we have

$$\begin{aligned} &-\frac{c^2}{\rho}\rho_z + \frac{(c_0 + 2c)\rho_z}{\rho^2}[\rho V_e(\rho) + b_1\rho_z + b_2\rho_{zz}] \\ &-\frac{\rho_z}{\rho^3}[\rho V_e(\rho) + b_1\rho_z + b_2\rho_{zz}]^2 - \frac{cc_0}{\rho}\rho_z \\ &= aV_e(\rho) - \frac{a}{\rho}[\rho V_e(\rho) + b_1\rho_z + b_2\rho_{zz}] + \frac{c\mu_0}{\rho}\rho_{zz} \\ &-\frac{\mu_0}{\rho^2}\rho_{zz}[\rho V_e(\rho) + b_1\rho_z + b_2\rho_{zz}]. \end{aligned} \quad (28)$$

The parameters b_1 and b_2 can be determined through balancing the terms ρ_z and ρ_{zz} in Eq. (28), which turns out to be

$$b_1 = \frac{(V_e - c)^2 + c_0(c - V_e)}{a}, \quad (29)$$

$$b_2 = \frac{\mu_0(c - V_e)}{a}. \quad (30)$$

Near the neutral stability condition, assuming $\rho = \rho_h + \hat{\rho}(x,t)$ and performing Taylor expansions to the second order of $\hat{\rho}$, then we have

$$\rho V_e(\rho) \approx \rho_h V_e(\rho_h) + (\rho V_e)_\rho\,|_{\rho=\rho_h}\,\hat{\rho} + \frac{1}{2}(\rho V_e)_{\rho\rho}\,|_{\rho=\rho_h}\,\hat{\rho}^2. \quad (31)$$

Inserting the above expression into Eq. (23), and the $\hat{\rho}$ in later formulas will be denoted by ρ, we obtain

$$-c\rho_z + [(\rho V_e)_\rho + (\rho V_e)_{\rho\rho}\rho]\rho_z + b_1\rho_{zz} + b_2\rho_{zzz} = 0. \quad (32)$$

In order to derive the regularized KdV–Burgers equation from Eq. (32), we perform the following transformation

$$U = -[(\rho V_e)_\rho + (\rho V_e)_{\rho\rho}\rho], \quad X = wx, \quad T = -wt. \quad (33)$$

And considering Eq. (22), we derive the equation

$$U_T + UU_X - wb_1U_{XX} - w^2b_2U_{XXX} = 0. \quad (34)$$

We suppose $w_1 = -wb_1$, $w_2 = -w^2 b_2$, then we get the standard KdV–Burgers equation

$$U_T + UU_X + w_1 U_{XX} + w_2 U_{XXX} = 0. \tag{35}$$

One of the solutions of the above KdV–Burgers equation is

$$U = -\frac{3w_1^2}{25w_2}\left[1 + 2\tanh\left(\pm\frac{w_1}{10w^2}\right)\left(X + \frac{6w_1^2}{25w_2}T + \xi_0\right) + \tanh^2\left(\pm\frac{w_1}{10w^2}\right)\left(X + \frac{6w_1^2}{25w_2}T + \xi_0\right)\right], \tag{36}$$

where ξ_0 is an arbitrary constant.

5. Numerical simulation

The new modified model can also describe the nonlinear theory of the local cluster effect in the traffic flow, namely, the influence of the appearance of a region of high density and low average speed of vehicles in the original uniform flow. In this section, we simulate the local cluster effect on local disturbance in an initial homogeneous condition.

We carry out the numerical simulation for the new anisotropic model with the finite difference method. The following difference equations are obtained by using the finite difference scheme on the set of partial differential Eqs. (9) and (10):

$$\rho_i^{j+1} = \rho_i^j + \frac{\Delta t}{\Delta x}\rho_i^j(v_i^j - v_{i+1}^j) + \frac{\Delta t}{\Delta x}v_i^j(\rho_{i-1}^j - \rho_i^j). \tag{37}$$

(i) If the traffic is heavy (i.e., $v_i^j < c_0$), we have

$$v_i^{j+1} = v_i^j + \frac{\Delta t}{\Delta x}(c_0 - v_i^j)(v_{i+1}^j - v_i^j) - \frac{\Delta t}{T}(v_i^j - V_e) + \mu_0\frac{\Delta t}{(\Delta x)^2}(v_{i+1}^j - 2v_i^j + v_{i-1}^j); \tag{38}$$

(ii) If the traffic is light (i.e., $v_i^j \geqslant c_0$), we obtain

$$v_i^{j+1} = v_i^j + \frac{\Delta t}{\Delta x}(c_0 - v_i^j)(v_i^j - v_{i-1}^j) - \frac{\Delta t}{T}(v_i^j - V_e) + \mu_0\frac{\Delta t}{(\Delta x)^2}(v_{i+1}^j - 2v_i^j + v_{i-1}^j), \tag{39}$$

where index i represents the road section and index j represents time. Here Δt and Δx are the grid sizes of the finite difference mesh in time and space dimensions, respectively. The above difference scheme can maintain the physical properties of traffic flow even under extreme conditions.

The following initial variation of the average density ρ_0 is given by Herrmann and Kerner[24]

$$\rho(x,0) = \rho_0 + \Delta\rho_0\left\{\cosh^{-2}\left[\frac{160}{L}\left(x - \frac{5L}{16}\right)\right] - \frac{1}{4}\cosh^{-2}\left[\frac{40}{L}\left(x - \frac{11L}{32}\right)\right]\right\}, \tag{40}$$

where $x \in [0, L]$, and $L = 32.2$ km is the length of the road section under consideration. The periodic boundary conditions are given by as follows:

$$\rho(L,t) = \rho(0,t), \quad v(L,t) = v(0,t). \tag{41}$$

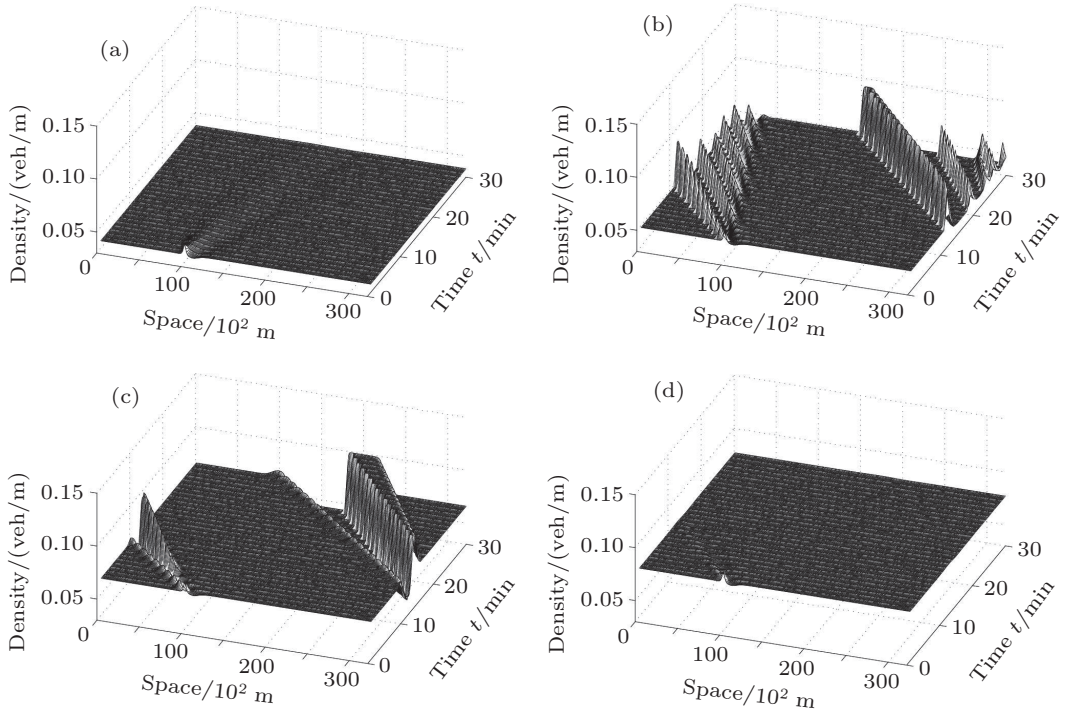


Fig. 1. The time evolution of traffic flow in a range of 32.2-km circumference with a uniform initial traffic and a localized perturbation of amplitude $\Delta\rho_0 = 0.01$ veh/m for (a) $\rho_0 = 0.038$ veh/m; (b) $\rho_0 = 0.049$ veh/m; (c) $\rho_0 = 0.066$ veh/m; (d) $\rho_0 = 0.077$ veh/m. The vehicle is abbreviated to veh in the figure.

We next apply the equilibrium speed-density relationship proposed by Kerner and Konhäuser[25]

$$V_e = v_f\left[\left(1+\exp\frac{\rho/\rho_m-0.25}{0.06}\right)^{-1}-3.72\times10^{-6}\right], \quad (42)$$

where v_f is the free-flow velocity, ρ_m is the maximum density. Assume that the initial flow is in local equilibrium everywhere, i.e., $v(x,0)=V_e(\rho(x,0))$. For the purpose of computation, the space domain is divided into equal intervals of length 100 m, and the time interval is 1 s. Other parameter values are set as follows:[5]

$$v_f = 30\ \text{m/s},\ \ \rho_m = 0.2\ \text{veh/m},\ \ T = 10\ \text{s},$$
$$c_0 = 11\ \text{m/s},\ \ \mu_0 = 550.$$

Substituting the values into the stable condition (18), and the simulation results are shown in Figs. 1(a)–1(d).

Through the numerical simulation, we find that the traffic will be unstable when $0.042<\rho_0<0.07$. In Fig. 1(a), the traffic flow density is below the critical density, the perturbation dissipates with time and the traffic flows is stable. As the initial density increases to 0.049, a complex localized structure consisting of two or more clusters forms, so Fig. 1(b) reappears the stop-and-go traffic phenomenon. If the traffic density increased even further, we observe an elliptical cosine wave-like structure as illustrated by Fig. 1(c). When the density above the up-critical density, a stable form is obtained again as shown in Fig. 1(d). The above results show good agreement with the properties of traffic flow found by Kerner and Konhäuser, Herrmann and Kerner. And these local cluster effects are also in accordance with the diverse nonlinear dynamics phenomena happened in real traffic.

6. Summary

We have proposed a new speed gradient continuum model with the Taylor expansion, which has the anisotropic property without any addition strategies. On the basis of the model, the stability condition and the KdV–Burgers equation describing the density wave of traffic congestion are obtained by using the linear stability theory and the nonlinear analysis method, respectively. The results indicate that the model is unstable with some conditions, which leads to the formation of complex traffic. The KdV–Burgers equation we have derived is seldom found in other traffic flow models and the corresponding soliton solution is given. Finally, the numerical tests indicate that the model can simulate complex traffic phenomena such as stop-and-go waves and local cluster effects. What is more, it coincides with the result obtained from nonlinear analysis.

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