A novel study on the impulsive synchronization of fractional-order chaotic systems*

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(Received 4 October 2012; revised manuscript received 8 December 2012)

The stability of impulsive fractional-order systems is discussed. A new synchronization criterion of fractional-order chaotic systems is proposed based on the stability theory of impulsive fractional-order systems. The synchronization criterion is suitable for the case of the order $0 < q \leq 1$. It is more general than those of the known results. Simulation results are given to show the effectiveness of the proposed synchronization criterion.

Keywords: fractional-order, chaotic systems, impulsive synchronization, Gronwall–Bellman's inequality

PACS: 05.45.Gg, 05.45.Xt **DOI:** 10.1088/1674-1056/22/6/060510

1. Introduction

In recent years, many researchers have been greatly attracted to fractional calculus due to its potential applications in physics and engineering.[1,2] Especially, some fractional-order dynamical systems behave chaotically, such as the fractional-order Chen system, the fractional-order Lü system, and the fractional-order Rössler system, etc.[3–6]

The synchronization of chaotic systems is a hot subject in the field of nonlinear science since Pecora and Carroll introduced a method to synchronize two identical chaotic systems with different initial conditions. And many different synchronization phenomena and control methods have been proposed, such as complete synchronization, lag synchronization, projective synchronization, the OGY method, adaptive control, active control, the backstepping technique, impulsive control, etc.[7–12] Among all kinds of control methods, impulsive control has received much attention as it has advantages of easy accomplishment, lower cost, and little wastage.[13–15] Although many studies on control and synchronization of fractional-order chaotic systems have been published, only the impulsive synchronization of fractional-order systems has been discussed.[16–19] Wang *et al.*[16] and Zhong *et al.*[17] studied impulsive synchronization of fractional-order systems by using the comparison theory of integer-order systems. The methods in the above studies are mainly approximations of integer-order impulsive control systems based on the comparison theory of impulsive systems, and only the impulsive controllers are employed to fractional-order systems, which lack theoretical basis. Ma *et al.*[18] and Fu *et al.*[19] investigated impulsive synchronization and modified impulsive synchronization of fractional-order systems by constructing a suitable response system method. The synchronization scheme for the fractional-order systems is converted into the stability analysis of the integer-order error systems. However, the method in Refs. [18] and [19] is still based on the comparison theory of integer-order systems. To the best of my knowledge, few theoretical studies on impulsive synchronization of fractional-order systems are reported in the literature. In this paper, the stability of impulsive fractional-order systems is investigated by employing Gronwall–Bellman's inequality. Furthermore, a new impulsive synchronization criterion of fractional-order chaotic systems is obtained.

The rest of this paper is organized as follows. In Section 2, the stability criterion of impulsive fractional-order systems is presented. In Section 3, the impulsive synchronization is investigated and the synchronization criterion is obtained. In Section 4, corresponding numerical simulations are performed to verify the effectiveness of the results. Finally, conclusions are drawn in Section 5.

2. Stability theory of impulsive fractional-order systems

Consider a class of nonlinear fractional-order systems

$$\begin{cases} \dfrac{\mathrm{d}^q \boldsymbol{x}(t)}{\mathrm{d}t^q} = f(\boldsymbol{x}(t)), \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0, \end{cases} \tag{1}$$

where $\boldsymbol{x} \in R^n$ is the state vector of the system; $f \in R^n$ is a continuous differentiable vector function. $\mathrm{d}^q \boldsymbol{x}(t)/\mathrm{d}t^q$ denotes the Caputo definition with $0 < q < 1$, which is defined as follows:

$$\frac{\mathrm{d}^q \boldsymbol{x}(t)}{\mathrm{d}t^q} = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-\tau)^{-q} \boldsymbol{x}'(\tau)\,\mathrm{d}\tau.$$

Suppose that a discrete time set $\{t_k\}$ satisfies $0 < t_1 < t_2 < \cdots < t_k < \cdots$, $\lim\limits_{k\to\infty} t_k = \infty$.

*Project supported by Scientific Research Foundation of Huaiyin Institute of Technology (Grant No. HGA1102).

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The impulsive fractional-order system is given by

$$\begin{cases} \dfrac{\mathrm{d}^q \boldsymbol{x}(t)}{\mathrm{d}t^q} = f(\boldsymbol{x}(t)), \quad t \neq t_k, \\ \Delta \boldsymbol{x} = U(k, \boldsymbol{x}), \quad t = t_k, \\ \boldsymbol{x}(t_0^+) = \boldsymbol{x}_0. \end{cases} \tag{2}$$

Definition 1 The trivial solution of system (1) is said to be stable, if $\forall \varepsilon > 0$, there exists $\delta(\varepsilon, t_0) > 0$ such that any solution $\boldsymbol{x}(t)$ of system (1) satisfying $\|\boldsymbol{x}(t_0)\| < \delta(\varepsilon, t_0)$ satisfies $\|\boldsymbol{x}(t)\| < \varepsilon$ for all $t \geq t_0$. The trivial solution of system (1) is said to be asymptotically stable if, in addition to being stable, $\|\boldsymbol{x}(t)\| \to 0$ as $t \to \infty$.

Lemma 1[20] The system (1) is equivalent to the nonlinear integral equation

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(\boldsymbol{x}) \mathrm{d}s, \tag{3}$$

where $0 < t_0 < t \leq b$. That is, every solution of the integral equation (3) is also a solution of the original IVP (1), and *vice versa*.

Lemma 2[21] Let $u(t)$ and $v(t)$ be nonnegative continuous functions on some interval $t_0 \leq t \leq b$. Also, let the function $h(t)$ be positive, continuous, and monotonically nondecreasing on $t_0 \leq t \leq b$, and satisfy the inequality

$$u(t) \leq h(t) + \int_{t_0}^{t} u(s) v(s) \mathrm{d}s,$$

then there exists

$$u(t) \leq h(t) \exp\{\int_{t_0}^{t} v(s) \mathrm{d}s\}, \quad t_0 \leq t \leq b.$$

Inspired by the comparison theory of integer-order impulsive systems,[22] the following theorems are presented to discuss the stability of the impulsive fractional-order systems.

Theorem 1 Suppose that

(i) $|f(\boldsymbol{x}(t))| \leq g(t, |\boldsymbol{x}|), t \neq t_k, (t, \boldsymbol{x}) \in R_+ \times S(\rho)$, where $g \in C(R_+ \times R_+, R)$ with $g(t,0) = 0$, $S(\rho) = \{\boldsymbol{x} \in R^n | |\boldsymbol{x}| < \rho\}$;

(ii) $|\boldsymbol{x} + U(k, \boldsymbol{x})| \leq \psi_k(|\boldsymbol{x}|), t = t_k$, where $\psi_k : [0, \rho_0) \to [0, \rho)$ with $\psi_k(0) = 0$, and $\psi_k(u)$ is strictly increasing in u.

Then the asymptotic stability of trivial solution $w = 0$ of

$$\begin{cases} \dfrac{\mathrm{d}^q w(t)}{\mathrm{d}t^q} = g(w(t)), \quad t \neq t_k, \\ w(t_k^+) = \psi_k(w(t_k)), \quad t = t_k, \\ w(t_0^+) = w_0 \geq 0, \end{cases} \tag{4}$$

implies the asymptotic stability of the trivial solution $x = 0$ of system (2).

Proof Suppose that the trivial solution $w = 0$ is asymptotically stable. From stability definition of the system, the trivial solution $w = 0$ is stable, let $\rho^* = \min\{\rho_0, \rho\}$, then for a given $0 < \varepsilon < \rho^*$, there exists a $\delta = \delta(t_0, \varepsilon)$ such that any solution $w(t)$ of system (4) satisfying $w_0 < \delta$ satisfies $w(t) < \varepsilon$. Next, prove that $\boldsymbol{x} = 0$ of system (2) is stable for this δ and ε. Suppose $\boldsymbol{x} = 0$ of system (2) is not stable. There would exist a solution $\boldsymbol{x}(t)$ of system (2) and $t^* > t_0$ such that for some k, satisfying $t_k < t^* \leq t_{k+1}$, $|\boldsymbol{x}(t^*, t_0, \boldsymbol{x}_0)| \geq \varepsilon$ and $|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)| < \varepsilon$ for $t_0 \leq t \leq t_k$.

By condition (ii), $|\boldsymbol{x}(t_k^+, t_0, \boldsymbol{x}_0)| \leq \psi_k(|\boldsymbol{x}(t_k, t_0, \boldsymbol{x}_0)|) < \psi_k(\varepsilon) < \psi(\rho_0) < \rho$ is obtained. We choose $t_k < \bar{t} \leq t^*$ satisfying $\varepsilon \leq |\boldsymbol{x}(\bar{t}, t_0, \boldsymbol{x}_0)| < \rho$. Let $\bar{w}(t, t_0, w_0)$ be the maximal solution of system (4), by the Comparison Theorem in Ref. [20], it can be obtained that $|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)| < \bar{w}(t, t_0, w_0), t_0 \leq t \leq \bar{t}$, which can lead to the contradiction $\varepsilon \leq |\boldsymbol{x}(\bar{t}, t_0, \boldsymbol{x}_0)| \leq \bar{w}(\bar{t}, t_0, w_0) < \varepsilon$, so we prove that $\boldsymbol{x} = 0$ of system (2) is stable.

Let $\varepsilon = \rho^*, \delta_0^* = \delta(t_0, \rho^*)$, we have that the $\boldsymbol{x}(t)$ of system (2) satisfying $|\boldsymbol{x}_0| < \delta^*$ satisfies $|\boldsymbol{x}(t)| < \rho, t \geq t_0$.

Since $w = 0$ is asymptotically stable, for given $0 < \varepsilon < \rho^*$ and $t_0 \in R_+$, there exists $\delta_{10} = \delta(t_0) > 0$ and $T = T(t_0, \varepsilon) > 0$ such that any solution $w(t)$ of system (4) satisfying $w_0 < \delta_{10}$ satisfies $w(t) < \varepsilon (t \geq t_0 + T)$. Choose $\delta_0 = \min\{\delta_0^*, \delta_{10}\}$ and $|\boldsymbol{x}_0| < \delta_0$, thus we have $|\boldsymbol{x}(t, t_0, \boldsymbol{x}_0)| \leq \bar{w}(t, t_0, w_0) < \varepsilon, t \geq t_0$.

Hence, the trivial solution $\boldsymbol{x} = 0$ of system (2) is asymptotically stable. This completes the proof.

Theorem 2 Let $g(w) = \lambda w, \lambda > 0$, $\psi_k(w) = d_k w, d_k \geq 0$ for all k, $w(t_0^+) = w_0 \geq 0$, then the trivial solution $w = 0$ of the system

$$\begin{cases} \dfrac{\mathrm{d}^q w(t)}{\mathrm{d}t^q} = \lambda w, \quad t \neq t_k, \\ w(t_k^+) = d_k w, \quad t = t_k, \\ w(t_0^+) = w_0 \geq 0, \end{cases} \tag{5}$$

is asymptotically stable if condition

$$\gamma d_k E_q(\lambda \tau_k^q) < 1 \tag{6}$$

is satisfied, where $\gamma > 1, \tau_k = t_{k+1} - t_k, k = 1, 2, \ldots$.

Proof From Lemma 2, we can obtain the following results.

For $t \in (t_{k-1}, t_k]$, it follows from Eq. (3) that

$$\begin{aligned} w(t) &= w(t_{k-1}^+) + \frac{1}{\Gamma(q)} \int_{t_{k-1}}^{t} (t-s)^{q-1} g(w) \mathrm{d}s \\ &= w(t_{k-1}^+) + \frac{1}{\Gamma(q)} \lambda \int_{t_{k-1}}^{t} (t-s)^{q-1} w \mathrm{d}s. \end{aligned} \tag{7}$$

Moreover we obtain

$$\begin{aligned} w(t) &= w(t_{k-1}^+) E_q(\lambda (t - t_{k-1})^q), \\ & t \in (t_{k-1}, t_k]. \end{aligned} \tag{8}$$

On the other hand, for $t = t_{k-1}$, we have $w(t_{k-1}^+) = d_{k-1} w(t_{k-1})$, therefore,

$$\begin{aligned} w(t) &= w(t_{k-1}^+) E_q(\lambda (t - t_{k-1})^q) \\ &= d_{k-1} w(t_{k-1}) E_q(\lambda (t - t_{k-1})^q). \end{aligned} \tag{9}$$

For $t \in (t_0, t_1]$, it follows from inequality (8) that

$$w(t) = w(t_0^+) E_q(\lambda (t - t_0)^q) = w_0 E_q(\lambda (t - t_0)^q),$$

which leads to

$$w(t_1) = w_0 E_q(\lambda(t_1 - t_0)^q).$$

Similarly, for $t \in (t_1, t_2]$,

$$w(t) = d_1 w_0 E_q(\lambda(t_1 - t_0)^q) E_q(\lambda(t - t_1)^q).$$

In general, for $t \in (t_{k-1}, t_k]$,

$$\begin{aligned} w(t) &= w_0 d_1 d_2 \cdots d_{k-1} E_q(\lambda(t_1 - t_0)^q) \\ &\quad \times E_q(\lambda(t_2 - t_1)^q) \cdots E_q(\lambda(t - t_{k-1})^q) \\ &\le w_0 d_1 d_2 \cdots d_{k-1} E_q(\lambda(t_1 - t_0)^q) \\ &\quad \times E_q(\lambda(t_2 - t_1)^q) \cdots E_q(\lambda(t_k - t_{k-1})^q). \end{aligned}$$

It follows from inequality (6) that

$$w(t) \le w_0 \frac{1}{\gamma^{k-1}} E_q(\lambda(t_1 - t_0)^q).$$

Obviously, $w(t) \to 0$ as $k \to \infty$. From stability theory in the sense of Lyapunov, the trivial solution $w = 0$ of system (5) is asymptotically stable. This completes the Proof.

Remark 1 The system (5) is a scalar impulsive fractional-order system. If we can construct the scalar impulsive system of impulsive fractional-order system, then the stability analysis can be simplified to that of a scalar impulsive system.

3. Impulsive synchronization of fractional-order chaotic systems

In the following, the sufficient condition of impulsive synchronization for system (1) is presented.

In order to achieve synchronization, we construct the controlled response system as follows:

$$\begin{cases} \dfrac{\mathrm{d}^q \boldsymbol{y}(t)}{\mathrm{d}t^q} = f(\boldsymbol{y}(t)), \quad t \ne t_k, \\ \Delta \boldsymbol{y} = U(k, \boldsymbol{e}) = \boldsymbol{C}_k(\boldsymbol{y} - \boldsymbol{x}), \quad t = t_k, \\ \boldsymbol{y}(t_0^+) = \boldsymbol{y}_0, \end{cases} \tag{10}$$

where $\boldsymbol{y} \in R^n$ is the state vector of the system, $\Delta \boldsymbol{y}(t_k) = \boldsymbol{y}(t_k^+) - \boldsymbol{y}(t_k^-), \boldsymbol{y}(t_k^+) = \lim\limits_{t \to t_k^+} \boldsymbol{y}(t_k)$, $\boldsymbol{y}(t_k^-) = \lim\limits_{t \to t_k^-} \boldsymbol{y}(t_k)$, and $U(k, \boldsymbol{e})$ is the control law, which is the "jump" in the state variable at the time instant t_k, and $\boldsymbol{y}(t_k^-) = \boldsymbol{y}(t_k)$ is assumed.

Let $\boldsymbol{e} = [e_1, e_2, \ldots, e_n]^{\mathrm{T}} = [y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n]^{\mathrm{T}}$. Then, subtracting Eq. (1) from Eq. (10) yields error dynamical system between Eqs. (1) and (10)

$$\begin{cases} \dfrac{\mathrm{d}^q \boldsymbol{e}(t)}{\mathrm{d}t^q} = f(\boldsymbol{y}(t)) - f(\boldsymbol{x}(t)), \quad t \ne t_k, \\ \Delta \boldsymbol{e} = U(k, \boldsymbol{e}) = \boldsymbol{C}_k \boldsymbol{e}, \quad t = t_k, \\ \boldsymbol{e}(t_0^+) = \boldsymbol{e}_0. \end{cases} \tag{11}$$

Our goal is to find some conditions on the control gains $\boldsymbol{C}_k$ and the impulsive distances $\tau_k = t_{k+1} - t_k < \infty (k = 1, 2, \ldots)$ guaranteeing the synchronization of the systems (1) and (10), that is, $\lim\limits_{t \to \infty} \|\boldsymbol{e}(t)\| = \lim\limits_{t \to \infty} \|\boldsymbol{y}(t) - \boldsymbol{x}(t)\| = 0$.

Theorem 3 Suppose that $f(\boldsymbol{x})$ satisfies the inequality $\|f(\boldsymbol{x}) - f(\boldsymbol{y})\| \le L\|\boldsymbol{x} - \boldsymbol{y}\|$. If there exists a positive number $\gamma > 1$ such that the following condition is satisfied

$$\gamma |\beta_k| E_q(L\tau_k^q) < 1, \tag{12}$$

then the impulsive synchronization between systems (1) and (10) is obtained, where L is a positive constant, β_k^2 is the largest eigenvalues of $(\boldsymbol{I} + \boldsymbol{C}_k)^{\mathrm{T}}(\boldsymbol{I} + \boldsymbol{C}_k)$, $\tau_k = t_{k+1} - t_k, k = 1, 2, \ldots$.

Proof Suppose that $\|f(\boldsymbol{x}) - f(\boldsymbol{y})\| \le L\|\boldsymbol{x} - \boldsymbol{y}\|$, condition (i) of Theorem 1 is satisfied with $g(t, w) = Lw$.

Given any $\rho_0 > 0$ and $\boldsymbol{e} \in S(\rho_0)$, we have

$$\begin{aligned} \|\boldsymbol{e} + U(k, \boldsymbol{e})\| &= \|(\boldsymbol{I} + \boldsymbol{C}_k)\boldsymbol{e}\| \le \|\boldsymbol{I} + \boldsymbol{C}_k\| \|\boldsymbol{e}\| \\ &\le |\beta_k| \|\boldsymbol{e}\| \le \|\boldsymbol{e}\|. \end{aligned} \tag{13}$$

From the last inequality, we have $\boldsymbol{e} + U(k, \boldsymbol{e}) \in S(\rho_0)$.

When $t = t_k$, we have $\|\boldsymbol{e} + U(k, \boldsymbol{e})\| = \|(\boldsymbol{I} + \boldsymbol{C}_k)\boldsymbol{e}\| \le |\beta_k| \|\boldsymbol{e}\|$, condition (ii) of Theorem 1 is satisfied with $\psi_k(w) = |\beta_k| w$. Thus, from Theorem 2, if there exists a $\gamma > 1$ such that

$$\gamma |\beta_k| E_q(L\tau_k^q) < 1$$

is satisfied, then the trivial solution $e = 0$ of system (11) is asymptotically stable, i.e., impulsive synchronization between systems (1) and (10) is obtained. This completes the Proof.

Remark 2 When $q = 1$, the synchronization criterion (12) is the results of integer-order system. So the synchronization criterion is more significant in general. It is suitable for the case of order $0 < q \le 1$.

4. Numerical examples

This section provides numerical simulations to verify the effectiveness of the proposed synchronization criterion. In order to reduce computational cost, currently most efficient numerical scheme, improved predictor–corrector approach in Ref. [23] is applied to perform a simulation-based analysis.

Consider the fractional-order Lü system

$$\begin{cases} \dfrac{\mathrm{d}^q x_1}{\mathrm{d}t^q} = 36(x_2 - x_1), \\ \dfrac{\mathrm{d}^q x_2}{\mathrm{d}t^q} = 20x_2 - x_1 x_3, \\ \dfrac{\mathrm{d}^q x_3}{\mathrm{d}t^q} = -3x_3 + x_1 x_2. \end{cases} \tag{14}$$

The fractional-order Lü system (14) exhibits chaotic behavior for order $q > 0.9156$. When $q = 0.95$, the chaotic attractor is shown in Fig. 1.

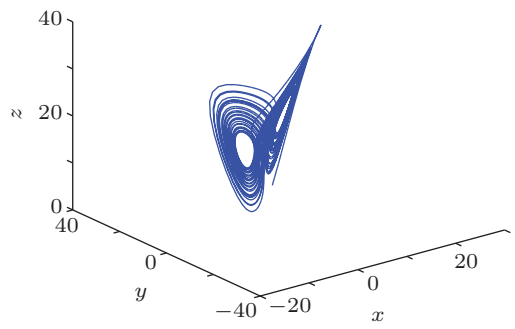


Fig. 1. (color online) Simulation result of system (14) in state space with order $q = 0.95$.

Let

$$C_k = \begin{bmatrix} -0.9 & 0 & 0 \\ 0 & -0.9 & 0 \\ 0 & 0 & -0.9 \end{bmatrix},$$

then $\beta_k = 0.1$. From inequality (12), the boundaries of the stable region can be determined as $0 \le \tau \le 0.0027$. The initial values $\boldsymbol{x}(0) = (5, -1, -3)^{\mathrm{T}}$, and $\boldsymbol{y}(0) = [2, 4, 1]^{\mathrm{T}}$. The simulation results are shown in Fig. 2.

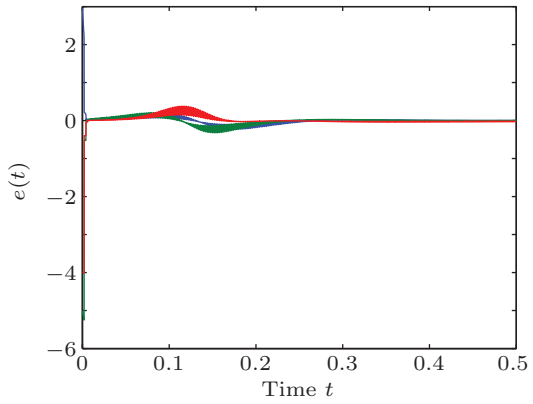


Fig. 2. (color online) Errors of the fractional-order Lü system with $\tau = 0.002$.

In order to illustrate the effectiveness of the proposed synchronization criterion, for the same parameters γ and C_k, the boundaries of the stable region can be determined as $0 \le \tau \le -(\ln\gamma\beta^2)/2L \le 0.0137$, which is based on the inpulsive synchronization criterion of integer-order systems. The simulation results are shown in Fig. 3. It is shown that the synchronous effect of the systems is greatly improved based on the proposed method in this paper for the same parameters γ and C_k.

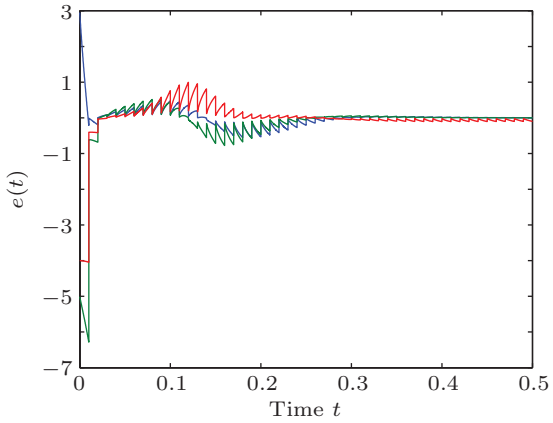


Fig. 3. (color online) Errors of the fractional-order Lü system with $\tau = 0.01$.

5. Conclusion

In this paper, the stability of impulsive fractional-order systems has been investigated. Furthermore, impulsive synchronization of fractional-order chaotic systems is discussed. A new synchronization criterion is proposed based on the stability theory of impulsive fractional-order systems. The effectiveness of the presented results has been illustrated by numerical simulations.

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