Network evolution driven by dynamics applied to graph coloring*

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An evolutionary network driven by dynamics is studied and applied to the graph coloring problem. From an initial structure, both the topology and the coupling weights evolve according to the dynamics. On the other hand, the dynamics of the network are determined by the topology and the coupling weights, so an interesting structure-dynamics co-evolutionary scheme appears. By providing two evolutionary strategies, a network described by the complement of a graph will evolve into several clusters of nodes according to their dynamics. The nodes in each cluster can be assigned the same color and nodes in different clusters assigned different colors. In this way, a co-evolution phenomenon is applied to the graph coloring problem. The proposed scheme is tested on several benchmark graphs for graph coloring.

Keywords: network dynamics, evolution of network, evolutionary strategies, graph coloring problem

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1. Introduction

It is well known that the dynamics of a network are closely related to its structure.[1–7] The influences of network structure on dynamics (function) have been intensively studied in the past decade along several directions: synchronization in small-world networks,[8–10] scale-free networks,[11–13] and modular networks;[14,15] synchronization in networks with time-varying topology or coupling strength;[16,17] epidemic spreading in small-world and scale-free networks[18–20] and the influence of the network topology on epidemic spreading;[21,22] opinion formation dynamics affected by the network structure;[23–25] etc.

On the other hand, the network's dynamics may affect the evolution of its structure, including the topology and the interaction weights.[26] For example, conflicting views may lead to the break-up or suppression of social contacts and common views may enhance the contacts. Individuals in social networks are often able to enhance or suppress the strength of interconnections and adjust the network topology.

The interaction between structure and function (dynamics) may be the original source which drives the evolution of real world networks.[27] Small-world and scale-free networks are common in nature, and it is well known that they are the result of long-term evolution. In theoretical study, the famous Watts–Strogatz (WS) small-world network evolves from a regular ring by rewiring a portion of the edges[2] and the scale-free (SF) model evolves by growth and preferential attachment.[28] Evolution is not driven by dynamics in either of these models.

Recently, several research efforts have looked into the evolutionary network driven by dynamics.[29–32] Chen *et al.* proposed a dynamical gradient network approach to synchronize the Kuramoto oscillator network,[29] where the couplings in the $(n+1)$-th time interval are decided by the oscillator dynamics in the n-th time interval. Lellis *et al.* presented two strategies (vertex-based and edge-based) for the synchronization of complex networks through local adaptive coupling driven by states mismatch.[30] In Ref. [31], two strategies to design the coupling functions between nodes are proposed: one is based on finding appropriate adaptation laws to evolve the coupling gain driven by synchronization error, and the other on the *ad hoc* design of a nonlinear coupling protocol based on contraction theory. In Ref. [32], a continuous second order model is proposed to describe the evolution of coupling gains, which brings about a novel network adaptation strategy. The model has a bi-stable steady state that leads to the formation of a self-emerging topology. A hierarchy of decentralized strategies to control a network onto a desired trajectory is proposed in Ref. [33], which allows adaptively evolving the network topology through both edge snapping and the coupling gains.

However, the evolution of those networks driven by their dynamics is only for synchronizing the networks. Li *et al.* presented an interesting model of evolutionary sub-networks, in which the link function (attractive or repulsive) is updated according to the local network dynamics. For a network of modular structure, the network evolves into a form in which each

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intralink (interlink) in the network owns an attractive (repulsive) coupling function.[34] Then the question is how a computational problem can be solved by the evolution of such a network driven by dynamics, which may help us to better understand the relation between network structure and its dynamics. In this paper, two purpose-designed evolutionary strategies are proposed, aimed at solving the graph coloring problem via a network's evolution driven by its dynamics.

Graph coloring is a classical combinational optimization problem coming from graph theory.[35–38] It is defined as coloring the nodes of a graph with the minimum number of colors and makes sure no two adjacent nodes have the same color. Coloring a graph is equivalent to segmenting the graph into k sub-graphs so that the nodes in each sub-graph have the same color and nodes in different sub-graphs have different colors. A network is usually described by a graph. For the graph-coloring problem, we use the network described by its complement graph. A graph and its complement graph have the same set of nodes, but for edges they are complementary. In other words, if an e_i=(v_p, v_q) is an edge of the graph, then it is not an edge of its complement, and *vice versa*.

Given a graph, in which two unconnected nodes can be assigned the same color, in its complement, two connected nodes can be assigned the same color. All the nodes of a complete sub-graph in the complement can be assigned the same color. Coloring a graph is equivalent to segmenting its complement graph into k complete sub-graphs. In Refs. [37] and [38], the synchronization and clustering phenomena of oscillator networks have been applied to the graph-coloring problem respectively.

In this paper, the phenomenon of network evolution is applied to graph coloring. Two evolutionary strategies are proposed according to the dynamics: one is for the removal of edges and the other is for the increase of coupling weights. After a number of iterations, the network is stable and has been split into several clusters (sub-graphs) by its dynamics. The nodes in each cluster are assigned the same color, and thus the graph is colored.

2. Description of the model

A network is usually described by a graph $G(E,V)$, where $V=\{v_1,\ldots,v_N\}$ is the set of N nodes and $E=\{e_{ij}|v_i, v_j \in V\}$ is the set of edges. The complement graph of $G(E,V)$ is usually described by $\bar{G}(\bar{E},V)$. The nodes set of G and $\bar{G}$ is the same, but their edge sets are different, if $e_{ij} \in E$ then $e_{ij} \bar{\in} \bar{E}$ and *vice versa*. In this paper, for convenience in being applied to the graph-coloring problem, the network is described by the complement of the graph to be colored. We say two nodes are connected in the network, which means that there is an edge in the complement graph.

2.1. Network model

The coupled Kuramoto oscillators are described by the following differential equation:[39,40]

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = \omega_i + \frac{k}{N}\sum_{j=1}^{N} a_{ij}\sin(\theta_j-\theta_i), \quad i=1,\ldots,N,$$

where θ_i and ω_i are the phase variable and intrinsic frequency of the i-th oscillator, respectively, k is the coupling strength, and $[a_{ij}]$ is the adjacency matrix. The Kuramoto oscillator model has been modified and applied in different situations.[41–43] In this paper, the Kuramoto oscillator model is modified as follows by using a difference equation:

$$\begin{aligned}&\theta_i(k+1) = \theta_i(k) + \frac{K}{K_{\max}}\sum_{j=1}^{N} a_{ij}(k)\sin(\theta_j(k)-\theta_i(k)),\\ &i=1,\ldots,N,\\ &\begin{cases} K=K_{\mathrm{p}}>0, & e_{ij}\in\bar{E},\\ K=K_{\mathrm{n}}<0, & e_{ij}\in E,\end{cases}\end{aligned} \tag{1}$$

where $\theta_i(k)$ is the phase variable of v_i (agent) in the k-th iteration. Initially, $\theta_i(0)$ of the N nodes are randomly and uniformly distributed in $[0, 2\pi]$. $K_{\max}$ is the maximum degree of all the nodes in the network. K is the coupling strength of the network, which may be attractive ($K=K_{\mathrm{p}}$, positive) or repulsive ($K=K_{\mathrm{n}}$, negative) depending on whether there is a connection between v_i and v_j. Obviously, the attractive coupling strength K_{p} is to make the phases of two connected nodes in the network evolve together; on the other hand, the repulsive coupling strength K_{n} is to make the phases of two unconnected nodes evolve divergently. The positive coupling weight $a_{ij}(k)$ is initialized to 1 ($a_{ij}(0)=1$) before evolution.

The nodes of the network in Eq. (1) can be considered as agents that are not oscillators, because each isolated node does not oscillate itself.

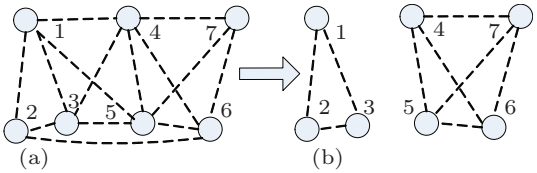


Fig. 1. Evolution of a graph: (a) the complement of a graph to be colored, (b) result of the graph's evolution by removing interference edges.

Model (1) can solve a very simple graph-coloring problem directly without evolution, the phases of the nodes which can be assigned the same color approach each other and the phases of nodes which cannot be assigned the same color evolve to be farther away. The principle is similar to that revealed in Ref. [34]. Eventually, all the nodes are separated into several sub-graphs by their phases; each is a complete sub-graph corresponding to one color. For example, figure 1(a) is

the complement of a graph to be colored, node set $S_1 = \{v_1, v_2, v_3\}$ and $S_2 = \{v_4, v_5, v_6, v_7\}$ construct two complete sub-graphs, so their nodes can be assigned the same color respectively. Figure 2(a) is the simulation result of Fig. 1(a) by model (1); S_1 and S_2 can be separated by their phases; the parameters are $K_p = 1$ and $K_n = -1$. But for a graph among the graph-coloring benchmarks,[44,45] model (1) is not effective in solving the graph-coloring problem. The reason is that the edges between two nodes in different sub-graphs make it difficult to separate them by their phases, and such edges are called interference edges in the following. For example, e_{14}, e_{15}, e_{34}, e_{35}, and e_{26} in Fig. 1(a) are interference edges. If the five interference edges are removed by network evolution (see Fig. 1(b)), then the two subsets can easily be separated by their phases. Figure 2(b) is the simulation result of model (1) on the graph of Fig. 1(b).

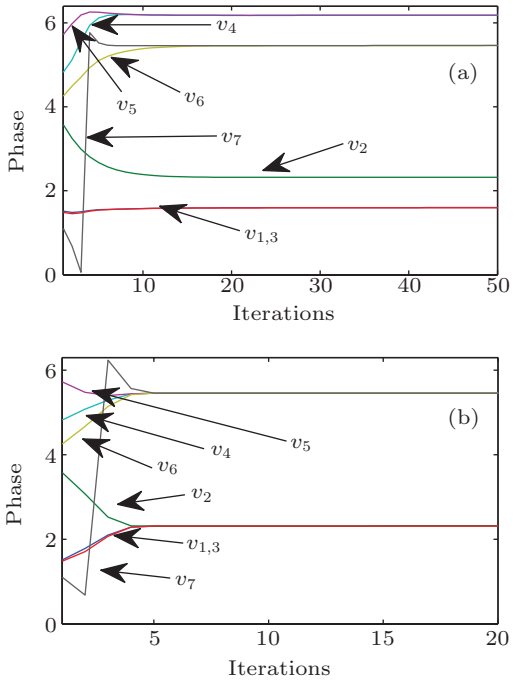


Fig. 2. (color online) Simulation result of model (1) on the graph shown in Fig. 1: (a) simulation result without evolution, (b) simulation result with evolution.

2.2. Evolutionary strategies

The evolutionary strategies include two parts: one is for the evolution of topology and the other is for the evolution of coupling weights $a_{ij}(k)$.

To clearly describe the evolutionary strategies, we need the concept of "segment graph" of a node: all the adjacent nodes of v_i and the edges among them construct the segment graph of v_i. The segment graph of node v_i is denoted as Seg_i in the following. A node may belong to two or more segment graphs, which is described by membership degree defined as

$$u_{i_j} = \frac{d_{j_\text{Seg}_i}}{d_j}, \tag{2}$$

where μ_{i_j} is the membership degree of v_j belonging to Seg_i, d_j is the degree of v_j in the whole complement graph, and $d_{j\text{-Seg}_i}$ is the degree of v_j in Seg_i.

Using the membership degree, the average phase of a node is defined by

$$\theta_{\text{avg}_i} = \frac{1}{u_i} \sum_{j \in \text{Seg}_i} u_{i_j}\theta_j, \quad u_i = \sum_{j \in \text{Seg}_i} u_{i_j}. \tag{3}$$

The evolutionary strategies are given by using the average phase of nodes. Compared with the phase of a node, its average phase is more stable and includes information about local structure.

Strategy 1 Edge e_{ij} exists in the complement graph (e_{ij}). If $|\theta_{\text{avg}_i} - \theta_{\text{avg}_j}| > \theta_{\text{Th1}}$, $u_{i_j} = \min\limits_{p \in \text{Seg}_i, p \neq i}(u_{i_p})$, and $u_{j_i} \neq 1$, then remove e_{ij} in the next iteration. Strategy 1 aims at removing the interference edges.

Strategy 2 Edge e_{ij} does not exist in the complement graph (e_{ij}). If $|\theta_{\text{avg}_i} - \theta_{\text{avg}_j}| < \theta_{\text{Th2}}$, then $a_{ij}(k+1) = a_{ij}(k) + \varepsilon$, $\varepsilon > 0$. Since e_{ij}, v_i, and v_j cannot be assigned into the same sub-graph corresponding to one color. Strategy 2 aims at driving their phases far apart.

3. Algorithm by network evolution

Using the network model and evolutionary strategies introduced in Section 2, a detailed algorithm is then obtained for the graph-coloring problem.

3.1. Algorithm

Graph coloring from network evolution driven by dynamics.

Step 1 For a given graph to be colored, obtain its complement graph and all nodes' segment graphs of the complement graph.

Step 2 Compute μ_{i_j} by Eq. (2), $v_j \in$ Seg_i. Obtain the average phases of each node by Eq. (3).

Step 3 Examine the conditions to remove an edge: (i) $e_{ij} \in \bar{E}$; (ii) $|\theta_{\text{avg}_i} - \theta_{\text{avg}_j}| > \theta_{\text{Th1}}$; (iii) $u_{i_j} = \min\limits_{p \in \text{Seg}_i, p \neq i}(u_{i_p})$; (iv) $u_{j_i} \neq 1$. If all four conditions hold, remove e_{ij}.

Step 4 Examine the conditions to increase the coupling weights a_{ij} ($v_j \in$ Seg_i): 1) $e_{ij} \in E$; 2) $|\theta_{\text{avg}_i} - \theta_{\text{avg}_j}| < \theta_{\text{Th2}}$. If the two conditions hold, $a_{ij}(k+1) = a_{ij}(k) + \varepsilon$. In the experiments of this paper, $\varepsilon = 1$.

Step 5 Compute the phases of all nodes by Eq. (1). After all the phases of nodes are updated, let $k = k$+1, initialize $a_{ij}(k)$ to 1, and restart from Step 2 again.

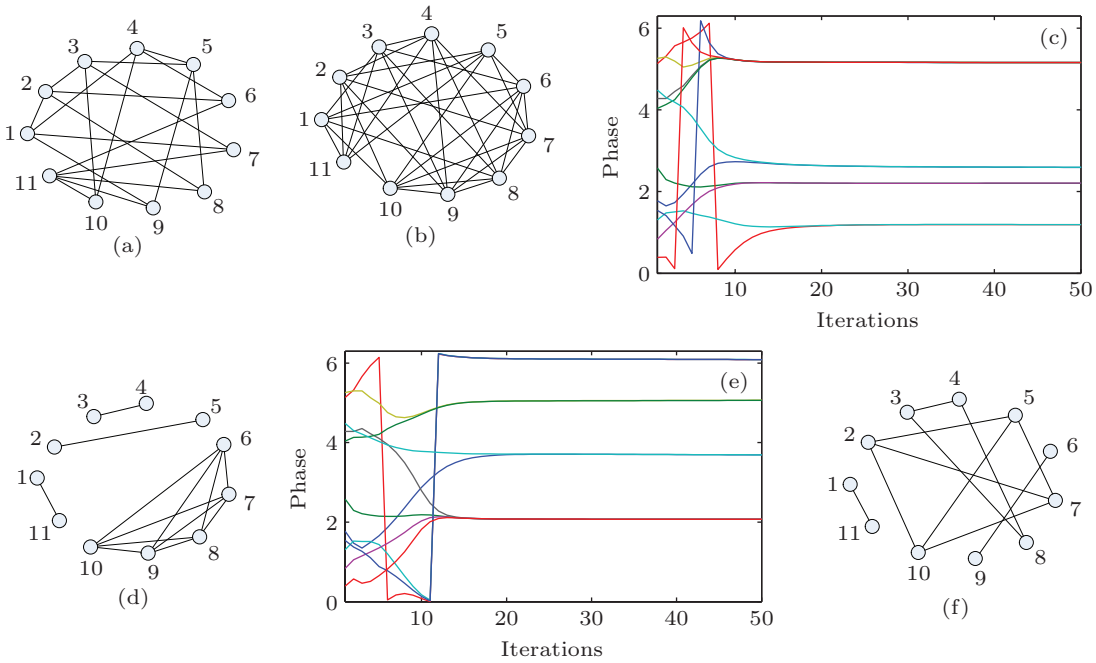


Fig. 3. (color online) Simulation results on myciel3.col, $\theta_{\mathrm{Th2}} = 0.1$, $K_{\mathrm{p}} = 1$, $K_{\mathrm{n}} = -1$, and $d = 0.99$. (a) The graph of myciel3.col. (b) The complement graph of myciel3.col. (c) The simulation result when $\theta_{\mathrm{Th1}} = 0.15$. (d) The complement graph after evolution when $\theta_{\mathrm{Th1}} = 0.15$. (e) The simulation result when $\theta_{\mathrm{Th1}} = 0.24$. (f) The complement graph after evolution when $\theta_{\mathrm{Th1}} = 0.24$.

If all the interference edges are removed through network evolution, the phases of nodes in the same sub-graph will be equal and the phases of nodes in different sub-graphs are different. In other words, synchronization will be realized in each sub-network and the entire network comes to cluster synchronization. In general, only some of the interference edges are removed, the phases of nodes in the same sub-network are only approximately synchronized. Thus, a threshold d ($d \in (0,1)$) is needed to distinguish the result of coloring. Precisely, node v_i and v_j are assigned the same color if inequality (4) holds; or else, they should be assigned different colors.

$$\cos(\theta_i - \theta_j) \geq d. \tag{4}$$

3.2. Parameters selection.

There are five parameters of the algorithm: K_{p}, K_{n}, θ_{Th1}, θ_{Th2}, and d. The value of θ_{Th1} relates to estimating whether an edge is an interference edge. Too small a value of θ_{Th1} may lead to removing an edge mistakenly; on the other hand, too large a value of θ_{Th1} may block the removal of many interference edges.

Here, an example is given to illustrate the effect of θ_{Th1}. Figure 3(a) is the graph of myciel3.col, which is a benchmark graph for coloring.[44] Myciel3.col has 11 nodes and 20 edges and its chromatic number is 4. The complement graph of myciel3.col is given in Fig. 3(b). Let $\theta_{\mathrm{Th2}} = 0.1$, $K_{\mathrm{p}} = 1$, $K_{\mathrm{n}} = -1$, and $d = 0.99$ be unchanged in the simulations. The initial phases of the 11 oscillators are 1.773, 2.577, 5.133, 1.308, 0.834, 5.260, 4.280, 1.536, 4.0361, 0.391, and 4.484 respectively. Figure 3(c) is the simulation result when $\theta_{\mathrm{Th1}} = 0.15$; all the nodes are correctly split into 4 clusters: S_1={1, 11}, S_2={2, 5}, S_3={3, 4}, and S_4={6, 7, 8, 9, 10}. The complement graph after network evolution is shown in Fig. 3(d), which shows that all the interference edges in the complement graph are correctly removed. Figure 3(e) is the simulation result when $\theta_{\mathrm{Th1}} = 0.24$; a different solution is obtained as follows: S_1={1, 11}, S_2={2, 5, 7, 10}, S_3={3, 4, 8}, and S_4={6, 9}. The complement graph after network evolution is shown in Fig. 3(f), and again, all the interference edges are correctly removed. More simulations show that all the interference edges can be correctly removed when $0.11 \leq \theta_{\mathrm{Th1}} \leq 0.24$. When $\theta_{\mathrm{Th1}} < 0.11$, erroneous removals occur. In a wide range of $\theta_{\mathrm{Th1}} \geq 0.25$, only part of the interference edges are removed, but we can also obtain proper solution by increasing the coupling weights a_{ij} or tuning the values of K_{p} and K_{n}.

As for K_{p} and K_{n}, if all the interference edges are removed through network evolution, a solution can be found in a very wide range of the parameters for the graph-coloring problem ($K_{\mathrm{p}} > 0$, $K_{\mathrm{n}} < 0$). In general, K_{p} and K_{n} should be properly selected. By experience (see Section 4), K_{p} and K_{n}, can adopt value in [0.1, 1.5] and [−0.1, −1.5] respectively.

The value of θ_{Th2} relates to when to increase the coupling weights a_{ij} to make $|\theta_{\mathrm{avg_}i} - \theta_{\mathrm{avg_}j}| \geq \theta_{\mathrm{Th2}}$ hold, avoiding the outcome that v_i and v_j are assigned the same color by inequality (4). Thus, we can adopt $\theta_{\mathrm{Th2}} \approx \arccos(d)$. The value of d relates to the phase difference between two different clusters (sub-graphs), which again relates to the number of colors

(clusters) required. In general, the phases of the clusters are not uniformly distributed in [0, 2π]. As an example, to estimate d, we consider that a network is separated into 10 clusters by their phases uniformly in [0, 2π], the phase difference between two neighboring clusters is about 0.628, so $d > 0.8092$ from inequality (4). Similarly, for a graph which needs 20 colors, $d > 0.9511$ from inequality (4). In the simulations of this paper, d=0.99.

4. Simulations

Simulations are done on four benchmark graphs for coloring:[44,45] myciel4.col, queen5_5.col, 2-Insertions_3.col, and 3-Insertions_3.col.

The graph of myciel4.col has 23 nodes and 71 edges and its chromatic number is 5.[44] When $K_p = 1$, $K_n = -1$, $\theta_{Th1} = 0.2$, and $\theta_{Th2} = 0.3$, a simulation result is given in Fig. 4(a), which shows that the 23 nodes are correctly split into 5 clusters by their phases: S_1={1, 3, 6, 8, 23}, S_2={2, 4, 11}, S_3={5, 21}, S_4={7, 9, 10, 17, 19, 20}, and S_5={12, 13, 14, 15, 16, 18, 22}. When we change the value of θ_{Th2} from 0.3 to 0.2 (the other parameters remain unchanged), another solution is obtained: S_1={1, 6, 10}, S_2={2, 5, 11, 23}, S_3={3, 4, 9}, S_4={7, 8}, and S_5={12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22}, the simulation result is not shown, for simplicity. Examination shows that both outcomes are proper 5-colorings of the graph.

The graph of queen5_5.col has 25 nodes and 320 edges and its chromatic number is 5, also.[45] Figure 4(b) shows a simulation result when $K_p = 1$, $K_n = -1$, $\theta_{Th1} = 0.3$, and $\theta_{Th2} = 0.2$. The 25 nodes are correctly split into 5 clusters: S_1={1, 8, 15, 17, 24}, S_2={2, 9, 11, 18, 25}, S_3={3, 10, 12, 19, 21}, S_4={4, 6, 13, 20, 22}, and S_5={5, 7, 14, 16, 23}. When θ_{Th2} changes from 0.2 to 0.3, another solution is obtained: S_1={1, 9, 12, 20, 23}, S_2={2, 10, 13, 16, 24}, S_3={3, 6, 14, 17, 25}, S_4={4, 7, 15, 18, 21}, and S_5={5, 8, 11, 19, 22}. Both solutions are proper 5-colorings of the graph.

The graph of 2-Insertions_3.col has 37 nodes and 72 edges and its chromatic number is 4.[45] Figure 4(c) shows a simulation result when $K_p = 1$, $K_n = -1$, $\theta_{Th1} = 0.2$, and $\theta_{Th2} = 0.2$. The obtained solution is: S_1={1, 5, 6, 9, 10, 14, 15, 18, 19, 27, 28, 30, 32, 33, 36}, S_2={2, 20, 21, 22, 24, 37}, S_3={3, 23}, and S_4={4, 7, 8, 11, 12, 13, 16, 17, 25, 26, 29, 31, 34, 35}. A different solution is found when the value of θ_{Th2} changes from 0.2 to 0.3, which is S_1={1, 5, 6, 9, 14, 18, 20}, S_2={2 }, S_3={3, 4, 7, 8, 19, 21, 22, 23, 24, 25, 26, 27, 37}, and S_4={10, 11, 12, 13, 15, 16, 17 28, 29, 30, 31, 32, 33, 34, 35, 36}. Both are proper 4-colorings of the graph.

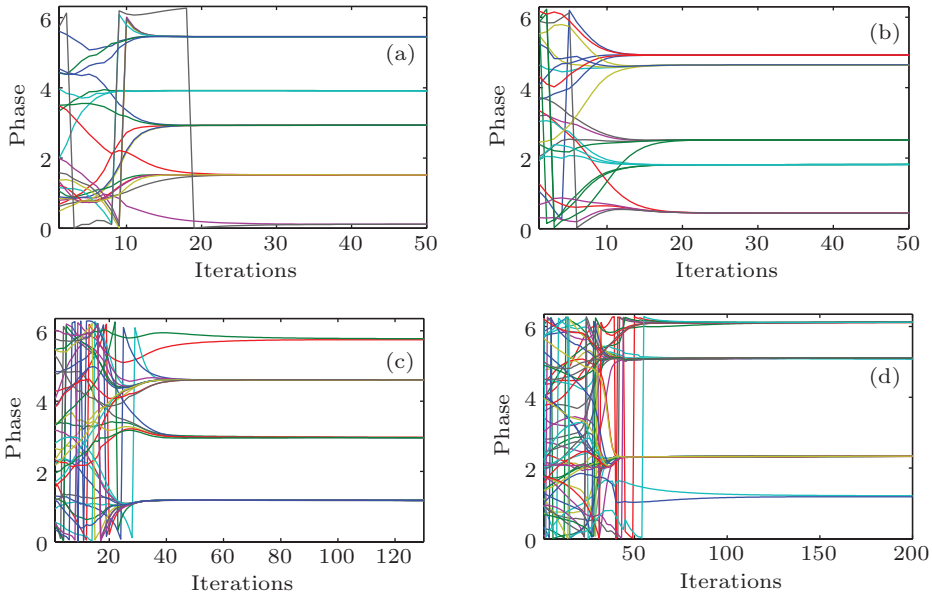


Fig. 4. (color online) Simulation results on several benchmark graphs for coloring. (a) Myciel4.col, $K_p = 1$, $K_n = -1$, $\theta_{Th1} = 0.2$, and $\theta_{Th2} = 0.3$. (b) Queen5_5.col, $K_p = 1$, $K_n = -1$, $\theta_{Th1} = 0.3$, and $\theta_{Th2} = 0.2$. (c) 2-Insertions_3.col, $K_p = 0.1$, $K_n = -1$, $\theta_{Th1} = 0.2$, and $\theta_{Th2} = 0.2$. (d) 3-Insertions_3.col, $K_p = 0.1$, $K_n = -1$, $\theta_{Th1} = 0.3$, and $\theta_{Th2} = 0.2$.

The 3-Insertions_3.col has 56 nodes and 110 edges, and its chromatic number is 4.[45] Figure 4(d) shows a simulation result when $K_p = 0.1$, $K_n = -1$, $\theta_{Th1} = 0.3$, and $\theta_{Th2} = 0.2$. The 56 nodes are split into 4 clusters: S_1={1, 3, 5, 7, 9, 12, 14, 16, 18, 20, 25, 27, 29, 34, 36, 38, 40, 42, 44, 56}, S_2={2, 4, 6, 8, 11, 13, 15, 17, 19, 22, 24, 26, 28, 30, 33, 35, 37, 39, 41, 46, 48, 50, 52, 54, 55}, S_3={10, 21, 23, 31, 45, 47, 49, 51, 53}, and S_4={32, 43}. If the value of θ_{Th2} is changed from 0.2 to

0.3, the solution obtained is S_1={1, 3, 7, 12, 14, 16, 18, 19, 20, 22, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 56}, S_2={2, 6, 9, 10, 23, 24, 26, 27, 28, 30, 31, 32, 33, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55}, S_3={4, 8, 11, 15, 25, 29}, and S_4={5, 13, 17, 21}. Both are proper 4-colorings of the graph.

5. Conclusion

Network evolution driven by dynamics is applied to the graph-coloring problem in this paper, which indicates that the dynamics-based evolution can be designed to solve some computational problems beyond synchronization.

Two evolutionary strategies are provided for the multi-agent network, one is for the evolution of edges and the other is for the coupling weights. The main ideas of the two strategies are: (I) remove some or all the interference edges among those complete sub-graphs by network evolution; (II) increase the coupling weight between two nodes when there is no edge and their phase difference is small. Then the network can be split into several clusters by their phases. All the nodes in each cluster are assigned the same color. The proposed scheme has been verified by simulations on graph-coloring benchmarks.

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