Explosive synchronization of complex networks with different chaotic oscillators*

Zhao Jun-Chan(赵军产)†

College of Mathematics and Computer Science, Wuhan Textile University, Wuhan 430073, *China*

(Received 16 August 2012; revised manuscript received 28 November 2012)

Recent studies have shown that explosive synchronization transitions can be observed in networks of phase oscillators [Gómez-Gardeñes J, Gómez S, Arenas A and Moreno Y 2011 *Phys. Rev. Lett.* **106** 128701] and chaotic oscillators [Leyva I, Sevilla-Escoboza R, Buldú J M, Sendiña-Nadal I, Gómez-Gardeñes J, Arenas A, Moreno Y, Gómez S, Jaimes-Reátegui R and Boccaletti S 2012 *Phys. Rev. Lett.* **108** 168702]. Here, we study the effect of different chaotic dynamics on the synchronization transitions in small world networks and scale free networks. The continuous transition is discovered for Rössler systems in both of the above complex networks. However, explosive transitions take place for the coupled Lorenz systems, and the main reason is the abrupt change of dynamics before achieving complete synchronization. Our results show that the explosive synchronization transitions are accompanied by the change of system dynamics.

Keywords: complex network, explosive synchronization transition, chaotic oscillator

PACS: 05.45.–a **DOI:** 10.1088/1674-1056/22/6/060506

1. Introduction

Complex networks are playing an increasingly important part in our lives, such as transportation and phone networks, Internet, wireless networks, and the power networks. Great progress has been made in studying complex networks since the discovery of their small-world[1,2] and scale-free[3] characteristics.

Synchronization of complex dynamical networks has received a great deal of attention from the systems science community in the last decade.[4–15] Using the master stability function,[4] Pecora and Carroll developed a general approach to the synchronization of identical dynamical systems. They discovered a bounded synchronization region for the coupling chaotic systems, and showed that if the coupling strength is greater than a threshold, the coupling chaotic systems cannot achieve synchronization. In Ref. [5], the authors investigated the synchronization region for different complex dynamical networks, and addressed unbounded and bounded synchronization regions. Li and Chen[6] further studied the relationship between the synchronization region and the coupling strength, and they found that a sufficiently large coupling strength will lead to synchronization with coupling through all the variables.

A large system is said to undergo a phase transition when one or more of its properties change abruptly after a slight change in a controlling variable. The transitions are called second-order if they are continuous (smooth), whereas discontinuous (abrupt) transitions are called first-order. Recently, some researchers analyzed the phase transitions in the complex dynamical networks, and discovered an abrupt percolation transition in complex networks.[16–18] Generally speaking, there are two key factors to influence phase transitions of complex networks. One is the structure of the network, and another is the dynamics of the system. More recently, Jesús *et al.* studied the phase transition in scale-free networks with Kuramoto oscillators, and showed that an explosive synchronization transition emerges when the natural frequency of dynamical units are positively correlated with the degree of the units.[19] Introducing a natural oscillation frequency for coupled Rössler systems, Leyva *et al.* discovered that the positive correlation between oscillation frequencies and structural heterogeneity is a necessary condition for such an explosive phenomenon.[20]

In this paper, we study different chaotic systems on two prototype complex networks: the Newman–Watts model and the Barabási–Albert model.[2,3] The first model is a small-world (SW) network with a homogeneous degree distribution, while the second one is a scale-free (SF) network with a power-law-degree distribution. The chaotic systems are the well-known Rössler system and Lorenz system: the one has a simple chaotic attractor and the other has a butterfly-shaped attractor. Recent studies suggested that the coupled Rössler systems have emerged explosive first-order transition by changing the oscillation frequencies. Here, we do not introduce new parameters to change the property of the chaotic system. Considering a model of N linearly coupled chaotic oscillators, we analyze the synchronization transitions in small-world networks and scale-free networks. Numerical simulations show

*Project supported by the National Natural Science Foundation of China (Grant Nos. 61203159, 61164020, 11271295, and 11071280) and the Foundation of Wuhan Textile University (Grant No. 113073).

†Corresponding author. E-mail: junchanzhao@163.com

© 2013 Chinese Physical Society and IOP Publishing Ltd

that explosive synchronization transitions are discovered for Lorenz systems in both complex networks. The main reason is the abrupt change of system state before achieving complete synchronization. However, the synchronization transitions are always of continuous type for coupled Rössler systems regardless of the network structures. Therefore, there is a positive correlation between the explosive synchronization transitions and the change of system dynamics.

2. Model and method

Consider an unweighted and undirected network of N coupled chaotic oscillators

$$\dot{x}_i = f(x_i) + \lambda \sum_{i=1}^{N} c_{ij} x_j, \quad i = 1, \ldots, N, \tag{1}$$

where $x_i \in R^m$ is the state variables of node i, $f : R^+ \times R^m \to R^m$ is a continuous vector function. The outer-coupling matrix $\boldsymbol{C} = (c_{ij})_{N\times N}$ is defined as $\boldsymbol{C} = \boldsymbol{A} - \boldsymbol{D}$, where $\boldsymbol{A} = (a_{ij})_{N\times N}$ is the adjacency matrix of the graph of the network, $\boldsymbol{D} = (d_{ij})$ is the diagonal matrix of vertex with $d_{ii} = \sum_{j=1}^{N} a_{ij}$ for all $i = 1, 2, \cdots, N$, and λ is the coupling strength.

The network (1) is said to achieve *synchronization* if, for all i, j, $\|x_i(t) - x_j(t)\| \to 0$ as $t \to \infty$.

Note that all the variables of chaotic systems are coupled, from the result in Ref. [6], we obtain that a sufficiently large coupling strength will lead to synchronization of complex network (1). In this study, we are interested in the synchronization transition before equation (1) achieves synchronization. To monitor such a synchronization transition as λ grows, the $p(t)$ is introduced by

$$p(t) = \frac{2}{N(N-1)} \sum_{j=1}^{N-1} \sum_{i=j+1}^{N} \Theta(r - \|x_i(t) - x_j(t)\|_2),$$

where Θ is Heaviside function

$$\Theta(x) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0, \end{cases}$$

and r is a small positive constant to detect whether the two systems synchronize. The order parameter, $p(t) \in [0, 1]$, measures the synchronization of the coupled chaotic systems, reaching the value $p = 1$ when the complex network is in complete synchronization, while $p = 0$ for the incoherent solution.

For each λ, we solve Eqs. (1) for a long enough time, then calculate the average distance between nodes i and j over the end of a period of time. If the distance is less than a small positive constant, we think the nodes i and j achieve synchronization. The well-known Lorenz system and Rössler system are used to analyze the influence of different systems.

The Rössler system is described by

$$\begin{cases} \dot{x} = -(y + z), \\ \dot{y} = x + ay, \\ \dot{z} = b + z(x - c), \end{cases} \tag{2}$$

When $a = 0.2$, $b = 0.2$, $c = 8.5$, the system has a chaotic attractor, as shown in Fig. 1(a). The equation of Lorenz system is

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = \rho x - xz - y, \\ \dot{z} = xy - \beta z, \end{cases} \tag{3}$$

where the parameters $\sigma = 10$, $\beta = 8/3$, $\rho = 28$. It is a three-dimensional structure corresponding to the long-term behavior of a chaotic flow, noted for its butterfly shape. See Fig. 1(b).

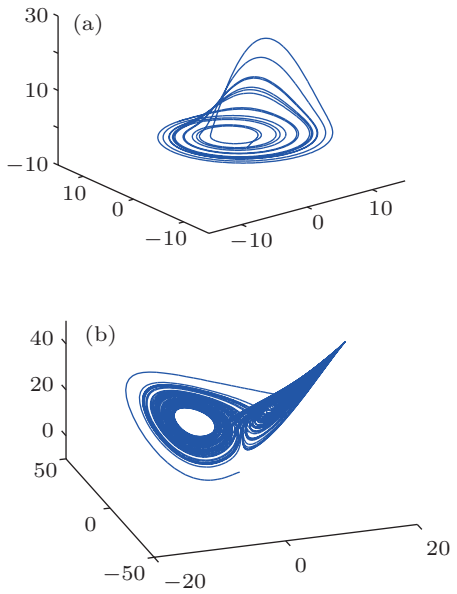


Fig. 1. (color online) The attractor of chaotic system (a) Rössler system and (b) Lorenz system.

3. Numerical simulations

To study the effect of the degree heterogeneity, we simulate model (1) for a small-world network and a scale-free network, which are generated according to Refs. [2] and [3] and construct the networks with the same number of nodes and edges. Figures 2(a) and 2(b) show that the degree sequence of the small-world network is homogeneous and the other network is heterogeneous. Considering the influence of initial conditions, we randomly select the initial values in $[-3, 3]$ and plot synchronization diagrams $p(\lambda)$ five times for each network. Setting $r = 0.03$ in the Heaviside function, the trajectories of the coupling chaotic systems are very similar as the time decreases.

Our simulation results show that the coupled Rössler systems achieve synchronization much faster than Lorenz systems in the same networks, so we use different time scales to investigate the synchronization transitions; the results are shown in Fig. 3. For the coupled Rössler systems, they display a clear second-order phase transition from $p = 0$ to $p = 1$ in both networks as the coupling strength λ increases, as shown in Figs. 3(a) and 3(b).

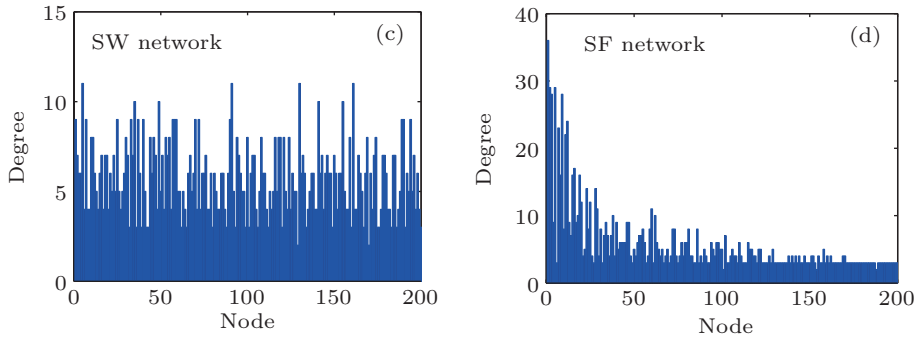


Fig. 2. (color online) The degree sequence of complex network: (c) small-world network, (d) scale-free network.

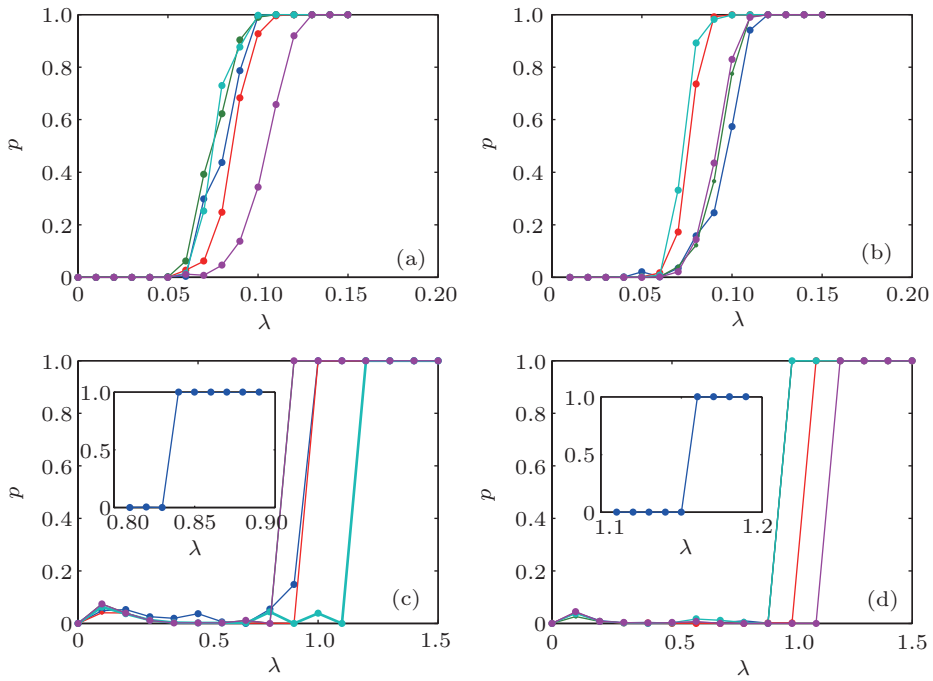


Fig. 3. (color online) The size of the networks is $N = 200$ and the number of the edges is $E = 577$. We select five different initial values for each network. Panels (a) and (b) are synchronization diagrams $p(\lambda)$ for small-world and scale-free networks with Rössler systems; panels (c) and (d) are synchronization diagrams $p(\lambda)$ for small world and scale-free networks with Lorenz systems. The inset plots show synchronization diagrams $p(\lambda)$, the time scale is the same as Rössler systems.

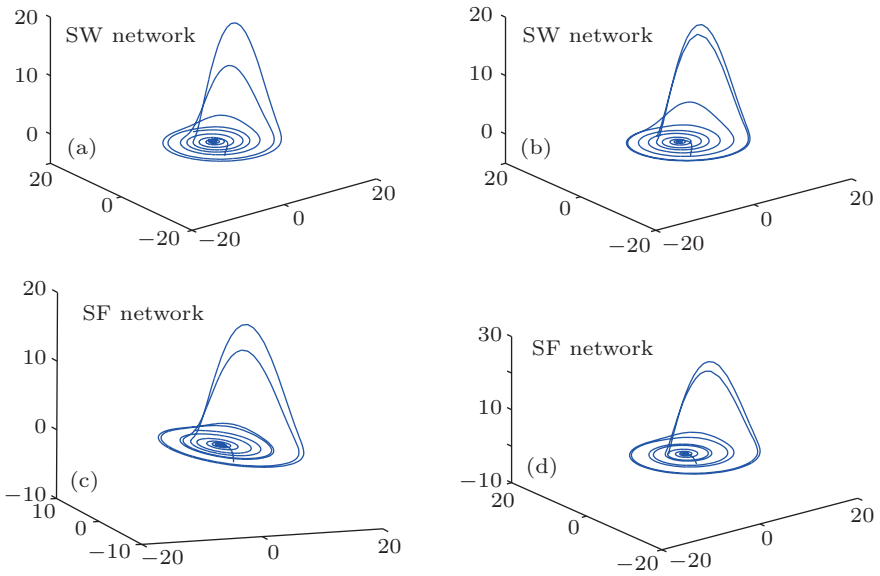


Fig. 4. (color online) The panels show one of the Rössler attractors in complex networks at different coupling strengths λ; (a) and (b) small-world network with $\lambda = 0.05, 0.1$; (c) and (d) scale-free network with $\lambda = 0.05, 0.1$.

The diagram shows a sharp transition from the incoherent state to full synchronization for the coupled Lorenz systems in both complex networks, as displayed in Figs. 3(c) and 3(d). The inset plots show the synchronization transition near the critical λ for smaller timescale. The conclusive information conveyed by Figs. 3(c) and 3(d) is that first-order transitions take place in the coupled Lorenz systems regardless of the network topologies.

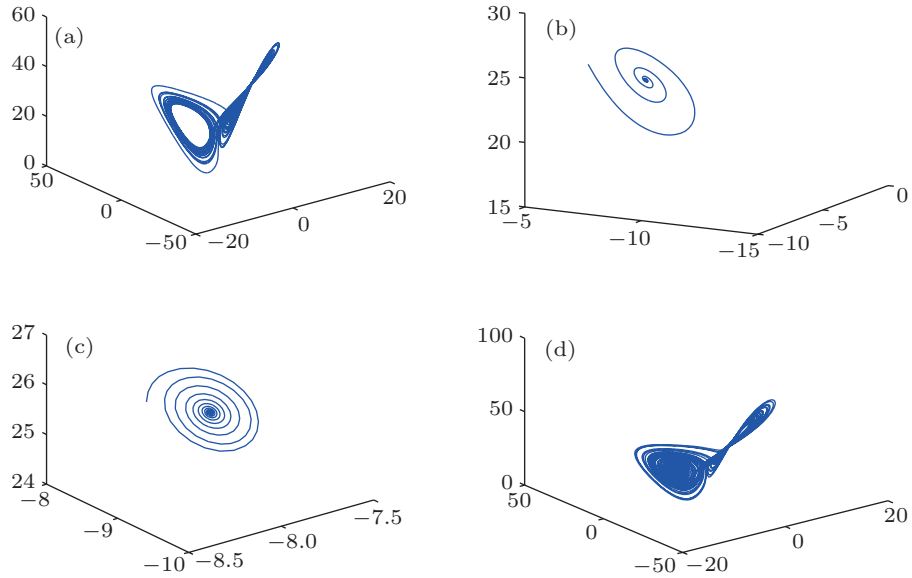


Fig. 5. (color online) One of Lorenz attractors with different coupling strengths λ in small-world network; (a) $\lambda = 0.05$; (b) $\lambda = 0.3$; (c) $\lambda = 0.6$; (d) $\lambda = 1.5$.

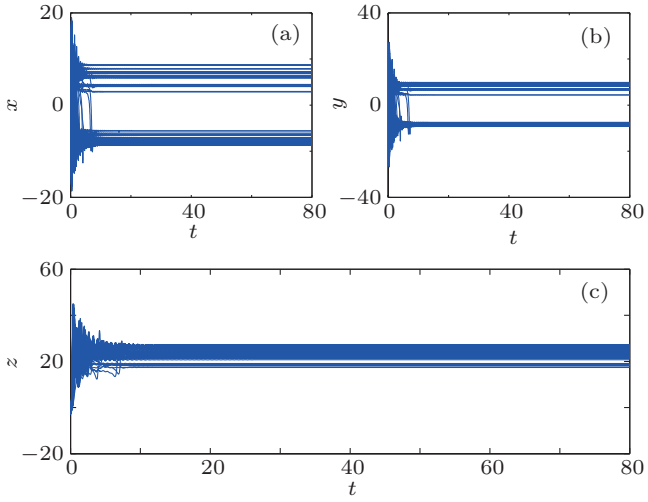


Fig. 6. (color online) The panels show the evolution of Lorenz systems in small-world network with $\lambda = 0.6$; panels (a), (b), and (c) show the system trajectories of Lorenz oscillator.

Then we conjecture that such transitions mainly depend on the dynamics of the system, and we observe the change of system dynamics during the synchronization process. It is difficult to analyze the stability of the large system (1) in theory, thus we mainly give the numerical results. Choosing two critical values of the coupling strength λ_c in Eqs. (1), we observe the change of Rössler systems in both complex networks. Figures 4(a) and 4(b) show that the coupled Rössler systems maintain chaotic attractors as the λ increases in the small-world network, and the same goes for the coupled Rössler systems in scale-free network, with the results shown in Figs. 4(c) and 4(d). Figure 5 displays one of Lorenz system dynamics in the small-world network as the coupling strength λ increases, and all the rest of them are very similar to this case. From the numerical simulations, we can discover an abrupt change of system dynamics before achieving complete synchronization. The coupled Lorenz systems will lose chaotic characteristic as λ increases, then reach some stable state, finally recovering the chaotic attractors. We can also obtain that there exist two critical coupling strengths λ_1, λ_2, when the coupling strength $\lambda \in [\lambda_1, \lambda_2]$, the coupled Lorenz systems are stable, but unsynchronized. When $\lambda > \lambda_2$, the transition from incoherent state to synchronization will occur. Once the network achieves synchronization, the second term of Eqs. (1) becomes zero, so each system will recover chaotic attractor. Figure 6 shows that the coupled systems (1) have stable equilibria at $\lambda = 0.6$ in the small-world network. Although the coupled systems (1) are stable during this stage, the equilibrium is different for each Lorenz system. Thus it is difficult to achieve synchronization during this stage. The dynamics of coupled Lorenz systems also goes through a very similar process in the scale-free network as shown in Figs. 5 and 6; here we do not give the detailed numerical results.

It is worth remarking that the change of system dynamics is different in the synchronization process for both chaotic

systems. The coupled Rössler systems maintain chaotic states during the whole process, so it is the major reason of the continuous synchronization transition. While for the coupled Lorenz systems, the states of systems are chaotic states, stable equilibria, and chaotic states in the process of synchronization. Thus the abrupt change of dynamics causes the explosive synchronization transition.

4. Conclusion

In this paper, we studied the phase transitions of the coupled chaotic systems in small-world networks and scale free networks. Explosive synchronization transitions take place for the coupled Lorenz systems. We found that the change of chaotic dynamics plays an important role in the synchronization transitions. Explosive transitions are accompanied by the changed of chaotic dynamics, but the dynamics is unchanged for the continuous transitions.

References

[1] Watts D J and Strogatz S H 1998 *Nature* **393** 440
[2] Newman M E J and Watts D J 1999 *Phys. Lett. A* **263** 341
[3] Barabasi A L and Albert R 1999 *Science* **286** 509
[4] Pecora L M and Carroll T L 1998 *Phys. Rev. Lett.* **80** 2109
[5] Kocarev L and Amato P 2005 *Chaos* **15** 024101
[6] Li X and Chen G 2003 *IEEE Trans. Circ. Syst. I* **50** 1381
[7] Wang X and Chen G 2002 *Int. J. Bifur. Chaos* **12** 187
[8] Lü J and Chen G 2005 *IEEE Trans. Autom. Control* **50** 841
[9] Zhang Y W, Bu S L and Wang B H 2006 *Chin. Phys. Lett.* **23** 2909
[10] Ma Z, Liu Z and Zhang G 2006 *Chaos* **16** 023103
[11] Zhou J and Chen T 2006 *IEEE Trans. Circuits Syst. I* **53** 733
[12] Zhou J, Lu J and Lü J 2008 *Automatica* **44** 996
[13] Yu W, Chen G and Lü J 2009 *Automatica* **45** 429
[14] Zhang Q J and Zhao J C 2012 *Chin. Phys. B* **21** 040502
[15] Sun W, Chen Z and Chen S 2012 *Chin. Phys. B* **21** 050509
[16] Achlioptas D, D'Souza R M and Spencer J 2009 *Science* **323** 1453
[17] Radicchi F and Fortunato S 2009 *Phys. Rev. Lett.* **103** 168701
[18] Cho Y S, Kim J S, Park J, Kahng B and Kim D 2009 *Phys. Rev. Lett.* **103** 135702
[19] Gómez-Gardeñes J, Gómez S, Arenas A and Moreno Y 2011 *Phys. Rev. Lett.* **106** 128701
[20] Leyva I, Sevilla-Escoboza R, Buldú J M, Sendiña-Nadal I, Gómez-Gardeñes J, Arenas A, Moreno Y, Gómez S, Jaimes-Reátegui R and Boccaletti S 2012 *Phys. Rev. Lett.* **108** 168702