Effects of non-Gaussian noise on the dynamical properties of a logistic system*

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The dynamical properties of a tumor cell growth system described by the logistic system with coupling between non-Gaussian and Gaussian noise terms are investigated. The effects of the nonextensive index q on the stationary properties and the transient properties are discussed, respectively. The results show that the nonextensive index q can induce the tumor cell numbers to decrease greatly in the case of $q > 1$. Moreover, the switch from the steady stable state to the extinct state is speeded up as the increases of q, and the tumor cell numbers can be more obviously restrained for a large value of q. The numerical results are found to be in basic agreement with the theoretical predictions.

Keywords: logistic growth system, non-Gaussian noise, dynamical properties

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1. Introduction

Biological processes subjected to noises have been the subject of increased interest recently. In many cases, noises play a positive role. Therefore, the noise effect has attracted extensive investigations. In nonlinear dynamical systems, plenty of investigations have been made, and some very interesting and meaningful results found. Such as, a classic optical bistable system driven by all kinds of noise sources has been studied, noises inducing transition,[1–3] stochastic resonance,[4–6] reentrance phenomena,[7] noise enhanced stability,[8,9] and so on, have been found. In biological processes, biochemical reactions play fundamental roles in numerous biological process. Random perturbations are inevitable in biochemical reactions. Based on the objective factors, the biological processes can be described more precisely by stochastic models, and some interesting phenomena are found. For example, in the gene transcriptional regulatory process, the correlated noise can induce reentrance phenomena,[10] and stochastic resonance happens in the weak noise region when the system is subjected to a seasonal external field.[11] When the genotype selection model is driven by noises, it undergoes a purely noise-induced transition, from a monostable regime to a bistable one, but when it is submitted to a color noise source, and for large values of the correlation time, the system undergoes a new transition to a monostable state, indicating a reentrance phenomenon.[12] The correlated noise and color noise break the balance of gene selection and induces the phase transition.[13–15] Noises in symmetric two species competition change the behavior of the system drastically and lead to new situations which have no counterpart in the deterministic case.[16] Noises induce suprathreshold circadian oscillations and coherent resonance in Drosophila.[17] Reference [18] presents the Michaelis–Menten mechanism for single-enzyme and multi-enzyme systems under stochastic noise and spatial diffusion.

The effect of noise on a tumor cell growth system which is generally described by a logistic model has been widely investigated. Ai *et al.*[19] found that correlated Gaussian white noise can restrain the tumor cell number to increase by analyzing the steady state characters of this system. Other cases including the mean first passage time and the relaxation time were investigated by our work groups.[20–26] Color noise and negative correlation were considered in Ref. [27]. Reference [28] showed pure multiplicative noise-induced stochastic resonance, which appears in an anti-tumor system modulated by a seasonal external field. Bose and Trimper[29] put forward a stochastic model for tumor growth with immunization, and analyzed the effects of the immunization rate. These studies on the effects of noise on the tumor cell growth system assume that the noise source has a Gaussian distribution (either white or colored). However, some recent experimental results for one kind of crayfish and for rat skin[30,31] offer strong indications that there could be non-Gaussian noise sources in these sensory systems. Thus, it is necessary to study the effects of the non-Gaussian nature of the noises.[32–39] Recently, reference [40] studied an enzyme dynamical model of tumor growth in an immune system with coupling between non-Gaussian and Gaussian noise terms, and found that the departure from the Gaussian noise can not only reduce the probability of tumor cells in the active state, induce the minimum of the average tumor-cell population to move toward a smaller non-Gaussian noise, but also decrease the mean first-passage

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time. It must be emphasized that our model is based on the logistic system, because the logistic model is more general than the enzyme dynamical model. The logistic model is used to describe the tumor cell growth, the population growth, etc.

In this paper, a tumor cell growth system based on the logistic model with coupling between non-Gaussian noise and Gaussian white noise is investigated. The nonextensive index q indicating the departure from the Gaussian noise is employed to analyze the non-Gaussian noise with q-dependent probability distribution. In Section 2, the stationary properties inducing the steady state probability distribution, the mean value, the second moment, and the normalized variance are obtained and discussed based on the approximate Fokker–Planck equation of the system, which is derived through a path integral approach. In Section 3, the expression of the mean first passage time is given and the effects of the nonextensive index q is analyzed. The numerical results are given in Section 4. Finally, a brief conclusion is given in Section 5.

2. Stationary properties

The the tumor cell growth system based on the logistic model is written as[41,42]

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - bx^2. \tag{1}$$

This is an ideal equation without any fluctuation, where x is the tumor mass, a the growth rate, and b the cell decay rate.

The deterministic potential of the tumor cell growth system is given by

$$V(x) = -\frac{a}{2}x^2 + \frac{b}{3}x^3. \tag{2}$$

It has one stable states at $x_s = a/b$, and an unstable state at $x_u = 0$. If $a \to 0$, the stable $x_s \to x_u$.

If the environmental fluctuation due to some external factors (such as temperature, radiotherapy, geological events) is considered, the parameter a should be modified as $a + \eta(t)$, and some factors such as internal fluctuation, gives rise to an additive noise $\xi(t)$. Considering all these factors, Eq. (1) can be rewritten as

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = ax(t) - bx^2(t) + x(t)\eta(t) + \xi(t), \tag{3}$$

where $\xi(t)$ is the Gaussian white noise, and the noise term $\eta(t)$ has a non-Gaussian distribution with

$$\frac{\mathrm{d}\eta(t)}{\mathrm{d}t} = \frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}\eta}V_q(\eta) + \frac{1}{\tau}\varepsilon(t), \tag{4}$$

and

$$V_q(\eta) = \frac{D}{\tau(q-1)}\ln\left[1 + \frac{\tau}{D}(q-1)\frac{\eta^2}{2}\right]. \tag{5}$$

The statistical properties of the Gaussian white noise terms $\xi(t)$ and $\varepsilon(t)$ are characterized by their mean and variance

$$\langle\xi(t)\rangle = \langle\varepsilon(t)\rangle = 0, \tag{6}$$

$$\langle\xi(t)\xi(t')\rangle = 2D\delta(t-t'), \tag{7}$$

$$\langle\varepsilon(t)\varepsilon(t')\rangle = 2Q\delta(t-t'), \tag{8}$$

$$\langle\varepsilon(t)\xi(t')\rangle = \langle\varepsilon(t')\xi(t)\rangle = 2\lambda\sqrt{DQ}\delta(t-t'), \tag{9}$$

where D and Q are the intensities of the Gaussian white noise terms $\eta(t)$ and $\varepsilon(t)$, respectively. The q in Eq. (5) is the nonextensive index, which denotes the departure from the Gaussian behavior. For $q = 1$, the process $\eta(t)$ is the (Gaussian) Ornstein–Uhlenbeck one (with correlation time equal to τ), while for $q \neq 1$ it is a non–Gaussian process. Here we simply summarize the stationary probability distribution of the noise $\eta(t)$, which has been studied in Ref. [43].

The stationary probability distribution is given by

$$P_q^{\mathrm{st}}(\eta) = \frac{1}{Z_q}\left[1 + \frac{\tau}{D}(q-1)\frac{\eta^2}{2}\right]^{1/(1-q)}, \tag{10}$$

where Z_q is the normalization factor. This distribution can be normalized only for $q < 3$. Since the above distribution function is an even function of $\eta(t)$, the first moment, $\langle\eta(t)\rangle$, is always equal to zero, and the second moment given by

$$\langle\eta^2\rangle = \frac{2D}{\tau(5-3q)} \tag{11}$$

is finite only for $q < 5/3$.

An effective Markovian approximation to the process η via a path integral procedure was obtained in Ref. [43], but the approximation renders a non-normalizable stationary probability distribution function $P_q^{\mathrm{st}}(\eta)$. Wio *et al.*[36] resorted to a simpler, albeit physically reasonable, approximation in the region $|q-1| \ll 1$ (both $q < 1$ and $q > 1$). In such a region the non-Gaussian noise will only slightly depart from the Gaussian behavior, but will show some of the main trends of the $q \neq 1$ region. Hence one has

$$\begin{aligned}\frac{1}{\tau}\frac{\mathrm{d}}{\mathrm{d}\eta}V_q &= \frac{\eta}{\tau}\left[1 + \frac{\tau}{D}(q-1)\frac{\eta^2}{2}\right]^{-1} \\ &\approx \frac{\eta}{\tau}\left[1 + \frac{\tau}{D}(q-1)\frac{\langle\eta^2\rangle}{2}\right]^{-1} \\ &= \frac{\eta}{\tau_{\mathrm{eff}}},\end{aligned} \tag{12}$$

where

$$\tau_{\mathrm{eff}} = \frac{2(2-q)}{5-3q}\tau. \tag{13}$$

Therefore, equation (2) can be reduced to a renormalized (Gaussian) Ornstein–Uhlenbeck noise with the effective correlation time τ_{eff} and the associated effective noise intensity

$$D_{\mathrm{eff}} = \left(\frac{2(2-q)}{5-3q}\right)^2 D.$$

Clearly, for $q \to 1$, we have $\tau_{\rm eff} \to \tau$ and $D_{\rm eff} \to D$, as could be expected. So equation (2) can be rewritten as

$$\frac{{\rm d}\eta(t)}{{\rm d}t} = \frac{1}{\tau_{\rm eff}}\eta + \frac{1}{\tau_{\rm eff}}\varepsilon(t), \tag{14}$$

and

$$\langle \varepsilon(t)\varepsilon(t')\rangle = 2D_{\rm eff}\delta(t-t'), \tag{15}$$

$$\langle \varepsilon(t)\xi(t')\rangle = \langle \varepsilon(t')\xi(t)\rangle = 2\lambda\sqrt{D_{\rm eff}Q}\delta(t-t'). \tag{16}$$

Applying the functional analysis, the Fokker–Planck equation corresponding to Eqs. (1)–(10) and with the steady-state value $x_{\rm s} = a/b$ can be written as

$$\frac{\partial P(x,t)}{\partial t} = L_{\rm FP}P(x,t), \tag{17}$$

$$L_{\rm FP} = -\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x), \tag{18}$$

where

$$A(x) = ax - bx^2 + \frac{D_{\rm eff}}{1+a\tau_{\rm eff}}x + \lambda\sqrt{D_{\rm eff}Q}, \tag{19}$$

$$B(x) = \frac{D_{\rm eff}}{1+a\tau_{\rm eff}}x^2 + 2\lambda\sqrt{D_{\rm eff}Q}x + Q. \tag{20}$$

Therefore, the steady-state probability distribution (SPD) function $P_{\rm st}(x)$ for $\lambda \neq 1$ is given by

$$P_{\rm st}(x) = NB(x)^{-1/2}\exp[-\phi(x)], \tag{21}$$

here N is the normalization constant, $\phi(x)$ is the modified potential function,

$$\phi(x) = \frac{bx}{m} - \left(\frac{a}{2m} + \frac{bn}{m^2}\right)\ln(mx^2+2nx+Q) - \frac{bQm - anm - 2n^2b}{m^2\sqrt{mQ-n^2}}\arctan\left(\frac{mx+n}{\sqrt{mQ-n^2}}\right), \tag{22}$$

where

$$m = \frac{D_{\rm eff}}{1+2a\tau_{\rm eff}}, \quad n = \lambda\sqrt{D_{\rm eff}Q}. \tag{23}$$

The extrema of the SPD (21) are determined by the following second-order equation

$$bx^2 + \left(\frac{D_{\rm eff}}{1+a\tau_{\rm eff}} - a\right)x + \lambda\sqrt{D_{\rm eff}Q} = 0, \tag{24}$$

the condition for extremum existence of $P_{\rm st}(x)$ is

$$\left(\frac{D_{\rm eff}}{1+a\tau_{\rm eff}} - a\right)^2 > 4b\lambda\sqrt{D_{\rm eff}Q}. \tag{25}$$

Under the condition of $\lambda > 0$ and Eq. (21), $P_{\rm st}(x)$ as the function of the tumor cell number x is plotted for different nonextensive index q, as shown in Fig. 1. From Fig. 1, we can see that there is a maximum in the curve of $P_{\rm st}(x)$. With q increasing, the maximum becomes small. However, it is clear that the position of the maximum of the $P_{\rm st}(x)$ shifts from large x to small x. It illuminates that q can induce the tumor cell numbers to decrease.

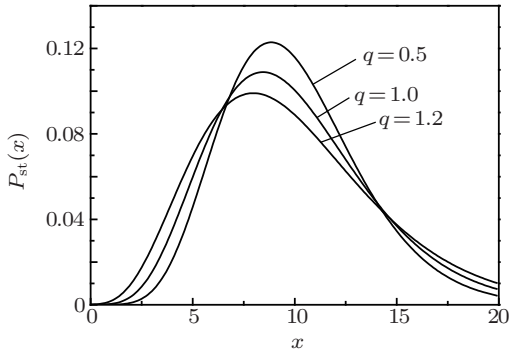


Fig. 1. The steady-state probability distribution function $P_{\rm st}(x)$ is plotted for different nonextensive indices $q = 0.5, 1.0, 1.5$, respectively. The other parameters are $D = 0.2$, $Q = 0.05$, $\lambda = 0.2$ and $\tau = 0.3$, $a = 1$, $b = 0.1$.

Further, in order to quantitatively investigate the stationary properties of the system, we introduce the moments of the the tumor cell number x, and it is given by,

$$\langle x^n\rangle_{\rm st} = \int_0^{+\infty} x^n P_{\rm st}(x)\,{\rm d}x. \tag{26}$$

The mean of the tumor cell number is given by

$$\langle x\rangle_{\rm st} = \int_0^{+\infty} xP_{\rm st}(x)\,{\rm d}x. \tag{27}$$

The mean value $\langle x\rangle_{\rm st}$ of the tumor cell number x is plotted as a function of the parameter q of the departure from the Gaussian noise in Fig. 2. It can be seen from Fig. 2 that, when the nonextensive index $q < 1$, the mean of the tumor cell number x is slowly decreased with the increasing of q, however, it is rapidly decreased with the increasing of q when the nonextensive index $q > 1$. These results show that the tumor cell number is sensitive to large value of q. For the case of $q > 1$, it is much more helpful to restrain the tumor cell number.

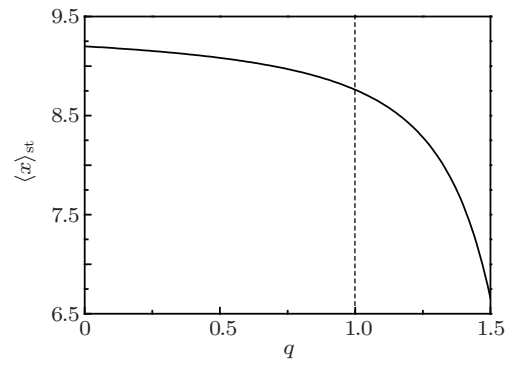


Fig. 2. The mean value $\langle x\rangle_{\rm st}$ is plotted as a function of q. The other parameters are $\lambda = 0.4$. $D = 0.5$, $Q = 0.5$, $\tau = 0.8$, $a = 1$, $b = 0.1$.

The second moment of tumor cell number x is given by

$$\langle x^2\rangle_{\rm st} = \int_0^{+\infty} x^2 P_{\rm st}(x)\,{\rm d}x. \tag{28}$$

The second moment of tumor cell number x is plotted as a function of the parameter q of the departure from the Gaussian noise in Fig. 3. It is clear that the varied trend of $\langle x^2\rangle_{\rm st}$ with q increasing is similar to the $\langle x\rangle_{\rm st}$. It means that $\langle x^2\rangle_{\rm st}$ is also sensitive to a large value of q, especially for $q > 1$.

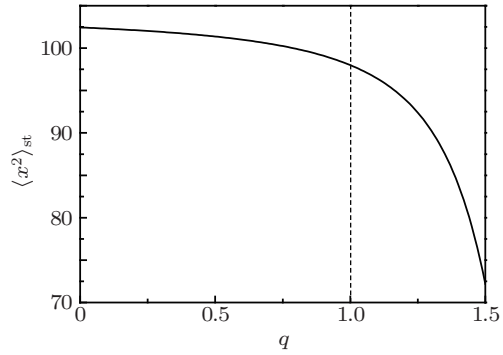


Fig. 3. $\langle x^2\rangle_{\rm st}$ is plotted as a function of q. The other parameter values are $\lambda = 0.4$. $D = 0.5$, $Q = 0.5$, $\tau = 0.8$, $a = 1$, $b = 0.1$.

The normalized variance of the tumor cell number x is given by

$$V_{\rm ar} = \frac{\langle (x - \langle x\rangle_{\rm st})^2\rangle}{\langle x\rangle_{\rm st}^2} = \frac{\langle x^2\rangle_{\rm st}}{\langle x\rangle_{\rm st}^2} - 1. \tag{29}$$

The $V_{\rm ar}$ as a function of q is shown in Fig. 4 for different correlated intensity λ. The normalized variance $V_{\rm ar}$ increases as the increase of q for the cases of the positive correlation and the negative correlation. However, for the case of $q < 1$, $V_{\rm ar}$ is slowly increased, and rapidly increases for the case of $q > 1$.

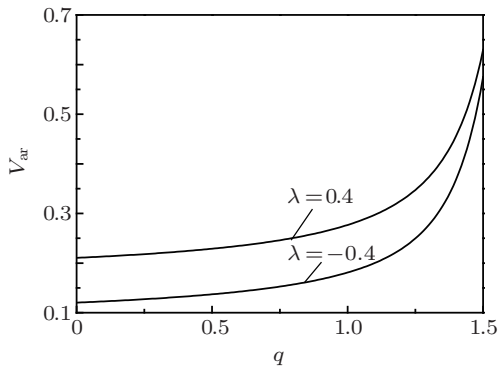


Fig. 4. $V_{\rm ar}$ is plotted as a function of q. The other parameter values are $\lambda = 0.4, -0.4$. $D = 0.5$, $Q = 0.5$, $\tau = 0.8$, $a = 1$, $b = 0.1$.

3. Transient properties

In order to quantify the effects of noises on the switch between the steady stable states. The mean first passage time (MFPT) is calculated. For the tumor cell growth process, the MFPT of the process $x(t)$ is defined to the average time to reach the extinct state $x_{\rm u} = 0$ with initial condition $x_{\rm s}(t = 0) = a/b$. For small D, α, and λ, applying the steepest-descent approximation, the analytic expression of MFPT is given by[44,45]

$$\begin{aligned} & T(x_{\rm s} \to x_{\rm u}) \\ \approx\ & 2\pi[|V''(x_{\rm s})V''(x_{\rm u})|]^{-1/2} \exp[\Phi(x_{\rm u}) - \Phi(x_{\rm s})] \\ =\ & \frac{2\pi}{a} \exp\left\{ -\frac{a}{m} + \left(\frac{a}{2m} + \frac{bn}{m^2}\right) \ln\left(\frac{m(a/b)^2 + 2nx + Q}{Q}\right) \right. \\ & + \frac{bQm - anm - 2n^2 b}{m^2\sqrt{mQ - n^2}} \left[\arctan\left(\frac{ma/b + n}{\sqrt{mQ - n^2}}\right) \right. \\ & \left. \left. - \arctan\left(\frac{n}{\sqrt{mQ - n^2}}\right) \right] \right\}. \end{aligned} \tag{30}$$

When the nonextensive index q of the departure from the Gaussian noise is equal to 1, the MFPT of the tumor cell growth system has been discussed in Refs. [20] and [24], and will not be recounted here. By virtue of numerical calculation of the MFPT [Eq. (30)]. MFPT T is plotted as a function of q and λ as shown in Fig. 5. From Fig. 5, it is found that the MFPT monotonously decreases with q increasing. Clearly, the switch from the steady stable states $x_{\rm u} = a/b$ to the extinct state $x_{\rm u} = 0$ is speeded up as the increases of q, and the tumor cell numbers can be more obviously restrained for large value of q. In the meantime, the MFPT exhibits different effects in the positive correlation and the negative correlation. In the case of the positive correlation, the MFPT decreases with the correlation intensity λ increasing, but, in the case of the negative correlation, the MFPT increases with $|\lambda|$ increasing.

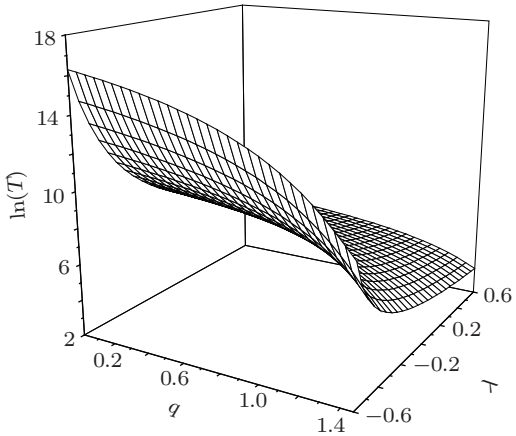


Fig. 5. The mean first passage time (MFPT) T is plotted as a function of q and λ. The parameters are dimensionless and are chosen as $D = 0.5$, $Q = 0.3$, $\tau = 0.6$, $a = 1$, $b = 0.1$.

4. Numerical results

In order to check the validity of the approximate method, we make use of the the second-order Runge–Kutta algorithm,[36,37] the SPD and the MFPT of the logistic system with coupling between non-Gaussian and Gaussian noise

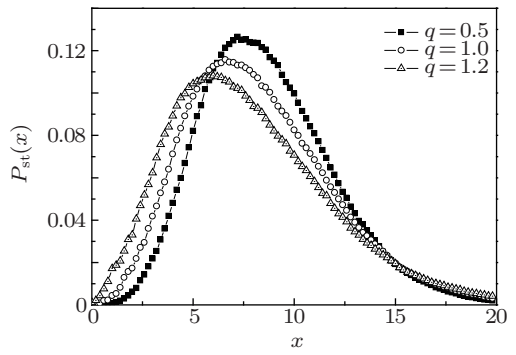


Fig. 6. The numerical results of the steady-state probability distribution $P_{\rm st}(x)$ of the logistic system, the parameters are the same as those in Fig. 1.

terms are calculated numerically according to Eqs. (3)–(9). The results are plotted in Figs. 6 and 7 with the same parameters as those in Figs. 1 and 5. The numerical results are found to be in basic agreement with the theoretical predictions.

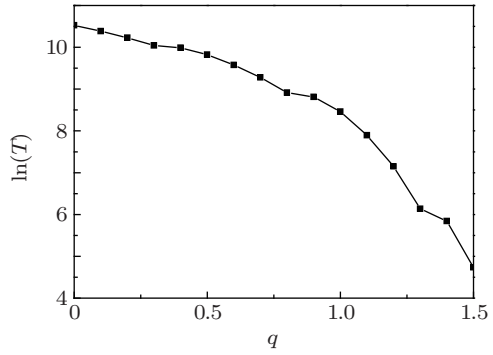


Fig. 7. The numerical results of the mean first passage time (MFPT) T of the logistic system, the parameters are the same as those in Fig. 5.

5. Conclusions

In summary, the dynamical properties of a tumor cell growth system with coupling between non-Gaussian and Gaussian noise terms are investigated based on the Fokker–Planck equation and a path integral approach. The effects of q on the dynamical properties of a tumor growth system described by a logistic model are discussed. The results show that the nonextensive index q can induce the tumor cell numbers to decrease greatly in the case of $q > 1$. Moreover, the switch from the steady stable state to the extinct state is speeded up with the increase of q, and the tumor cell numbers can be more obviously restrained for large value of q. The numerical results are found to be in basic agreement with the theoretical predictions.

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