Relativistic symmetries with the trigonometric Pöschl–Teller potential plus Coulomb-like tensor interaction

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(Received 18 November 2012; revised manuscript received 16 December 2012)

The Dirac equation is solved to obtain its approximate bound states for a spin-1/2 particle in the presence of trigonometric Pöschl–Teller (tPT) potential including a Coulomb-like tensor interaction with arbitrary spin–orbit quantum number κ using an approximation scheme to substitute the centrifugal terms $\kappa(\kappa\pm1)r^{-2}$. In view of spin and pseudo-spin (p-spin) symmetries, the relativistic energy eigenvalues and the corresponding two-component wave functions of a particle moving in the field of attractive and repulsive tPT potentials are obtained using the asymptotic iteration method (AIM). We present numerical results in the absence and presence of tensor coupling A and for various values of spin and p-spin constants and quantum numbers n and κ. The non-relativistic limit is also obtained.

Keywords: Dirac equation, trigonometric Pöschl–Teller potential, tensor interaction, approximation schemes, asymptotic iteration method

PACS: 03.65.Ge, 03.65.Fd, 03.65.Pm, 02.30.Gp **DOI:** 10.1088/1674-1056/22/6/060305

1. Introduction

The concept of pseudo-spin (p-spin) and spin symmetries in nucleons and antinucleons, respectively, introduced to reveal the dynamical nature of quantum systems and these symmetries, play a critical role in the shell structure and its evolution. Within the framework of the Dirac equation, the p-spin symmetry is used to feature the deformed nuclei and the superdeformation to establish an effective shell-model,[1–4] whereas spin symmetry is relevant for mesons.[5] The spin symmetry can be regarded as a relativistic symmetry. For the p-spin, in nuclei, its origin has not been fully clarified until now.[6–8] The exact spin symmetry occurs when the scalar $S(r)$ and vector $V(r)$ potentials have nearly the same size but same sign, i.e., $\Delta(r)=S(r)-V(r)=C_{\rm s}=0$ and the exact p-spin symmetry occurs when the scalar and vector potentials have the same size but opposite sign, i.e., $\Sigma(r)=S(r)+V(r)=C_{\rm ps}=0$.[6–8] Later it was found that spin and p-spin are exact $\mathrm{d}\Delta(r)/\mathrm{d}r$ and $\mathrm{d}\Sigma(r)/\mathrm{d}r$, respectively, under a less strict condition and to what extent the spin and p-spin are conserved are related to the competition between the centrifugal barrier and the spin or p-spin orbital potential.[6–8] The condition cannot be met in realistic nuclei. For more details, see the recent works[6–20] and references therein. The p-spin symmetry refers to a quasi-degeneracy of single nucleon doublets with non-relativistic quantum number $(n,\ell,j=\ell+1/2)$ and $(n-1,\ell+2,j=\ell+3/2)$, where n, ℓ, and j are single nucleon radial, orbital, and total angular quantum numbers, respectively.[21–24] The total angular momentum is $j=\tilde{\ell}+\tilde{s}$, where $\tilde{\ell}=\ell+1$ pseudo-angular momentum and $\tilde{s}$ is p-spin angular momentum.[25] Very recently, the tensor potential was introduced into the Dirac equation by simply substituting $\boldsymbol{P}\to\boldsymbol{P}-\mathrm{i}m\omega\beta.\hat{r}U(r)$ and a spin–orbit coupling is added to the Dirac Hamiltonian.[26–38]

The unbound solutions of generalized asymmetrical Hartmann potentials under the condition of the p-spin symmetry have been presented by mapping the wave functions of bound states in the complex momentum plane via the continuation method.[9] Very recently, Guo explored the p-spin symmetry by using the similarity renormalization group and showed explicitly the relativistic origin of the symmetry.[7] Chen and Guo[8] investigated the evolution of the spin and pseudo-spin symmetries from the relativistic to the nonrelativistic case and explored the relativistic relevance of the symmetries. By examining the zeros of Jost functions corresponding to the small components of Dirac wave functions and phase shifts of continuum states, Lu *et al.*[6] showed that the pseudo-spin symmetry in single particle resonant states in nuclei is conserved when the attractive scalar and repulsive vector potentials have the same magnitude but opposite sign.

In the present work, we intend to investigate the trigonometric Pöschl–Teller (tPT) potential proposed for the first time by Pöschl and Teller[39] in 1933 to describe the diatomic molecular vibration. Chen[40] and Zhang *et al.*[41] have studied the relativistic bound state solutions for the tPT potential and hyperbolical PT (second PT) potential, respectively. Liu *et al.*[42] studied the tPT potential within the framework of the Dirac theory. Recently, Candemir investigated the analytic s-wave solution of the Dirac equation for tPT potential under the p-spin symmetry condition.[43] Further, Hamzavi and Rajabi studied the exact s-wave $(\ell=0)$ solution of the

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Schrödinger equation for the vibration tPT potential.[44] Very recently, Hamzavi *et al.*[45] investigated the approximate analytic bound state eigensolutions of the Dirac equation with the tPT potential including the centrifugal (pseudo-centrifugal) term for any spin–orbit quantum number κ in view of spin and p-spin symmetries. This potential takes the following form

$$V(r) = \frac{V_1}{\sin^2(\alpha r)} + \frac{V_2}{\cos^2(\alpha r)}, \quad 0 < \alpha r < \pi/2, \tag{1}$$

where the parameters V_1 and V_2 describe the property of the potential well while the parameter α is related to the range of this potential.[42] In Figs. 1 and 2, we draw the tPT potential (1) for parameter values $V_1 = 5.0$ fm^{-1} (1 fm $= 10^{-15}$ m), $V_2 = 3.0$ fm^{-1}, $\alpha = 0.02$ fm^{-1}, and $\alpha = 0.30$ fm^{-1}.

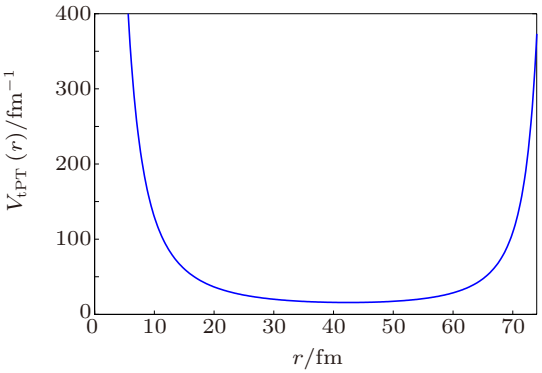


Fig. 1. (color online) Plot of the tPT as a function of r for $\alpha = 0.02$ fm^{-1}.

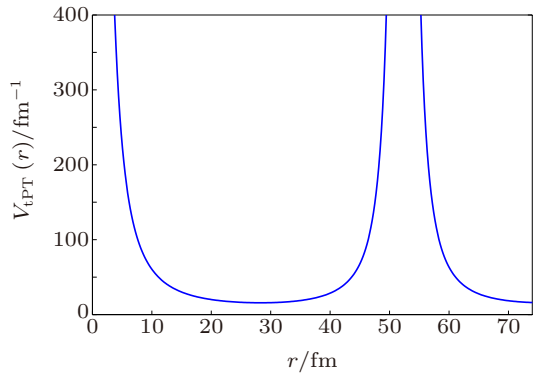


Fig. 2. (color online) Plot of the tPT as a function of r for $\alpha = 0.30$ fm^{-1}.

The goal of the present work is to extend previous works[43–46] to the relativistic case, $\kappa \neq \pm 1$ (rotational case) and also in the presence of a Coulomb-like tensor potential via the asymptotic iteration method.[3,46–48] We introduce a convenient approximation scheme to deal with the strong singular centrifugal term. The ansatz of this approximation possesses the same form of the potential and is singular as the centrifugal term r^{-2}.[49]

The structure of the paper is as follows. In Section 2, we briefly introduce the asymptotic iteration method (AIM). In Section 3, in view of spin and p-spin symmetries, we briefly introduce the Dirac equation with scalar and vector tPT potentials plus a Coulomb-like tensor interaction for an arbitrary spin–orbit quantum number. In Section 4, the approximate energy eigenvalue equations and corresponding two-component wave functions of the Dirac–tPT problem including tensor interaction are obtained. The non-relativistic limit of our solution is also obtained. We present our numerical results in the presence and absence of tensor coupling A for various n and κ quantum numbers. We end with our concluding remarks in Section 5.

2. Method of analysis

One of the calculational tools utilized in solving the Schrödinger-like equation including the centrifugal barrier and/or the spin–orbit coupling term is called as the asymptotic iteration method. For a given potential the idea is to convert the Schrödinger-like equation to the homogenous linear second-order differential equation of the form:[47]

$$y''(x) = \lambda_0(x)y'(x) + s_0(x)y(x), \tag{2}$$

where $\lambda_0(x)$ and $s_0(x)$ have sufficiently many continous derivatives and defined in some interval which are not necessarily bounded. The differential equation (2) has a general solution[47,48]

$$y(x) = \exp\left(-\int^x \alpha(x')\,{\rm d}x'\right) \times \left[C_2 + C_1 \int^x \exp\left(\int^{x'} \left[\lambda_0(x'') + 2\alpha(x'')\right]{\rm d}x''\right){\rm d}x'\right]. \tag{3}$$

If $k > 0$, for sufficiently large k, we obtain the $\alpha(x)$

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha(x), \quad k = 1,2,3,\ldots, \tag{4}$$

where

$$\begin{aligned} \lambda_k(x) &= \lambda'_{k-1}(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x), \\ s_k(x) &= s'_{k-1}(x) + s_0(x)\lambda_{k-1}(x), \quad k = 1,2,3,\ldots, \end{aligned} \tag{5}$$

The energy eigenvalues are obtained from the quantization condition of the method together with Eq. (5) and can be written as follows:

$$\delta_k(x) = \lambda_k(x)s_{k-1}(x) - \lambda_{k-1}(x)s_k(x) = 0, \quad k = 1,2,3,\ldots. \tag{6}$$

The energy eigenvalues are then obtained from Eq. (6), if the problem is exactly solvable. If not, for a specific n principal quantum number, we choose a suitable x_0 point, determined generally as the maximum value of the asymptotic wave function or the minimum value of the potential and the aproximate energy eigenvalues are obtained from the roots of this equation for sufficiently large values of k with iteration.

3. Dirac equation with scalar and vector tPT potentials and Coulomb-like tensor interaction

In spherical coordinates, the Dirac equation for fermonic massive spin-1/2 particles interacting with arbitrary scalar potential $S(r)$, the time-component $V(r)$ of a four-vector potential, and the tensor potential $U(r)$ can be expressed as[5,21]

$$[\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta(M + S(r)) - {\rm i}\beta\boldsymbol{\alpha}\cdot\hat{\boldsymbol{r}}U(r)]\psi(\boldsymbol{r}) = [E - V(r)]\psi(\boldsymbol{r}), \tag{7}$$

where E, $\boldsymbol{p}$, and M denote the relativistic energy of the system, the momentum operator, and mass of the particle respectively. α and β are 4×4 Dirac matrices given by

$$\bar{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix},\quad \beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix},\quad \sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$
$$\sigma_2=\begin{pmatrix}0 & -1\\ \mathrm{i} & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{8}$$

where I is the 2×2 unitary matrix and $\boldsymbol{\sigma}$ are the three-vector Pauli spin matrices. The eigenvalues of the spin–orbit coupling operator are $\kappa=(j+1/2)>0$ and $\kappa=-(j+1/2)<0$ for unaligned spin $j=\ell-1/2$ and the aligned spin $j=\ell+1/2$ respectively. The set (H^2, K, J^2, J_Z) can be taken as the complete set of conservative quantities with $\boldsymbol{J}$ being the total angular momentum operator and $K=(\boldsymbol{\sigma}\cdot\boldsymbol{L}+1)$ is the spin–orbit where $\boldsymbol{L}$ is the orbital angular momentum of the spherical nucleons that commutes with the Dirac Hamiltonian. Thus, the spinor wave functions can be classified according to their angular momentum j, the spin–orbit quantum number κ and the radial quantum number n. Hence, they can be written as follows:

$$\psi_{n\kappa}(\boldsymbol{r})=\begin{pmatrix}f_{n\kappa}(\boldsymbol{r})\\ g_{n\kappa}(\boldsymbol{r})\end{pmatrix}=\frac{1}{r}\begin{pmatrix}F_{n\kappa}(\boldsymbol{r})\mathrm{Y}^{\ell}_{jm}(\theta,\phi)\\ \mathrm{i}G_{n\kappa}(\boldsymbol{r})\mathrm{Y}^{\tilde{\ell}}_{jm}(\theta,\phi),\end{pmatrix}, \tag{9}$$

where $F_{n\kappa}(\boldsymbol{r})$ and $G_{n\kappa}(\boldsymbol{r})$ are the radial wave functions of the upper- and lower-spinor components respectively and $\mathrm{Y}^{\ell}_{jm}(\theta,\phi)$ and $\mathrm{Y}^{\tilde{\ell}}_{jm}(\theta,\phi)$ are the spherical harmonic functions coupled to the total angular momentum j and its projection m on the z axis. Substitution of Eq. (7) into Eq. (2) yields the following coupled differential equations

$$\Big[\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\Big]F_{n\kappa}(r)=(M+E_{n\kappa}-\Delta(r))G_{n\kappa}(r),$$
$$\Big[\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\Big]G_{n\kappa}(r)=(M-E_{n\kappa}+\Sigma(r))F_{n\kappa}(r), \tag{10}$$

where $\Delta(r)=V(r)-S(r)$ and $\Sigma(r)=V(r)+S(r)$ are the difference and sum potentials respectively. On solving Eq. (10), we obtain the following Schrödiger-like differential equation with coupling to the r^{-2} singular term and satisfying $F_{n\kappa}(r)$:

$$\Big[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa+1)}{r^2}+\frac{2\kappa}{r}U(r)-U^2(r)-\frac{\mathrm{d}U(r)}{\mathrm{d}r}$$
$$+\frac{\mathrm{d}\Delta(r)/\mathrm{d}r}{M+E_{n\kappa}-\Delta(r)}\Big(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa}{r}-U(r)\Big)\Big]F_{n\kappa}(r)$$
$$=[(M+E_{n\kappa}-\Delta(r))(M-E_{n\kappa}+\Sigma(r))]F_{n\kappa}(r), \tag{11}$$

where $E_{n\kappa}\neq-M$ when $\Delta(r)=0$ and $\kappa(\kappa+1)=\ell(\ell+1)$. Since $E_{n\kappa}=+M$ is an element of the positive energy spectrum of the Dirac Hamiltonian, this relation with upper spinor component is not valid for the negative energy spectum solution. Furthermore, a similar equation satisfying $G_{n\kappa}(r)$ can be obtained as

$$\Big[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\kappa(\kappa-1)}{r^2}+\frac{2\kappa}{r}U(r)-U^2(r)+\frac{\mathrm{d}U(r)}{\mathrm{d}r}$$
$$+\frac{\mathrm{d}\Sigma(r)/\mathrm{d}r}{M-E_{n\kappa}+\Sigma(r)}\Big(\frac{\mathrm{d}}{\mathrm{d}r}-\frac{\kappa}{r}+U(r)\Big)\Big]G_{n\kappa}(r)$$
$$=[(M+E_{n\kappa}-\Delta(r))(M-E_{n\kappa}+\Sigma(r))]G_{n\kappa}(r), \tag{12}$$

where $E_{n\kappa}\neq+M$ when $\Sigma(r)=0$ and $\kappa(\kappa-1)=\tilde{\ell}(\tilde{\ell}+1)$. Since $E_{n\kappa}=-M$ is an element of the negative energy spectrum of the Dirac Hamiltonian, this relation with the lower spinor component is not valid for the positive energy spectrum solution.

3.1. P-spin symmetry limit

The p-spin symmetry occurs when

$$\frac{\mathrm{d}[V(r)+S(r)]}{\mathrm{d}r}=\frac{\mathrm{d}\Sigma(r)}{\mathrm{d}r}=0$$

or $\Sigma(r)=C_{\mathrm{ps}}=$ constant. Here we take $\Delta(r)$ as the tPT potential and the tensor as the Coulomb-like potential, i.e.,

$$\Delta(r)=\frac{V_1}{\sin^2\alpha r}+\frac{V_2}{\cos^2\alpha r}\quad\text{and}\quad U(r)=-\frac{A}{r},\ r\geq R_{\mathrm{c}}, \tag{13}$$

with

$$A=\frac{Z_{\mathrm{a}}Z_{\mathrm{b}}e^2}{4\pi\epsilon_0}, \tag{14}$$

where R_{c} is the Coulomb radius, Z_{a} and Z_{b} respectively, denote the charges of the projectile a and the target nuclei b.[2,3] Under this symmetry, Eq. (12) can easily be transformed to

$$\Big[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\bar{\delta}}{r^2}-\frac{\bar{\gamma}_1}{\sin^2\alpha r}-\frac{\bar{\gamma}_2}{\cos^2\alpha r}-\bar{\beta}^2\Big]G_{n\kappa}(r)=0, \tag{15}$$

where $\kappa=-\tilde{\ell}$ and $\kappa=\tilde{\ell}+1$ for $\kappa<0$ and $\kappa>0$ respectively and

$$\begin{aligned}\bar{\gamma}_1&=(E_{n\kappa}-M-C_{\mathrm{ps}})V_1,\\ \bar{\gamma}_2&=(E_{n\kappa}-M-C_{\mathrm{ps}})V_2,\\ \bar{\beta}&=\sqrt{(M+E_{n\kappa})(M-E_{n\kappa}+C_{\mathrm{ps}})},\\ \bar{\delta}&=(\kappa+A)(\kappa+A-1),\end{aligned} \tag{16}$$

have been introduced for mathematical simplicity. The variation of energy as a function of the variable α for p-spin symmetry is shown in Fig. 3.

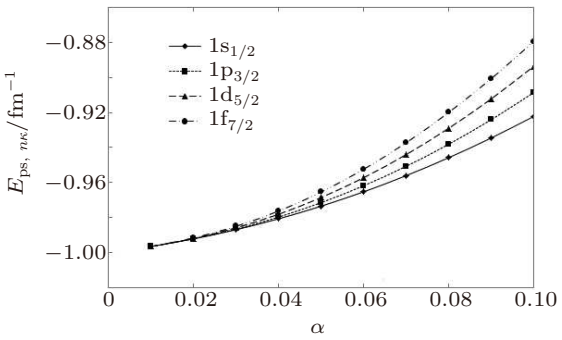


Fig. 3. The variations of energy as a function of the variable α for p-spin symmetry are displayed with parameters $M=1.0$ fm^{-1}, $A=1.0$, $C_{\mathrm{ps}}=0$, $V_1=-0.002$ fm^{-1}, and $V_2=0.003$ fm^{-1}.

3.2. Spin symmetry limit

In the spin symmetry limit, $\mathrm{d}\Delta(r)/\mathrm{d}r=0$ or $\Delta(r)=C_{\mathrm{s}}=$ constant.[5,21] Similarly to Subsection 3.1, we consider

$$\Sigma(r)=\frac{V_1}{\sin^2\alpha r}+\frac{V_2}{\cos^2\alpha r}\quad\text{and}\quad U(r)=-\frac{A}{r},\ r\geq R_{\mathrm{c}}. \tag{17}$$

Using Eq. (17), we can rewrite Eq. (11) as

$$\Big[\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\delta}{r^2}-\frac{\gamma_1}{\sin^2\alpha r}-\frac{\gamma_2}{\cos^2\alpha r}-\beta^2\Big]F_{n\kappa}(r)=0, \quad (18)$$

where $\kappa=\ell$ and $\kappa=-\ell-1$ for $\kappa<0$ and $\kappa>0$, respectively. We have also introduced the following parameters

$$\begin{aligned}\gamma_1 &= (M+E_{n\kappa}-C_{\rm s})V_1,\\ \gamma_2 &= (M+E_{n\kappa}-C_{\rm s})V_2,\\ \beta &= \sqrt{(M-E_{n\kappa})(M+E_{n\kappa}-C_{\rm s})},\\ \delta &= (\kappa+A)(\kappa+A+1),\end{aligned} \quad (19)$$

for mathematical simplicity. The variation of energy as a function of the variable α for spin symmetry is shown in Fig. 4.

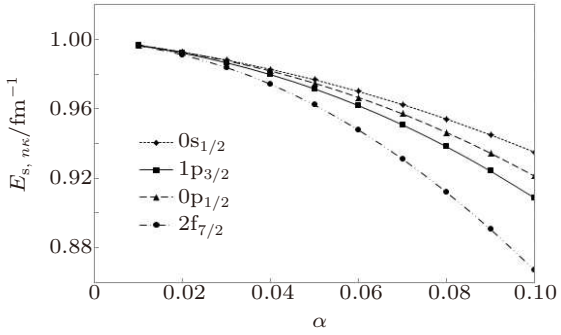


Fig. 4. The variation of energy as a function of the variable α for spin symmetry are displayed with parameters $M=1.0$ fm^{-1}, $A=1.0$, $C_{\rm s}=0$, $V_1=0.002$ fm^{-1}, and $V_2=-0.003$ fm^{-1}.

4. Approximate relativistic bound-states of a particle in the field of the tPT field including tensor interaction

In this section, we present the approximate bound state solutions of the Dirac equation with the tPT potential in the presence of the tensor interaction using the AIM.

4.1. Approximate bound-states in view of the p-spin symmetric limit

It is clearly seen that Eq. (15) is not analytically solvable because of the coupling term. We therefore must include an approximation to deal with the coupling term. It is found that for short range potential, the following formula is a good approximation to the coupling term:[49]

$$\frac{1}{r^2}\approx\frac{\alpha^2}{\sin^2\alpha r}. \quad (20)$$

By inserting approximation expression (20) and introducing a new variable of the form $z=\sin^2\alpha r$ into Eq. (15), we can easily find

$$\begin{aligned}&\frac{\mathrm{d}^2G_{n\kappa}(z)}{\mathrm{d}z^2}+\frac{1-2z}{2z(1-z)}\frac{\mathrm{d}G_{n\kappa}}{\mathrm{d}z}\\ &-\Big(\frac{\bar\delta\alpha^2+\bar\gamma_1}{4\alpha^2z^2(1-z)}+\frac{\bar\gamma_2}{4\alpha^2z(1-z)^2}+\frac{\bar\beta^2}{4\alpha^2z(1-z)}\Big)G_{n\kappa}=0.\end{aligned} \quad (21)$$

Now we reach a position that the differential equation is appropriate for applying AIM. Therefore Eq. (21) should have a solution in the form of normalized wave functions which should respect the boundary conditions that is $G_{n\kappa}(0)=0$ at $z=0$ for $r\to\infty$ and $G_{n\kappa}(1)=0$ at $z=1$ for $r\to0$. As a result, we take the wave functions of the form

$$G_{n\kappa}(z)=z^{\bar p}(1-z)^{\bar q}g_{n\kappa}(z), \quad (22)$$

where we have intoduced the following notations for mathematical simplicity

$$\begin{aligned}\bar q &= \frac{1}{4}+\frac{1}{4}\sqrt{1+4\frac{\bar\gamma_2}{\alpha^2}},\\ \bar p &= \frac{1}{4}+\frac{1}{4}\sqrt{1+4\Big(\bar\delta+\frac{\bar\gamma_1}{\alpha^2}\Big)}.\end{aligned} \quad (23)$$

Substituting Eq. (22) into Eq. (21) yields the following second-order differential equation

$$\begin{aligned}&g''_{n\kappa}(z)-\Big[\frac{4z(\bar p+\bar q+1/2)-(4\bar p+1)}{2z(1-z)}\Big]g'_{n\kappa}(z)\\ &-\Big[\frac{\bar\beta^2/\alpha^2+4(\bar p+\bar q)^2}{4z(1-z)}\Big]g_{n\kappa}(z)=0,\end{aligned} \quad (24)$$

which is suitable to an AIM solutions. Thus, by comparing Eq. (24) with Eq. (2), we can obtain $\lambda_0(z)$ and $s_0(z)$ and by using the recursion relation (5), we calculate $\lambda_k(z)$ and $s_k(z)$. Combining these results with the termination condition (6), we have the following expressions for eigenvalues,

$$\begin{aligned}&s_0(z)\lambda_1(z)=s_1(z)\lambda_0(z)\Rightarrow\bar\beta^2=-4\alpha^2(\bar p+\bar q)^2,\\ &s_1(z)\lambda_2(z)=s_2(z)\lambda_1(z)\Rightarrow\bar\beta^2=-4\alpha^2(\bar p+\bar q+1)^2,\\ &s_2(z)\lambda_3(z)=s_2(z)\lambda_0(z)\Rightarrow\bar\beta^2=-4\alpha^2(\bar p+\bar q+2)^2,\\ &\cdots.\end{aligned} \quad (25)$$

If we generalize the above equations, the relativistic energy spectrum equation becomes

$$\bar\beta^2+4\alpha^2(\bar p+\bar q+n)^2=0. \quad (26)$$

By using the notations in Eqs. (20) and (23), we obtain a more explicit expression for the relativistic energy spectrum equation as

$$\begin{aligned}&(M+E_{n\kappa})(M-E_{n\kappa}+C_{\rm ps})+4\alpha^2\Big[n+\frac{1}{2}\\ &+\frac{1}{4}\Big(\sqrt{1+\frac{4V_2}{\alpha^2}\left(E_{n\kappa}-M-C_{\rm ps}\right)}+\bar\eta\Big)\Big]^2=0,\end{aligned} \quad (27)$$

where

$$\bar\eta=\sqrt{1+4\bar\delta+\frac{4V_1}{\alpha^2}\left(E_{n\kappa}-M-C_{\rm ps}\right)}. \quad (28)$$

Now, let us study the wave function of this system. Generally speaking, the differential equation we wish to solve should be transformed to the form:[5]

$$y''(x)=2\Big(\frac{\Lambda x^{N+1}}{1-bx^{N+2}}-\frac{m+1}{x}\Big)y'(x)-\frac{Wx^N}{1-bx^{N+2}}, \quad (29)$$

where a, b, and m are constants. The general solution of Eq. (29) is found as[43]

$$y_n(x)=(-1)^nC_2(N+2)^n(\sigma)_n{}_2\mathrm{F}_1(-n,t+n;\sigma;bx^{N+2}), \quad (30)$$

where the following parameters have been used

$$(\sigma)_n=\frac{\Gamma(\sigma+n)}{\Gamma(\sigma)},\ \sigma=\frac{2m+N+3}{N+2}\ \text{and}\ t=\frac{(2m+1)b+2\Lambda}{(N+2)b}. \quad (31)$$

By comparing Eq. (29) with Eq. (24), we have $\Lambda=\bar{q}+1/4$, $b=1$, $N=-1$, $m=\bar{p}-3/4$, and $\sigma=2\bar{p}+1/2$. Thus the solution of Eq. (26) can easily be obtained as

$$G_{n\kappa}(z)=\bar{C}_2\frac{\Gamma\left(2\bar{p}+n+\frac{1}{2}\right)}{\Gamma\left(2\bar{p}+\frac{1}{2}\right)}z^{\bar{p}}(1-z)^{\bar{q}} \times {}_2\mathrm{F}_1\left(-n,2(\bar{p}+\bar{q});2\bar{p}+\frac{1}{2};z\right), \quad (32)$$

where $\bar{C}_2$ is the normalization factor. In order to determine this factor, we need to express the solution to Eq. (32) in terms of Jacobi polynomial[50]

$$G_{n\kappa}(z)=\bar{C}_2 z^{\bar{p}}(1-z)^{\bar{q}}\mathrm{P}_n^{(u,v)}(1-2z), \quad (33)$$

where

$$u=2\bar{p}-\frac{1}{2},\ \ v=2\bar{q}-\frac{1}{2}. \quad (34)$$

Having done this, we can easily determine the normalization factor using the normalization condition

$$\int_0^1|G_{n\kappa}(z)|^2\frac{\mathrm{d}z}{2\sin\theta\cos\theta}=1,$$

so that

$$\frac{\bar{C}_2^2}{2^{u+v+1}}\int_{-\infty}^{\infty}(1-x)^{u+1/2}(1+x)^{v+1/2}|\mathrm{P}_n^{(u,v)}|^2\mathrm{d}x=1. \quad (35)$$

Now using the following relation of the hypergeometric function[50]

$$\mathrm{P}_n^{(u,v)}=\frac{1}{2^n}\sum_{m=0}^{\infty}\binom{n+u}{m}\binom{n+v}{n-m}(x-1)^{n-m}(1+x)^m, \quad (36)$$

and integral[21]

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\mathrm{P}_n^{u,v}(x)\mathrm{d}x = \frac{2^{\alpha+\beta+1}\Gamma(\beta+1)\Gamma(\alpha+1)\Gamma(n+1+u)}{n!\Gamma(\alpha+\beta+2)\Gamma(1+\alpha)} \times {}_3\mathrm{F}_2(-n,u+v+n+1,\alpha+1;\beta+1,u+1,\alpha+\beta+2;1), \quad (37)$$

we obtain the normalization constant as

$$\bar{C}_2=\frac{1}{\sqrt{\bar{I}_n(u,v)}}, \quad (38)$$

where

$$\bar{I}_n(u,v) = \frac{1}{2}\sum_{m=0}^{\infty}(-1)^{n-m+1}\binom{n+u}{m}\binom{n+v}{n-m} \times\frac{\Gamma\left(n-m+u+\frac{3}{2}\right)\Gamma\left(m+v+\frac{3}{2}\right)\Gamma(n+1+u)}{n!\Gamma(n+u+v+3)\Gamma(1+u)} \times {}_3\mathrm{F}_2\left(-n,u+v+n+1,n-m+u+\frac{3}{2};m+v+\frac{3}{2},u+1,n+u+v+3;1\right). \quad (39)$$

The upper component spinor of the Dirac equation for the tPT potential plus a Coulomb-like tensor potential can be calculated from the relation

$$F_{n\kappa}(r)=\frac{1}{M-E_{n\kappa}+C_{\mathrm{ps}}}\left[\frac{\mathrm{d}}{\mathrm{d}r}-\frac{(\kappa+A)}{r}\right]G_{n\kappa}(r). \quad (40)$$

Using the derivative of the hypergeometric function

$$\frac{\mathrm{d}}{\mathrm{d}x}[\,{}_2\mathrm{F}_1(a;b;c;x)]=\frac{ab}{c}\,{}_2\mathrm{F}_1(a+1;b+1;c+1;x)$$

given in Ref. [41], we obtain the upper spinor component, in the p-spin symmetric case, from Eq. (40) as

$$\begin{aligned}&F_{n\kappa}(r)\\&=\frac{2\alpha}{(M-E_{n\kappa}+C_{\mathrm{ps}})}\left[\bar{p}\cot(\alpha r)+\bar{q}\tan(\alpha r)-\frac{(\kappa+A)}{r}\right]G_{n\kappa}(r)\\&-\frac{4n\alpha\bar{C}_2(\bar{p}+\bar{q})}{(M-E_{n\kappa}+C_{\mathrm{ps}})}\frac{\Gamma(2\bar{p}+n+\frac{1}{2})}{\Gamma(2\bar{p}+\frac{3}{2})}(\sin\alpha r)^{2\bar{p}+1}(\cos\alpha r)^{2\bar{q}+1}\\&\times{}_2\mathrm{F}_1\left(1-n,2(\bar{p}+\bar{q})+1;2\bar{p}+\tfrac{3}{2};\sin^2(\alpha r)\right).\end{aligned} \quad (41)$$

4.2. Approximate bound-states in view of the spin symmetric limit

In this symmetry, we obtain the solution of Eq. (18) by using a transformation of the form $z=\sin^2\alpha r$ so that Eq. (18) can be recast into a similar form of Eq. (21)

$$\frac{\mathrm{d}^2F_{n\kappa}(z)}{\mathrm{d}z^2}+\frac{1-2z}{2z(1-z)}\frac{\mathrm{d}F_{n\kappa}(z)}{\mathrm{d}z}-\Big(\frac{\delta\alpha^2+\gamma_1}{4\alpha^2z^2(1-z)} +\frac{\gamma_2}{4\alpha^2z(1-z)^2}+\frac{\beta}{4\alpha^2z(1-z)}\Big)F_{n\kappa}(z)=0. \quad (42)$$

It is clearly seen that Eq. (42) is identical to Eq. (21), therefore, the relativistic energy spectrum equation is obtained as

$$(M-E_{n\kappa})(M+E-C_{\mathrm{ps}})+4\alpha^2\Big[n+\frac{1}{2} +\frac{1}{4}\Big(\eta+\sqrt{1+\frac{4}{\alpha^2}(E+M-C_s)V_2}\Big)\Big]^2=0, \quad (43)$$

where

$$\eta=\sqrt{1+4\delta+\frac{4\gamma_1}{\alpha^2}}. \quad (44)$$

The corresponding lower component spinor $\mathrm{F}_{nk}(z)$ is obtained as

$$\begin{aligned}F_{n\kappa}(z) &= C_2\frac{\Gamma(2p+n+\frac{1}{2})}{\Gamma\left(2p+\frac{1}{2}\right)}z^p(1-z)^q \times{}_2\mathrm{F}_1\left(-n,2(p+q);2p+\frac{1}{2};z\right)\\&=C_2z^p(1-z)^q\mathrm{P}_n^{(c,d)}(1-2z),\end{aligned} \quad (45)$$

where

$$\begin{aligned}c &= 2p-\frac{1}{2}=\frac{1}{2}\sqrt{1+4(\delta+\frac{\gamma_1}{\alpha^2})},\\d &= 2q-\frac{1}{2}=\frac{1}{2}\sqrt{1+16p+4\gamma_2}.\end{aligned} \quad (46)$$

The normalization constant C_2 can easily be obtained as

$$C_2=\frac{1}{\sqrt{I_n(c,d)}}, \quad (47)$$

where

$$I_n(c,d) = \frac{1}{2}\sum_{m=0}^{\infty}(-1)^{n-m+1}\binom{n+c}{m}\binom{n+d}{n-m} \times \frac{\Gamma\left(n-m+c+\frac{3}{2}\right)\Gamma(m+d+\frac{3}{2})\Gamma(n+1+c)}{n!\Gamma(n+c+d+3)\Gamma(1+c)} \times {}_3\mathrm{F}_2\left(-n,c+d+n+1,n-m+c+\frac{3}{2};m+d+\frac{3}{2},c+1,n+c+d+3;1\right). \quad (48)$$

Again, the lower component spinor of the Dirac equation for the tPT potential plus a Coulomb-like tensor potential can be calculated from the relation

$$G_{n\kappa}(r) = \frac{1}{M+E_{n\kappa}-C_{\mathrm{s}}}\left[\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\kappa+A}{r}\right]F_{n\kappa}(r). \quad (49)$$

Following the calculations presented in the previous subsection, we obtain the lower spinor component, in the spin symmetric case, as

$$G_{n\kappa}(r) = \frac{2\alpha}{(M+E_{n\kappa}-C_{\mathrm{s}})}\left[p\cot(\alpha r)+q\tan(\alpha r)+\frac{(\kappa+A)}{r}\right]F_{n\kappa}(r) - \frac{4n\alpha C_2(p+q)}{(M+E_{n\kappa}-C_{\mathrm{s}})}\frac{\Gamma(2p+n+\frac{1}{2})}{\Gamma(2p+\frac{3}{2})}(\sin\alpha r)^{2p+1}(\cos\alpha r)^{2q+1} \times {}_2\mathrm{F}_1\left(1-n,2(p+q)+1;2p+\frac{3}{2};{}^2(\alpha r)\right). \quad (50)$$

Now, let us discuss the non-relativistic limit of the energy eigenvalues of our solution. If we take $C_{\mathrm{s}}=0$ and put $S(r)=V(r)=\Sigma(r)$, the non-relativistic limit of energy equation (43) under the following appropriate transformations $M+E_{n\kappa}\to 2\mu$ and $M-E_{n\kappa}\to -E_{n\ell}$ becomes

$$E_{n\ell} = \frac{\alpha^2}{8\mu}\left[2+\sqrt{1+\frac{8\mu V_2}{\alpha^2}}+\sqrt{(1+2\ell)^2+\frac{8\mu V_1}{\alpha^2}}+4n\right]^2. \quad (51)$$

This is found in excellent agreement with Refs. [36] and [39], and when $\ell=0$ it is identical to Ref. [34].

5. Numerical results

In Table 1, in view of p-spin symmetry, for the given parameter values and when $C_{\mathrm{ps}}=-5.0$ fm^{-1} we noticed the degenerate doublets $(ns_{1/2}, (n-1)d_{3/2})$, $(np_{3/2}, (n-1)f_{5/2})$, $(nd_{5/2}, (n-1)g_{7/2})$, and $(nf_{7/2}, (n-1)h_{9/2})$ in the absence of tensor interaction $(A=0)$. Also, the energy levels are negative, i.e., the system becomes less attractive when the quantum numbers increase. In the presence of the tensor interaction, it removes the degeneracy in the above doublet states and new degenerate states appear as follows: $(1p_{3/2}, 0d_{3/2})$, $(1d_{5/2}, 0f_{5/2})$, and $(1f_{7/2}, 0g_{7/2})$. Further, increasing the strength of the tensor interaction as $A=1.0$, the degenerate doublets become $(1d_{5/2}, 0d_{3/2})$ and $(1f_{7/2}, 0f_{5/2})$. In Table 2, when $C_{\mathrm{ps}}=0$, the energy states become positive, i.e., the system is repulsive.

In Table 3, in view of spin symmetry, for the given parameter values and when $C_{\mathrm{s}}=5$ fm^{-1} we noticed that the energy levels are repulsive and becoming weakly repulsive with increasing quantum numbers n and κ. The degenerate doublets are $(ns_{1/2}, nd_{3/2})$ and $(np_{3/2}, nf_{5/2})$ in the absence of tensor interaction $(A=0)$. In the presence of the tensor interaction, it removes the degeneracy in the above doublet states and new degenerate states appear as follows: $(ns_{1/2}, np_{1/2})$, $(np_{3/2}, nd_{3/2})$, and $(nd_{5/2}, nf_{5/2})$. Increasing the strength of the tensor interaction as $A=1.0$, the degenerate doublets become $(np_{3/2}, np_{1/2})$, $(nd_{5/2}, nd_{3/2})$, and $(nf_{7/2}, nf_{5/2})$. In Table 4, when $C_{\mathrm{s}}=0$ (exact spin symmetry), the energy levels turn to negative and system becomes strongly attractive as the quantum numbers n and κ increase.

Table 1. The bound state energy eigenvalues (fm^{-1}) of the p-spin symmetry tPT potential for various values of n and κ with $M=1.0$ fm^{-1}, $C_{\mathrm{ps}}=-5.0$ fm^{-1}, $V_1=-0.002$ fm^{-1}, $V_2=0.003$ fm^{-1}, and $\alpha=0.01$.

$\tilde{\ell}$	n	κ	(ℓ,j)	$E_{n\kappa}$ (fm^{-1}) $(A=0)$	$E_{n\kappa}$ (fm^{-1}) $(A=0.5)$	$E_{n\kappa}$ (fm^{-1}) $(A=1)$
1	1	−1	$1s_{1/2}$	−4.00084675171 −0.99651749280	−4.00074396128 −0.99652483503	−0.99652927726 −4.00062839464
2	1	−2	$1p_{3/2}$	−4.00103061187 −0.99649448980	−0.99650733888 −4.00094153754	−0.99651749280 −4.00084675171
3	1	−3	$1d_{5/2}$	−4.00119658319 −0.99646127635	−0.99647908416 −4.00111533113	−4.00103061187 −0.99649448980
4	1	−4	$1f_{7/2}$	−0.99641911259 −4.00135100100	−0.99644122998 −4.00127498841	−4.00119658319 −0.99646127635
1	2	−1	$2s_{1/2}$	−4.00097957046 −0.99638440732	−4.00087663841 −0.99639174872	−0.99639619042 −4.00076063954
2	2	−2	$2p_{3/2}$	−4.00116354206 −0.99636140695	−0.99637425458 −4.00107442646	−4.00097957046 −0.99638440732
3	2	−3	$2d_{5/2}$	−0.99632819728 −4.00132955810	−0.99634600310 −4.00124828793	−4.00116354206 −0.99636140695
4	2	−4	$2f_{7/2}$	−0.99628603824 −4.00148399768	−4.00140797604 −0.99630815318	−4.00132955810 −0.99632819728
1	1	2	$0d_{3/2}$	−0.99651749280 −4.00084675171	−0.99650733888 −4.00094153754	−0.99649448980 −4.00103061187

Table 1. (continued)

2	1	3	$0f_{5/2}$	−4.00103061187 −0.99649448980	−0.99647908416 −4.00111533113	−0.99646127635 −4.00119658319
3	1	4	$0g_{7/2}$	−0.99646127635 −4.00119658319	−0.99644122998 −4.00127498841	−4.00135100100 −0.99641911259
4	1	5	$0h_{9/2}$	−0.99641911259 −4.00135100100	−4.00142496554 −0.99639509068	−4.00149715088 −0.99636932608
1	2	2	$1d_{3/2}$	−0.99638440732 −4.00097957046	−4.00107442646 −0.99637425458	−0.99636140695 −4.00116354206
2	2	3	$1f_{5/2}$	−4.00116354206 −0.99636140695	−4.00124828793 −0.99634600310	−4.00132955810 −0.99632819728
3	2	4	$1g_{7/2}$	−4.00132955810 −0.99632819728	−4.00140797604 −0.99630815318	−0.99628603824 −4.00148399768
4	2	5	$1h_{9/2}$	−4.00148399768 −0.99628603824	−0.99626201898 −4.00155796870	−0.99623625721 −4.00163015861

Table 2. The bound state energy eigenvalues (fm^{-1}) of the spin symmetry tPT potential for various values of n and κ with $M = 1.0\ \mathrm{fm}^{-1}$, $C_\mathrm{s} = 5.0\ \mathrm{fm}^{-1}$, $V_1 = -0.002\ \mathrm{fm}^{-1}$, $V_2 = 0.003\ \mathrm{fm}^{-1}$, and $\alpha = 0.01$.

$\tilde{\ell}$	n	κ	(ℓ, j)	$E_{n\kappa}$ (fm^{-1}) ($A = 0$)	$E_{n\kappa}$ (fm^{-1}) ($A = 0.5$)	$E_{n\kappa}$ (fm^{-1}) ($A = 1$)
1	1	-1	$1s_{1/2}$	1.00128001185	1.00112781103	1.00096022670
2	1	-2	$1p_{3/2}$	1.00155412543	1.00142116103	1.00128001185
3	1	-3	$1d_{5/2}$	1.00180227570	1.00168074734	1.00155412543
4	1	-4	$1f_{7/2}$	1.00203337671	1.00191959944	1.00180227570
1	2	-1	$2s_{1/2}$	1.00148070407	1.00132886223	1.00116221307
2	2	-2	$2p_{3/2}$	1.00175447962	1.00162165120	1.00148070407
3	2	-3	$2d_{5/2}$	1.00200244763	1.00188100010	1.00175447962
4	2	-4	$2f_{7/2}$	1.00223342319	1.00211970399	1.00200244763
1	1	2	$0d_{3/2}$	1.00128001185	1.00142116103	1.00155412543
2	1	3	$0f_{5/2}$	1.00155412543	1.00168074734	1.00180227570
3	1	4	$0g_{7/2}$	1.00180227570	1.00191959944	1.00203337671
4	1	5	$0h_{9/2}$	1.00203337671	1.00214411048	1.00225219519
1	2	2	$1d_{3/2}$	1.00148070407	1.00162165120	1.00175447962
2	2	3	$1f_{5/2}$	1.00175447962	1.00188100010	1.00200244763
3	2	4	$1g_{7/2}$	1.00200244763	1.00211970399	1.00223342319
4	2	5	$1h_{9/2}$	1.00223342319	1.00234410569	1.00245214426

Table 3. The bound state energy eigenvalues (fm^{-1}) of the spin symmetry tPT potential for various values of n and κ with $M = 1.0\ \mathrm{fm}^{-1}$, $C_\mathrm{s} = 5.0\ \mathrm{fm}^{-1}$, $V_1 = 0.002\ \mathrm{fm}^{-1}$, $V_2 = -0.003\ \mathrm{fm}^{-1}$, and $\alpha = 0.01$.

$\tilde{\ell}$	n	κ	(ℓ, j)	$E_{n\kappa}$ (fm^{-1}) ($A = 0$)	$E_{n\kappa}$ (fm^{-1}) ($A = 0.5$)	$E_{n\kappa}$ (fm^{-1}) ($A = 1$)
0	0	−1	$0s_{1/2}$	0.99665059004 4.00071392089	4.00061127157 0.99665793312	0.99666237586 4.00049613269
1	0	−1	$1s_{1/2}$	4.00084675171 0.99651749280	4.00074396128 0.99652483503	4.00062839464 0.99652927726
2	0	−1	$2s_{1/2}$	4.00097957046 0.99638440732	4.00087663841 0.99639174872	0.99639619042 4.00076063954
3	0	−1	$3s_{1/2}$	4.00111237713 0.99625133365	0.99625867420 4.00100930296	4.00089286732 0.99626311539
0	1	−2	$0p_{3/2}$	4.00089766980 0.99662758439	4.00080863669 0.99664043494	4.00071392089 0.99665059004
1	1	−2	$1p_{3/2}$	0.99649448980 4.00103061187	0.99650733888 4.00094153754	0.99651749280 4.00084675171
2	1	−2	$2p_{3/2}$	0.99636140695 4.00116354206	0.99637425458 4.00107442646	4.00097957046 0.99638440732
3	1	−2	$3p_{3/2}$	4.00129646038 0.99622833594	0.99624118210 4.00120730344	4.00111237713 0.99625133365
0	2	−3	$0d_{5/2}$	4.00106359645 0.99659436718	4.00098236248 0.99661217702	4.00089766980 0.99662758439
1	2	−3	$1d_{5/2}$	0.99646127635 4.00119658319	0.99647908416 4.00111533113	0.99649448980 4.00103061187
2	2	−3	$2d_{5/2}$	0.99632819728 4.00132955810	4.00124828793 0.99634600310	4.00116354206 0.99636140695
3	2	−3	$3d_{5/2}$	4.00146252118 0.99619513000	0.99621293382 4.00138123289	4.00129646038 0.99622833594

Table 3. (continued)

0	3	−4	$0f_{7/2}$	4.00121799251 0.99655219872	0.99657431858 4.00114198898	0.99659436718 4.00106359645
1	3	−4	$1f_{7/2}$	0.99641911259 4.00135100100	4.00127498841 0.99644122998	4.00119658319 0.99646127635
2	3	−4	$2f_{7/2}$	4.00148399768 0.99628603824	0.99630815318 4.00140797604	4.00132955810 0.99632819728
3	3	−4	$3f_{7/2}$	4.00161698256 0.99615297566	4.00154095184 0.99617508814	0.99619513000 4.00146252118
0	1	1	$0p_{1/2}$	0.99666237586 4.00049613269	0.99665793312 4.00061127157	4.00071392089 0.99665059004
1	1	1	$1p_{1/2}$	4.00062839464 0.99652927726	4.00074396128 0.99652483503	4.00084675171 0.99651749280
2	1	1	$2p_{1/2}$	4.00076063954 0.99639619042	0.99639174872 4.00087663841	4.00097957046 0.99638440732
3	1	1	$3p_{1/2}$	0.99626311539 4.00089286732	4.00100930296 0.99625867420	0.99625133365 4.00111237713
0	2	2	$0d_{3/2}$	4.00071392089 0.99665059004	4.00080863669 0.99664043494	4.00089766980 0.99662758439
1	2	2	$1d_{3/2}$	4.00084675171 0.99651749280	4.00094153754 0.99650733888	0.99649448980 4.00103061187
2	2	2	$2d_{3/2}$	4.00097957046 0.99638440732	0.99637425458 4.00107442646	0.99636140695 4.00116354206
3	2	2	$3d_{3/2}$	4.00111237713 0.99625133365	4.00120730344 0.99624118210	4.00129646038 0.99622833594
0	3	3	$0f_{5/2}$	4.00089766980 0.99662758439	0.99661217702 4.00098236248	4.00106359645 0.99659436718
1	3	3	$1f_{5/2}$	0.99649448980 4.00103061187	4.00111533113 0.99647908416	4.00119658319 0.99646127635
2	3	3	$2f_{5/2}$	0.99636140695 4.00116354206	0.99634600310 4.00124828793	0.99632819728 4.00132955810
3	3	3	$3f_{5/2}$	4.00129646038 0.99622833594	0.99621293382 4.00138123289	4.00146252118 0.99619513000

Table 4. The bound state energy eigenvalues (fm^{-1}) of the spin symmetry tPT potential for various values of n and κ with $M = 1.0\ \mathrm{fm}^{-1}$, $C_s = 0\ \mathrm{fm}^{-1}$, $V_1 = 0.002\ \mathrm{fm}^{-1}$, $V_2 = -0.003\ \mathrm{fm}^{-1}$, and $\alpha = 0.01$.

$\tilde{\ell}$	n	κ	(ℓ, j)	$E_{n\kappa}$ (fm^{-1}) ($A=0$)	$E_{n\kappa}$ (fm^{-1}) ($A=0$)	$E_{n\kappa}$ (fm^{-1}) ($A=1$)
0	0	−1	$0s_{1/2}$	−1.00107927801	−1.00092671625	−1.00075818292
1	0	−1	$1s_{1/2}$	−1.00128001185	−1.00112781103	−1.00096022670
2	0	−1	$2s_{1/2}$	−1.00148070407	−1.00132886223	−1.00116221307
3	0	−1	$3s_{1/2}$	−1.00168135469	−1.00152986990	−1.00136414241
0	1	−2	$0p_{3/2}$	−1.00135373059	−1.00122062989	−1.00107927801
1	1	−2	$1p_{3/2}$	−1.00155412543	−1.00142116103	−1.00128001185
2	1	−2	$2p_{3/2}$	−1.00175447962	−1.00162165120	−1.00148070407
3	1	−2	$3p_{3/2}$	−1.00195479317	−1.00182210043	−1.00168135469
0	2	−3	$0d_{5/2}$	−1.00160206341	−1.00148045411	−1.00135373059
1	2	−3	$1d_{5/2}$	−1.00180227570	−1.00168074734	−1.00155412543
2	2	−3	$2d_{5/2}$	−1.00200244763	−1.00188100010	−1.00175447962
3	2	−3	$3d_{5/2}$	−1.00220257924	−1.00208121240	−1.00195479317
0	3	−4	$0f_{7/2}$	−1.00183329004	−1.00171945463	−1.00160206341
1	3	−4	$1f_{7/2}$	−1.00203337671	−1.00191959944	−1.00180227570
2	3	−4	$2f_{7/2}$	−1.00223342319	−1.00211970399	−1.00200244763
3	3	−4	$3f_{7/2}$	−1.00243342951	−1.00231976829	−1.00220257924
0	1	1	$0p_{1/2}$	−1.00075818292	−1.00092671625	−1.00107927801
1	1	1	$1p_{1/2}$	−1.00096022670	−1.00112781103	−1.00128001185
2	1	1	$2p_{1/2}$	−1.00116221307	−1.00132886223	−1.00148070407
3	1	1	$3p_{1/2}$	−1.00136414241	−1.00152986990	−1.00168135469
0	2	2	$0d_{3/2}$	−1.00107927801	−1.00122062989	−1.00135373059
1	2	2	$1d_{3/2}$	−1.00128001185	−1.00142116103	−1.00155412543
2	2	2	$2d_{3/2}$	−1.00148070407	−1.00162165120	−1.00175447962
3	2	2	$3d_{32}$	−1.00168135469	−1.00182210043	−1.00195479317
0	3	3	$0f_{5/2}$	−1.00135373059	−1.00148045411	−1.00160206341
1	3	3	$1f_{5/2}$	−1.00155412543	−1.00168074734	−1.00180227570
2	3	3	$2f_{5/2}$	−1.00175447962	−1.00188100010	−1.00200244763
3	3	3	$3f_{5/2}$	−1.00195479317	−1.00208121240	−1.00220257924

In Table 5, in the presence of p-spin symmetry with parameter values $C_{ps} = -5.0$ fm^{-1} and $A = 1.0$, the energy levels are negative and subject to restriction $|E| < M$. The system becomes less attractive as quantum numbers increase. In Table 6, in the presence of spin symmetry with parameter values $C_s = 5.0$ fm^{-1} and $A = 1.0$, the energy levels are positive and subject to restriction $|E| < M$ (the energy is less than mass but very near). The system becomes less repulsive with increasing quantum numbers. In Table 7, when $A = 1.0$ and $M = 1.0$ fm^{-1}, the system is weakly attractive when C_{ps} increases. The energy pushes up toward positive energy as C_{ps} becomes less negative. In Table 8, when $A = 1.0$ and $M = 1.0$ fm^{-1} the system is more repulsive when C_s increases. The energy pushes up toward positive energy as C_{ps} becoming more positive.

Table 5. The bound state energy eigenvalues (fm^{-1}) of the p-spin symmetry tPT potential for various values of n and κ with $A = 1.0$, $V_1 = -0.002$ fm^{-1}, $V_2 = 0.003$ fm^{-1}, $C_{ps} = -5.0$ fm^{-1}, and $\alpha = 0.01$ for various M values.

M	$E_{n\kappa}$/fm^{-1}				
	$1s_{1/2}$	$1p_{3/2}$	$1d_{5/2}$	$1f_{7/2}$	$2p_{3/2}$
0.1	−4.90039392640 −0.09732700200	−4.90052995635 −0.09732146762	−4.90064482152 −0.09731056761	−0.09729461539 −4.90074856469	−4.90061310055 −0.09723820785
0.2	−0.19726363906 −4.80041096535	−4.80055294367 −0.19725771395	−4.80067280728 −0.19724605202	−0.19722900196 −4.80078106094	−0.19717083955 −4.80063969332
0.3	−4.70042954473 −0.29719572965	−4.70057801533 −0.29718936662	−0.29717685178 −4.70070333204	−0.29715857470 −4.70081650610	−4.70066869719 −0.29709854959
0.4	−0.39712271208 −4.60044988331	−0.39711585572 −4.60060546823	−0.39710238108 −4.60073675770	−0.39708272579 −4.60085532055	−4.60070045558 −0.39702072145
0.5	−4.50047224329 −0.49704392346	−4.50063565847 −0.49703650814	−0.49702194739 −4.50077351833	−0.49700073559 −4.50089800858	−0.49693662590 −4.50073538040
0.6	−0.59695857526 −4.40049694173	−4.40066901686 −0.59695052254	−4.40081413912 −0.59693472524	−0.59691174485 −4.40094518034	−0.59684539381 −4.40077396991
0.7	−4.30052436563 −0.69686572091	−0.69685693638 −4.30070606962	−4.30085926163 −0.69683972144	−4.30099758120 −0.69681471839	−4.30081683274 −0.69674597988
0.8	−4.20055499237 −0.79676421325	−0.79675458195 −4.20074746629	−4.20090967790 −0.79673572951	−0.79670839608 −4.20105613138	−4.20086472012 −0.79663711411
0.9	−4.10058941764 −0.89665264655	−0.89664202692 −4.10079401812	−4.10096637726 −0.89662126706	−4.10112198039 −0.89659122671	−4.10091857028 −0.89651723662
1.0	−4.00062839464 −0.99652927726	−4.00084675171 −0.99651749280	−4.00103061187 −0.99649448980	−4.00119658319 −0.99646127635	−0.99638440732 −4.00097957046
1.1	−3.90067288966 −1.0963919127	−1.09637874016 −3.90090698555	−1.09635307082 −3.90110399015	−3.90128180875 −1.09631609904	−3.90104924536 −1.09623617950
1.2	−3.80072416324 −1.19623775209	−3.80097644160 −1.19622290459	−1.19619402672 −3.80118861311	−3.80138009857 −1.19615255070	−3.80112958615 −1.19606941778
1.3	−4.70042954473 −0.29719572965	−4.70057801533 −0.29718936662	−0.29717685178 −4.70070333204	−4.70081650610 −0.29715857470	−4.70066869719 −0.29709854959
1.4	−1.39586328048 −3.60085435156	−3.60115301847 −1.39584382808	−3.6014037953 −1.39580617694	−3.60163005154 −1.39575247698	−3.60133382545 −1.39566256488
1.5	−1.49563155612 −3.50093871986	−3.50126761921 −1.49560885888	−1.49556506384 −3.50154348618	−1.49550287629 −3.50179232744	−1.49540957039 −3.50146637118
1.6	−3.40104156115 −1.59535877115	−1.59533185124 −3.40140749459	−1.59528010388 −3.40171402081	−3.40199044576 −1.59520700999	−1.59511056581 −3.40162813925
1.7	−3.30116968106 −1.69503157659	−3.30158203270 −1.69499899825	−1.69493666456 −3.30192686849	−1.69484917754 −3.30223773705	−1.69475027299 −3.30182997821
1.8	−3.20133369067 −1.79462977645	−3.20180592116 −1.79458933368	−1.79451240641 −3.20219997935	−3.20255506406 −1.79440528715	−3.20208885552 −1.79430542808
1.9	−3.10155109245 −1.89412112087	−1.89406922289 −3.10210349324	−3.10256310344 −1.89397126174	−1.89383621716 −3.10297700153	−1.89373860058 −3.10243286936
2.0	−1.9934503596 −3.00185293961	−3.00251817685 −1.99338070398	−3.00306935812 −1.99325059061	−3.00356526814 −1.99307357655	−3.00291213384 −1.99298508396

Table 6. The bound state energy eigenvalues (fm^{-1}) of the spin symmetry tPT potential for various values of n and κ with $A = 1.0$, $V_1 = 0.002$ fm^{-1}, $V_2 = -0.003$ fm^{-1}, $C_s = 5.0$ fm^{-1}, and $\alpha = 0.01$ for various M values.

M	$E_{n\kappa}$/fm^{-1}				
	$1s_{1/2}$	$1p_{3/2}$	$0p_{1/2}$	$2f_{7/2}$	$1f_{5/2}$
0.1	4.90031100228 0.09741026489	0.09740473026 4.90044680919	0.09722730832 4.90072801156	0.09737787681 4.90066535215	4.90069624178 0.09715495100
0.2	0.19735051704 4.80032445514	0.19734459162 4.80046619067	4.80075960665 0.19715917820	0.19731587830 4.80069423709	0.19708396838 4.80072643960

Table 6. (continued)

0.3	4.70033912433	4.70048732963	4.70079406804	0.29724939385	4.70075937520
	0.29728655078	0.29728018736	0.29708603541	4.70072574339	0.29700773632
0.4	0.39721785105	4.60051047646	0.39700724759	4.60076024477	0.39692559145
	4.60035518260	0.39721099430	4.60083180422	0.39717786242	4.60079543853
0.5	4.50037283704	0.49713639534	4.50087330516	4.50079818954	0.49683674861
	0.49714381118	4.50053593143	0.49692206612	0.49710062052	4.50083509722
0.6	4.40039233803	4.40056405785	0.59682959762	0.59701687660	4.40087891700
	0.59706371036	0.59705565706	4.40091916370	4.40084012014	0.59674027090
0.7	0.69697668496	0.69696789975	4.30097010397	0.69692567839	0.69663503020
	4.30041399122	4.30059529950	0.69672876634	4.30088669926	4.30092758886
0.8	0.79688169001	0.79687205789	0.79661826339	4.20093874505	0.79651965441
	4.20043817364	4.20063020415	4.20102701994	0.79682586789	4.20098196565
0.9	4.10046535582	4.10066945600	0.89649647884	4.10099727956	4.10104311249
	0.89677744766	0.89676682692	4.10109102824	0.89671602164	0.89639245604
1.0	4.00049613269	4.00071392089	4.00116354206	0.99659436718	4.00111237713
	0.99666237586	0.99665059004	0.99636140695	4.00106359645	0.99625133365
1.1	1.09653448962	3.90076471090	3.90124637660	1.09645866601	1.09609363330
	3.90053126759	1.09652131538	1.09621051352	3.90113935763	3.90119149033
1.2	1.19639125924	3.80082327853	3.80134190280	3.80122673419	3.80128271220
	3.80057175638	1.19637640944	1.19604054434	1.19630604489	1.19591594906
1.3	3.70061892325	3.70089155767	3.70145327762	3.70132861751	1.29571382942
	1.29622940344	1.29621250199	1.29584724308	1.29613274660	3.70138905128
1.4	3.60067456805	1.39602512106	3.60158479885	3.60144894482	3.60151460198
	1.39604457766	3.60097218101	1.39562492178	1.39593375071	1.39548133145
1.5	1.49583089015	1.49580818698	1.49536578661	3.50159321853	3.50166508269
	3.50074120092	3.50106882676	3.50174247115	1.49570217736	1.49521032146
1.6	1.59558011948	3.40118679454	3.40193494582	1.59542831045	1.59488933452
	3.40082243066	1.59555319089	1.59505883485	3.40176936913	3.40184872855
1.7	1.69528039211	1.69524780054	1.69468796424	3.30198925857	1.69450162464
	3.30092363917	3.30133400857	3.30217515729	1.69509792051	3.30207784517
1.8	3.20105322137	1.79487335380	3.20248334274	1.79468921256	1.79402163686
	1.79491381806	3.20152286996	1.79422854094	3.20227145667	3.20237167316
1.9	3.10122503324	3.10177393271	3.10289302854	1.89416685828	3.10276206133
	1.89445196156	1.89440002611	1.89364070856	3.10264675123	1.89340815892
2.0	3.00146368278	1.99377663414	1.99285510169	3.00317013653	3.00330577578
	1.99384636185	3.00212390413	3.00346403560	1.99346920195	1.99258977338

Table 7. The bound state energy eigenvalues (fm^{-1}) of the p-spin symmetry tPT potential for various values of n and κ with $A = 1.0$, $V_1 = -0.002\ \mathrm{fm}^{-1}$, $V_2 = 0.003\ \mathrm{fm}^{-1}$, $M = 1.0\ \mathrm{fm}^{-1}$, and $\alpha = 0.01$ for various C_{ps} values.

C_{ps}	$E_{n\kappa}/\mathrm{fm}^{-1}$				
	$1s_{1/2}$	$1p_{3/2}$	$1d_{5/2}$	$1f_{7/2}$	$2p_{3/2}$
−40	−0.99921219950	−0.99921205140	−49.0000729061	−49.0000749459	−0.99920239080
	−49.0000395646	−49.0000530984	−0.99920342210	−0.99921131220	−49.0000929333
−35	−44.0000441627	−44.0000592712	−0.99915625190	−0.99916502910	−44.0001037383
	−0.99916608040	−0.99916590460	−44.0000813822	−44.0000836594	−0.99915503010
	−0.99911091390	−0.99911070110	−0.99909975020	−0.99910964120	−0.99909827220
	−39.0000499701	−39.0000670680	−39.0000920884	−39.0000946656	−39.0001173863
−30	−34.0000575361	−0.99904305190	−0.99903040260	−0.99904173360	−0.99902856520
	−0.99904331680	−34.0000772268	−34.0001060382	−34.0001090063	−34.0001351694
−25	−29.0000678020	−0.99895748920	−29.0001249687	−0.99895578880	−0.99894015410
	−0.99895783120	−29.0000910124	−0.99894252240	−29.0001284677	−29.0001593024
−20	−0.99884494550	−24.0001107891	−24.0001521274	−0.99884217290	−24.0001939257
	−24.0000825269	−0.99884448100	−0.99882616480	−24.0001563884	−0.99882295410
−15	−19.0001054219	−19.0001415468	−0.99866193120	−0.99868212840	−19.0002477784
	−0.99868619020	−0.99868550860	−19.0001943680	−19.0001998154	−0.99865723720
	−0.99843932820	−14.0001959457	−0.99840516810	−14.0002766319	−0.99839738960
	−14.0001458971	−0.99843819160	−14.0002690827	−0.99843257320	−14.0003430387
−10	−9.00023681890	−0.99797366094	−0.99791882675	−0.99796163488	−0.99790228590
	−0.99797611064	−9.00031825471	−9.00043709995	−9.00044939125	−9.00055728650
−5	−0.99652927726	−0.99651749280	−0.99636140695	−0.99646127635	−4.00148399768
	−4.00062839464	−4.00084675171	−4.00116354206	−4.00119658319	−0.99628603824

Table 8. The bound state energy eigenvalues (fm^{-1}) of the spin symmetry tPT potential for various values of n and κ with $A = 1.0$, $V_1 = -0.002\ \mathrm{fm}^{-1}$, $V_2 = 0.003\ \mathrm{fm}^{-1}$, $M = 1.0\ \mathrm{fm}^{-1}$ and $\alpha = 0.01$ for various C_s values.

C_s	$E_{n\kappa}/\mathrm{fm}^{-1}$				
	$0\mathrm{s}_{1/2}$	$0\mathrm{p}_{3/2}$	$2\mathrm{d}_{5/2}$	$0\mathrm{f}_{7/2}$	$3\mathrm{p}_{1/2}$
5	0.99652927726	4.00084675171	4.00071392089	4.0013295581	4.00132955810
	4.00062839464	0.99651749280	0.99665059004	0.99632819728	0.99632819728
10	9.00023681890	0.99797366094	9.00026831929	9.00049935066	0.99791165490
	0.99797611064	9.00031825471	0.99802364178	0.99791165490	9.00049935066
15	0.99843932820	14.0001959457	0.99846895540	0.99840180990	14.0003073866
	14.0001458971	0.99843819160	14.0001652004	14.0003073866	0.99840180990
20	19.0001054219	0.99868550860	0.99870772840	0.99865990880	0.99865990880
	0.99868619020	19.0001415468	19.0001193369	19.0002220302	19.0002220302
25	24.0000825269	0.99884448100	24.0000934054	0.99882478300	24.0001737753
	0.99884494550	24.0001107891	0.99886187090	24.0001737753	0.99882478300
30	0.99895783120	0.99895748920	29.0000767317	0.99894150390	0.99894150390
	29.0000678020	29.0000910124	0.99897177420	29.0001427505	29.0001427505
35	0.99904331680	0.99904305190	0.99905517250-	0.99902961290	34.0001211254
	34.0000575361	34.0000772268	34.0000651092	34.0001211254	0.99902961290
40	39.0000499701	0.99911070110	0.99912122700	0.99909911520	39.0001051903
	0.99911091390	39.0000670680	39.0000565444	39.0001051903	0.99909911520
45	44.0000441627	0.99916590460	0.99917520690	0.99915572700	0.99915572700
	0.99916608040	44.0000592712	44.0000499710	44.0000929605	44.0000929605
50	0.99921219950	49.0000530984	0.99922038440	49.0000832783	49.0000832783
	49.0000395646	0.99921205140	49.0000447668	0.99920297920	0.99920297920

6. Concluding remarks

In this paper, we have studied the relativistic bound state solutions of a particle influenced by the scalar and vector tPT potentials including a Coulomb-like tensor interaction for any spin–orbit quantum number κ. We introduce a suitable approximation to substitute the spin–orbit centrifugal (p-centrifugal) coupling term to obtain an approximate energy eigenvalue equation and the normalized two components of the radial wave functions. The spin and p-spin symmetric cases were studied for any wave states. Our numerical results were presented in the presence/absence of tensor interaction and using various values of spin/p-spin constant. Further, the non-relativistic limit of our solution, in the absence of tensor coupling $A = 0$, can be found by simply making an appropriate transformation.[47]

Acknowledgment

B. J. Falaye dedicates this work to his father on his 81st birthday anniversary. S. M. Ikhdair acknowledges the partial financial support by the Scientific and Technological Research Council of Turkey.

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