Deformed oscillator algebra for quantum superintegrable systems in two-dimensional Euclidean space and on a complex two-sphere

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In this work, we study superintegrable quantum systems in two-dimensional Euclidean space and on a complex two-sphere with second-order constants of motion. We show that these constants of motion satisfy the deformed oscillator algebra. Then, we easily calculate the energy eigenvalues in an algebraic way by solving of a system of two equations satisfied by its structure function. The results are in agreement to the ones obtained from the solution of the relevant Schrödinger equation.

Keywords: superintegrable systems, constants of motion, deformed oscillator algebra, structure function

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1. Introduction

It is known from classical and quantum mechanics that a system with N degrees of freedom is completely integrable, if there are N functionally independent constants of motion.[1]

From the mathematical and physical point of view, these systems play a fundamental role in description of physical systems due to their many interesting properties. If one could obtain more than N constants of motion, the idea of a superintegrable system can be helpful. If there exist $2N$-1 constants of motion, the system is maximally superintegrable or just superintegrable,[2–7] provided that the commutator of operators as constants of motion are zero with the Hamiltonian of the system. Recently the study of superintegrable systems has been considered for different potentials and the corresponding spectra have been calculated by different methods. For example, in Refs. [8]–[10], the spectra of some two-dimensional and three-dimensional superintegrable systems were calculated via the algebraic method using the realization of Lie groups.[11,12]

For a two-dimensional integrable system, it is obvious that there is always one operator like A_1 which commutates with the Hamiltonian of the system. For a superintegrable system, one should define another operator like A_2 which is a linear combination of H and A_1, and it commutates with the Hamiltonian of the system. In other words, for a two-dimensional superintegrable system, there are two constants of the motion (A_1, A_2) in addition of the Hamiltonian. A detailed review of the superintegrable systems has been given in Ref. [13] and some examples of superintegrable systems in three dimensions can be found in Refs. [14]–[18].

For physical systems with degenerated energy levels, the symmetry algebra of the system has to be determined, which is a finite-dimensional Lie algebra containing ladder operators,[19,20] and connect all eigenstates with a given equal energy. The superintegrability properties of classical and quantum physical systems coincide with the existence of a symmetry algebra for which the levels of system can be determined in a purely algebraic method.[21]

The N-dimensional isotropic harmonic oscillator and the Kepler systems are examples of the physical systems, with symmetry algebra.[22,23] Higgs[24] and Leemon[25] have shown that these systems, which move in a space with constant curvature, are still maximally superintegrable, both in classical and quantum mechanics. Furthermore, they have shown that the N-dimensional isotropic harmonic oscillator and the Kepler quantum systems can be described by symmetry algebras isomorphic to su(N) and so($N+1$), respectively. However the identification of a symmetry algebra is not always easy.

On the other hand, the quantum algebras are nonlinear deformations of Lie algebras, when the deformation parameter q is equal to 1.[26–30] It is known that the classical (non-deformed) algebras can be constructed using the harmonic oscillator algebra $\{a, a^\dagger, N\}$ as the basic underlying structure, and for quantum algebras in general, the deformed oscillator $\{\mathbf{1}, \mathscr{A}, \mathscr{A}^\dagger, \mathscr{N}\}$ has to be used and so an appropriate Fock basis can be constructed for each energy eigenvalue.[31–37]

In this paper, we study two-dimensional quantum superintegrable systems that have three functionally independent constants of motion in the Euclidean space[38] with Hamiltonian $H = p_x^2 + p_y^2 + V(x,y)$ and the system with Hamiltonian $H = J_1^2 + J_2^2 + J_3^2 + V(x,y,z)$ on the complex two-sphere where $x^2 + y^2 + z^2 = 1$.[39] These systems are second-order in the momenta and are obtained from their classical counterparts by a symmetrization procedure. We then use the deformed oscillator algebra[31] to calculate the spectrum of the system, which is a purely algebraic method without solv-

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ing the Schrödinger equation, and the results are in agreement with the ones obtained from the solution of the relevant Schrödinger equation.[40–42] Hence in Section 2, we consider two-dimensional quantum superintegrable systems and the constants of motion are introduced. In Section 3, we show that the relevant commutator algebra (non-Abelian algebras) has a structure similar to the deformed oscillator algebra and then a working hypothesis for the quantum superintegrable systems in two dimensions is proposed, which leads to calculation of the energy eigenvalues by purely algebraic means. In Section 4, we use the deformed oscillator algebra to study the Kepler potential and present one example to show how this method works. In Sections 5 and 6, we apply this formalism to study two Hamiltonians on the complex two-sphere and present some examples in applications of our method. The paper will end with a brief conclusion in Section 7.

2. Two-dimensional quantum integrable and superintegrable systems

Each quantum system with a discrete energy spectrum can be considered as a quantum integrable system,[43] which is described by a Hamiltonian H acting on a Hilbert space $\mathscr{H}$. All the operators defined in this system with two degrees of freedom are nonlinear combinations of the basic algebra of generators x, p_x, y, p_y and so the Hamiltonian is considered as

$$H = H(x, y, p_x, p_y). \tag{1}$$

We also consider that the generators of algebra ($\hbar = 1$) satisfy the usual commutation relations

$$[x, p_x] = [y, p_y] = \mathrm{i}, \tag{2}$$

other commutators equal zero. For an integrable system, there is always an operator A_1 commuting with the Hamiltonian

$$[H, A_1] = 0, \tag{3}$$

where the operators H and A_1 are linearly independent. Now, if we find a third operator A_2, linearly independent of H and A_1 which commute with H but not commuting with A_1

$$[H, A_2] = 0, \quad [A_1, A_2] \neq 0, \tag{4}$$

then we will say that the system is superintegrable, and these operators are as constants of motion.

According to Ref. [44], the Hamiltonian for all quantum superintegrable systems in two-dimensional complex Euclidean space are classified as $H = p_x^2 + p_y^2 + V(x, y)$ in which its constants of motion are quadratic in momenta. It is also shown that there are at least two constants of motion for different potentials in two dimensions

$$\begin{aligned} A_j \;=\; & a_j(x,y)p_x^2 + b_j(x,y)p_y^2 + c_j(x,y)p_x p_y \\ & + d_j(x,y), \quad j = 1, 2, \end{aligned} \tag{5}$$

and so, we have $2N - 1 = 3$ constants of motion $H = A_0, A_1, A_2$, i.e., the system will be maximally superintegrable.

3. Deformed oscillator algebra

For each quantum superintegrable system there exists a quantum algebra, and according to Ref. [44] by using constants of motion, one can make a quantum algebra compatible with deformed oscillator algebra hypothesis[31,45] as follows:

$$\begin{aligned} & \mathscr{N} = \mathscr{N}(H, A_1, A_2), \;\; \mathscr{N}^\dagger = \mathscr{N}, \\ & \mathscr{A} = \mathscr{A}(H, A_1, A_2), \;\; [\mathscr{N}, \mathscr{A}] = -\mathscr{A}, \;\; [\mathscr{N}, \mathscr{A}^\dagger] = \mathscr{A}^\dagger, \\ & [\mathscr{A}^\dagger \mathscr{A}, \mathscr{A}\mathscr{A}^\dagger] = 0, \;\; \mathscr{A}^\dagger \mathscr{A} = \Phi(H, \mathscr{N}), \end{aligned} \tag{6}$$

where $\Phi(E, n)$, named as structure function, is a real positive-definite function and satisfies the following conditions

$$\begin{aligned} & \Phi(E, n) = 0, \quad \text{for} \quad n = 0 \quad \text{and} \quad n = N + 1, \\ & \Phi(E, n) > 0, \;\; \text{for} \quad n = 1, 2, \ldots \end{aligned} \tag{7}$$

If this construction is possible, we can then define the Fock space representation for each energy eigenvalue as

$$\begin{aligned} & H|E, n\rangle = E|E, n\rangle, \quad \mathscr{N}|E, n\rangle = n|E, n\rangle, \quad \mathscr{A}|E, 0\rangle = 0, \\ & |E, n\rangle = \left(\frac{1}{\sqrt{[n]!}}\right)(\mathscr{A}^\dagger)^{\mathrm{n}}|E, 0\rangle, \end{aligned} \tag{8}$$

where the dimensionality of the corresponding Fock space should be equal to $N_d + 1$ which is equivalent to the condition

$$\Phi(E, N_d + 1) = 0. \tag{9}$$

We shall see in some examples that the conditions of Eqs. (7) and (9), as well as the positiveness of the structure function $\Phi(E, n)$ are sufficient to determine the energy spectra of the quantum maximally superintegrable systems. Indeed, if the relations (6) and (7) are satisfied then the system possesses the structure of the deformed oscillator algebra, and we could therefore calculate the spectra of the system from the corresponding creation and annihilation operators.

In the following section, we apply the idea to a quantum superintegrable two-dimensional system using purely algebraic methods.

4. Kepler problem in two-dimensional space with constant curvature

In Ref. [46], using the stackel transform of a superintegrable system on a constant curvature space, the three-dimensional 4-parameter extended Kepler–Coulomb system has been studied. This transformation was also applied to the geodesic motion on Euclidean space of the Kepler potential.[47] Now, in this section as an example, we use the deformed oscillator algebra and study the Kepler potential with superintegrable potential $V = \alpha/(\sqrt{x^2 + y^2})$, its Hamiltonian in two-dimensional space is

$$H = p_x^2 + p_y^2 + \frac{\alpha}{\sqrt{x^2 + y^2}}. \tag{10}$$

In Ref. [44] for case $E18$, the constants of motion have been taken for two-dimensional classical Kepler potential. Now, in

our quantum system, we use those operators by a symmetrization procedure, hence, we have

$$A_1 = \frac{1}{2}(Mp_x + p_xM) - \frac{\alpha}{2}\frac{y}{\sqrt{x^2+y^2}},$$
$$A_2 = \frac{1}{2}(Mp_y + p_yM) + \frac{\alpha}{2}\frac{x}{\sqrt{x^2+y^2}},$$
$$\chi = M = xp_y - yp_x, \quad A_0 = H, \tag{11}$$

where

$$[A_1, A_2] = -\mathrm{i}\chi A_0, \quad [\chi, A_1] = \mathrm{i}A_2, \quad [\chi, A_2] = -\mathrm{i}A_1. \tag{12}$$

After some calculations, it is seen that the above constants of motion satisfy the following equation as the Casimir relation

$$A_1^2 + A_2^2 - \chi^2 H - \frac{\alpha^2}{4} - \frac{H}{4} = 0, \tag{13}$$

which is used in calculation of the structure function. Using the mentioned hypothesis, we define the deformed oscillator algebra for our potential by the following operators

$$\mathscr{N} = \chi + u1, \quad \mathscr{A}^\dagger = -A_2 + \mathrm{i}A_1, \quad \mathscr{A} = -A_2 - \mathrm{i}A_1. \tag{14}$$

Using the Casimir relation (13) into the structure function, we obtain:

$$\Phi(H, \mathscr{N}) = \mathscr{A}^\dagger \mathscr{A} = (\mathscr{N} - u)^2 H - (\mathscr{N} - u)H + \frac{\alpha^2}{4} + \frac{H}{4},$$
$$\Phi(H, \mathscr{N}+1) = \mathscr{A}\mathscr{A}^\dagger = (\mathscr{N} - u + 1)^2 H - (\mathscr{N} - u + 1)H + \frac{\alpha^2}{4} + \frac{H}{4}. \tag{15}$$

It is also obvious that by using of the Fock space representation, the structure function satisfies the following relations

$$\Phi(E, n) = (n-u)^2 E - (n-u)E + \frac{\alpha^2}{4} + \frac{E}{4}. \tag{16}$$

According to conditions (7), the structure function should be zero for $n = 0$ and $n = N+1$ which simply gives

$$u = \frac{N}{2}, \quad E = \frac{-\alpha^2}{(N+1)^2}, \tag{17}$$

which is in exact agreement with explicit solution of the Schrödinger equation.[48]

In order to obtain the general wave function from Eqs. (8), we first solve the equation $\mathscr{A}|E,0\rangle = 0$. For simplicity, we use the polar coordinate variables and get the following second-order deferential equation

$$\mathscr{A}|E,0\rangle = \frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}\left\{\frac{2}{r}\frac{\partial^2}{\partial\theta^2} + 2\mathrm{i}\frac{\partial}{\partial r}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial r} - \frac{\mathrm{i}}{r}\frac{\partial}{\partial\theta} - \alpha\right\}|E,0\rangle = 0. \tag{18}$$

We consider the angular part of the ground state $|E,0\rangle$ as $\mathrm{e}^{\mathrm{i}m\theta}$ and by solving the first-order differential equation for the radial part, we obtain the ground state wave function as

$$|E,0\rangle = \left(\frac{1}{r}\right)^m \exp\left(\frac{-\alpha r}{2m-1}\right)\exp(\mathrm{i}m\theta). \tag{19}$$

It is easily to see that the energy eigenvalue equation is satisfied for $m = 1$ and so we get

$$|E,0\rangle = \frac{1}{r}\exp(-\alpha r)\exp(\mathrm{i}\theta). \tag{20}$$

In order to obtain the excited states, we apply the creation operator $\mathscr{A}^\dagger$ over Eq. (19). The first excited state is calculated as follows:

$$\begin{aligned}|E,1\rangle &= \mathscr{A}^\dagger|E,0\rangle \\ &= \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}\left\{\frac{2}{r}\frac{\partial^2}{\partial\theta^2} - 2\mathrm{i}\frac{\partial}{\partial r}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial r} + \frac{\mathrm{i}}{r}\frac{\partial}{\partial\theta} - \alpha\right\}|E,0\rangle \\ &= \frac{1}{2}\left(\frac{1}{r}\right)^m \exp\left(\frac{-\alpha r}{2m-1}\right) \\ &\quad \times \exp[\mathrm{i}(m+1)\theta]\left[-\frac{4m\alpha}{2m-1} - \frac{-2m(2m+1)}{r}\right],\end{aligned} \tag{21}$$

where $m = 3/2$. The second excited state is also obtained as:

$$\begin{aligned}|E,2\rangle &= \mathscr{A}^\dagger|E,1\rangle = \mathscr{A}^\dagger\mathscr{A}^\dagger|E,0\rangle \\ &= \frac{1}{4}\left(\frac{1}{r}\right)^m \exp\left(\frac{-\alpha r}{2m-1}\right) \\ &\quad \times \exp[\mathrm{i}(m+2)\theta]\left[\frac{8m\alpha^2(2m+1)}{(2m-1)^2}\right] \\ &\quad + \frac{16m\alpha(m+1)(2m+1)\alpha}{(2m-1)r} \\ &\quad + \frac{4m(m+1)(2m+1)(2m+3)}{r^2},\end{aligned} \tag{22}$$

where $m = 2$. It is necessary to notice that the parameter m must be as $m = N/2+1$ for a given state $|E,N\rangle$. The excited eigenfunction $|E,N\rangle$ is also obtained by applying consecutive of the operator $\mathscr{A}^\dagger$ over ground state as $|E,N\rangle = C(\mathscr{A}^\dagger)^N|E,0\rangle = C\mathscr{A}^\dagger\mathscr{A}^\dagger\cdots\mathscr{A}^\dagger|E,0\rangle$, where after a tedious calculation we get

$$\begin{aligned}|E,N\rangle &= C\exp\left(\frac{-\alpha r}{N+1}\right)\left(\frac{1}{r}\right)^{1+3N/2} \\ &\quad \times \exp\left[\mathrm{i}\left(1+\frac{3N}{2}\right)\theta\right]{}_1\mathrm{F}_1\left(N, 3N+1, \frac{2\alpha r}{N+1}\right),\end{aligned} \tag{23}$$

where C is the normalization constant and ${}_1\mathrm{F}_1(a,b,x)$ is the confluent hypergeometric function.

5. Superintegrability on the complex two-sphere

In Ref. [49], equivalent to superintegrable potentials in complex Euclidean two-space or on the complex two-sphere, a large class of the superintegrable systems in Darboux space have been obtained via Stackel multiplier transformations. Now, in this section, we want to solve the quantum superintegrable potentials on the complex sphere and obtain the energy spectrum of these systems by the previous formalism. In this space, the Hamiltonian for two-sphere is as follows:[44,50,51]

$$H = J_1^2 + J_2^2 + J_3^2 + V(x,y,z), \tag{24}$$

where $x^2+y^2+z^2 = 1$, $J_1 = yp_z - zp_y$, $J_2 = zp_x - xp_z$, and $J_3 = xp_y - yp_x$.

According to Ref. [44], one can consider the following constants of motion:

$$A_1 = J_-^2 + d_1(x,y,z),$$
$$A_2 = AJ_+^2 + BJ_3J_- + CJ_3J_- + DJ_3^2 + d_2(x,y,z),$$
$$A_3 = \chi = J_3, \tag{25}$$

where $J_\pm = J_1 \pm \mathrm{i}J_2$, and d_1 and d_2 are chosen in such that the above operators satisfy a closed algebra. In the follwoing section, we study the superintegrable Hamiltonians on the complex two-sphere for potential $V = \alpha z/\sqrt{x^2+y^2}$ given in Ref. [49] as the S4 when $\alpha = \beta = \delta = 0$ and $V = \alpha/z^2$. We will show that for these potentials, a pair of second-order constants of motion can be obtained which satisfy the deformed oscillator algebra.

6. Applications by examples

Example 1

In the first example, we study the deformed oscillator algebra for the one-parameter potential $V = \alpha z/\sqrt{x^2+y^2}$ on complex two-sphere with the Hamiltonian

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{\alpha z}{\sqrt{x^2+y^2}}. \tag{26}$$

For convenience, we modify the constants of motion (25) and after some calculations, the quantum constants of the motion are obtained as follows:

$$\begin{aligned} A_1 &= \frac{1}{2}(J_1J_3 + J_3J_1) - \frac{\alpha x}{2\sqrt{x^2+y^2}}, \\ A_2 &= \frac{1}{2}(J_2J_3 + J_3J_2) - \frac{\alpha y}{2\sqrt{x^2+y^2}}, \\ A_3 &= \chi = J_3, \quad A_0 = H. \end{aligned} \tag{27}$$

These constants of motion can be also obtained from their corresponding classical integrals by a symmetrization procedure, and it is easy to see that there is a closed algebra between them

$$\begin{aligned} &[\chi, A_1] = \mathrm{i}A_2, \quad [\chi, A_2] = -\mathrm{i}A_1, \\ &[A_1, A_2] = -\mathrm{i}A_0\chi + 2\mathrm{i}\chi^3 + \frac{1}{4}\chi. \end{aligned} \tag{28}$$

Similar to previous section, we can obtain a relation between constants of the motion (Casimir relation) with tedious calculation

$$A_1^2 + A_2^2 = H\chi^2 - \chi^4 + \frac{H}{4} + \frac{\alpha^2}{4} - \frac{5}{4}\chi^2. \tag{29}$$

Hence, for this system, the deformed oscillator algebra has the form

$$\mathscr{N} = \chi - u1, \quad \mathscr{A} = A_2 + \mathrm{i}A_1, \quad \mathscr{A}^\dagger = A_2 - \mathrm{i}A_1, \tag{30}$$

where u is a constant and it must be determined such that

$$\begin{aligned} &[\mathscr{N}, \mathscr{A}^\dagger] = \mathscr{A}^\dagger, \quad [\mathscr{N}, \mathscr{A}] = -\mathscr{A}, \\ &\Phi(H, \mathscr{N}) = \mathscr{A}^\dagger\mathscr{A} = A_2^2 + A_1^2 - \mathrm{i}[A_1, A_2]. \end{aligned} \tag{31}$$

Using of the Casimir relation within the structure function, we obtain

$$\begin{aligned} \Phi(H,\mathscr{N}) &= H(\mathscr{N}+u)^2 - (\mathscr{N}+u)^4 + \frac{H}{4} + \frac{\alpha^2}{4} - \frac{5}{4}(\mathscr{N}+u)^2 \\ &\quad - H(\mathscr{N}+u) + 2(\mathscr{N}+u)^3 + \frac{\mathscr{N}+u}{4}, \\ \Phi(H,\mathscr{N}+1) &= \mathscr{A}\mathscr{A}^\dagger \\ &= H(\mathscr{N}+u+1)^2 - (\mathscr{N}+u+1)^4 + \frac{H}{4} \\ &\quad + \frac{\alpha^2}{4} - \frac{5}{4}(\mathscr{N}+u+1)^2 - H(\mathscr{N}+u+1) \\ &\quad + 2(\mathscr{N}+u+1)^3 + \frac{\mathscr{N}+u+1}{4}, \end{aligned} \tag{32}$$

where, after using the conditions (7), we get

$$\begin{aligned} &\Phi(E,0) = Eu^2 - u^4 + 2u^3 + Eu + \frac{E}{4} + \frac{\alpha^2}{4} - \frac{5}{4}u^2 + \frac{u}{4} = 0, \\ &\Phi(E,N+1) = 0. \end{aligned} \tag{33}$$

With a tedious calculation of the above two equations, we calculate the unknown parameter u and thereby obtain the energy of the system

$$u = \frac{-N}{2}, \quad E = \frac{N(N+2)}{4} - \frac{\alpha^2}{(N+1)^2}. \tag{34}$$

Similar to the previous section, in order to obtain the ground state wave function from Eqs. (8), we first-solve the differential equation $\mathscr{A}|E,0\rangle = 0$. Using the spherical polar coordinates system we get

$$\begin{aligned} \mathscr{A}|E,0\rangle &= \mathrm{i}\,\mathrm{e}^{-\mathrm{i}\varphi}\Big(2\cot\theta\frac{\partial^2}{\partial\varphi^2} + 2\mathrm{i}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi} - \mathrm{i}\cot\theta\frac{\partial}{\partial\varphi} \\ &\quad + \frac{\partial}{\partial\theta} - \alpha\Big)|E,0\rangle = 0. \end{aligned} \tag{35}$$

We consider the azimuth angle part of the ground state as $\mathrm{e}^{\mathrm{i}m\varphi}$ and by solving the first-order differential equation for the polar angle part, we obtain the ground state wave function as

$$|E,0\rangle = \frac{1}{\sin^m\theta}\exp\Big(\frac{-\alpha\theta}{2m-1}\Big)\exp(\mathrm{i}m\varphi). \tag{36}$$

The energy eigenvalue equation leads to $m = 1$ for the ground state wave function, that is

$$|E,0\rangle = \frac{1}{\sin\theta}\exp(-\alpha\theta)\exp(\mathrm{i}\varphi). \tag{37}$$

The excited eigenfunctions can be calculated by applying consecutive operator $\mathscr{A}^\dagger$ over the ground state (36) which can be written in a closed form. We have calculated the first excited state as

$$\begin{aligned} |E,1\rangle &= \mathscr{A}^\dagger|E,0\rangle \\ &= \mathrm{i}\exp[\mathrm{i}(m+1)\varphi]\exp\Big[\frac{-\alpha\theta}{2m-1}\Big] \\ &\quad \times\Big\{\frac{\cos\theta}{\sin^{m+1}\theta}[4m(2m+1)] + \frac{1}{\sin^m\theta}\Big(\frac{4m\alpha}{2m-1}\Big)\Big\}, \end{aligned} \tag{38}$$

where $m = 2$. In this case, the parameter m must be $m = N/2+1$, too.

Example 2

As an another example, we consider the following Hamiltonian on the complex two-sphere:[52,53]

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{\alpha}{z^2}. \tag{39}$$

This potential has been studied in Ref. [52], and the connection between the obtained results with a position-dependent mass Hamiltonian given by Quesne[54] has been pointed out. Now, in this example, we study Eq. (39) by deformed oscillator algebra, and similar to previous example, we modify the constants of motion (25) and after some calculations, we ob-

tain the following quantum constants of motion

$$A_1 = J_1^2 + \frac{\alpha y^2}{z^2}, \quad A_2 = \frac{1}{2}(J_1J_2 + J_2J_1) - \frac{\alpha xy}{z^2},$$
$$A_3 = \chi = J_3, \quad A_0 = H. \tag{40}$$

It is easy to see that these constants of motion can be obtained from their corresponding classical integrals by a symmetrization procedure with the closed algebra

$$[\chi, A_1] = +2\mathrm{i}A_2, \quad [\chi, A_2] = -\mathrm{i}\chi^2 - 2\mathrm{i}A_1 + \mathrm{i}A_0 - \mathrm{i}\alpha,$$
$$[A_1, A_2] = \mathrm{i}(A_1\chi + \chi A_1) + \mathrm{i}\Big(2\alpha + \frac{1}{2}\Big)\chi. \tag{41}$$

The Casimir relation in this case reads:

$$A_1^2 + A_2^2 = -\frac{1}{3}(\chi^2 A_1 + \chi A_1 \chi + A_1\chi^2) - \Big(\alpha + \frac{11}{12}\Big)\chi^2 + \frac{H}{6} - \Big(\alpha - \frac{2}{3}\Big)A_1 + HA_1 + \frac{5\alpha}{6}. \tag{42}$$

Hence, for this system, the deformed oscillator algebra is

$$\mathscr{N} = \frac{\chi}{2} - u\mathbf{1}, \quad \mathscr{A} = A_1 - \frac{A_0}{2} + \frac{\chi^2}{2} + \frac{\alpha}{2} - \mathrm{i}A_2,$$
$$\mathscr{A}^\dagger = A_1 - \frac{A_0}{2} + \frac{\chi^2}{2} + \frac{\alpha}{2} + \mathrm{i}A_2, \tag{43}$$

where the constant term u is determined such that

$$[\mathscr{N}, \mathscr{A}^\dagger] = \mathscr{A}^\dagger, \quad [\mathscr{N}, \mathscr{A}] = -\mathscr{A},$$
$$\Phi(H, \mathscr{N}) = \mathscr{A}^\dagger\mathscr{A} = A_2^2 + A_1^2 - A_1A_0 + \frac{1}{2}(A_1\chi^2 + \chi^2A_1) + A_1\alpha - \mathrm{i}(A_1A_2 - A_2A_1) + \frac{A_0}{4} - \frac{A_0\chi^2}{2} - \frac{A_0\alpha}{2} + \frac{\chi^4}{4} + \frac{\chi^2\alpha}{2} + \frac{1}{2}(A_2\chi^2 - \chi^2A_2) + \frac{\alpha^2}{4}. \tag{44}$$

Using of the Casimir relation witin the structure function, we obtain

$$\begin{aligned}\Phi(H, \mathscr{N}) &= 2\Big(2\alpha + \frac{1}{2}\Big)(\mathscr{N} + u) + \frac{H^2}{4} - 2H(\mathscr{N} - u)^2 \\ &\quad - \frac{1}{2}H\alpha + 4(\mathscr{N} - u)^4 + 2(\mathscr{N} - u)^2\alpha \\ &\quad - 8(\mathscr{N} - u)^3 + 2H(\mathscr{N} - u) - 2\alpha(\mathscr{N} - u) \\ &\quad + \frac{\alpha^2}{4} - 4\Big(\alpha + \frac{11}{12}\Big)(\mathscr{N} - u)^2 + \frac{H}{2} + \frac{5\alpha}{6} \\ &\quad - \frac{4}{3}(\mathscr{N} - u)^2 - \frac{\alpha}{3}, \\ \Phi(H, \mathscr{N} + 1) &= 2\Big(2\alpha + \frac{1}{2}\Big)(\mathscr{N} + u + 1) + \frac{H^2}{4} \\ &\quad - 2H(\mathscr{N} - u + 1)^2 - \frac{1}{2}H\alpha \\ &\quad + 4(\mathscr{N} - u + 1)^4 + 2(\mathscr{N} - u + 1)^2\alpha \\ &\quad - 8(\mathscr{N} - u + 1)^3 + 2H(\mathscr{N} - u) \\ &\quad - 2\alpha(\mathscr{N} - u + 1) + \frac{\alpha^2}{4} \\ &\quad - 4\Big(\alpha + \frac{11}{12}\Big)(\mathscr{N} - u + 1)^2 + \frac{H}{2} \\ &\quad + \frac{5\alpha}{6} - \frac{4}{3}(\mathscr{N} - u + 1)^2 - \frac{\alpha}{3}. \end{aligned} \tag{45}$$

After using the conditions (7), we obtain the unknown parameter u and then we calculate the energy of system

$$u = \frac{+N}{2},$$
$$E = (N-1)^2 + (\alpha - 2) \pm (\sqrt{28N^2 + 4 + 8N - 16\alpha + 16N^2\alpha - 32N\alpha})/2. \tag{46}$$

Similar to previous example, the ground state and exited state can be obtained where we have not calculated them.

7. Conclusion

In this paper, we have studied some quantum potentials in two-dimensional Euclidean space and on the complex two-sphere which have more than one constant of motion. We have shown that these quantum superintegrable systems can be described in terms of a deformed oscillator algebra and so we have calculated the spectra of the system by solving two equations obtained from the structure function. In all of the examples, the quantum superintegrability can be induced by the classical superintegrability and the quantum integrals of motion are the symmetrized versions of the corresponding classical integrals and the hypothesis that to each superintegrable system corresponds a deformed oscillator algebra.

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