The Yukawa potential in semirelativistic formulation via supersymmetry quantum mechanics

S. Hassanabadi[a)], M. Ghominejad[a)], S. Zarrinkamar[b)], and H. Hassanabadi[c)†]

[a)] *Physics Department, Semnan University, Semnan, Iran*
[b)] *Department of Basic Sciences, Garmsar Branch, Islamic Azad University, Garmsar, Iran*
[c)] *Department of Basic Sciences, Shahrood Branch, Islamic Azad University, Shahrood, Iran*

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We apply an approximation to the centrifugal term and solve the two-body spinless-Salpeter equation (SSE) with the Yukawa potential via the supersymmetric quantum mechanics (SUSYQM) for arbitrary quantum numbers. Useful figures and tables are also included.

Keywords: spinless-Salpeter equation, two-body-system, supersymmetry quantum mechanics, Yukawa potential

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1. Introduction

The Bethe–Salpeter equation (BSE),[1–6] despite all its advantages and capability, has a very complicated mathematical structure. This equation of relativistic field theory, after some approximations and neglecting the spin degree of freedom, is normally known as the spinless Salpeter equation (SSE)[7,8] in the literature. The SSE is in fact a generalization of the Schrödinger equation into the relativistic regime. Although the mathematical structure of the SSE is simpler than that of its origin, i.e., the BSE, it cannot be immediately solved using the common techniques of mathematical physics due to its nonlocal nature. In simpler words, and as the forthcoming equation (1) reveals, the kinetic term in the Hamiltonian goes under an inverse square term and this is the place where the difficulty arises from. To get rid of such a term, various scenarios of quantum mechanics including the operator inequalities, variation techniques, and the envelope theory have been brilliantly applied to the equation.[9–12] Another proposition, which we describe here, is based on the expansion of the Hamiltonian. The resulting equation, which is a good approximation for heavy interacting particles, is a Schrödinger-like equation with an effective potential. In our study, bearing in mind the high number of two-body systems, we work on the two-body formulation of the semi-relativistic SSE.

For the choice of the potential, we pick up the Yukawa interaction,[13] which stands as a successful phenomenological interaction in many fields of physics,[14–16] such as atomic, molecular, and chemical physics. The Yukawa potential is alternatively called Debye–Huckel, Thomas–Fermi or screened Coulomb potential. Here, we intend to combine these two interesting topics of quantum mechanics and solve the two-body SSE under the Yukawa potential. We first review the two-body SSE. Then, by considering a Pekeris-type approximation as well as some transformations, we bring the problem into a form, which can be solved by the powerful quantum mechanical idea of supersymmetry quantum mechanics (SUSYQM).

2. Two-body Hamiltonian

The SSE for two particles interacting in a spherically symmetric potential in the center of mass system appears as[17,18]

$$\Big[\sum_{i=1,2}\Big(\sqrt{-\Delta+m_i^2}-m_i\Big)+V(r)-E_{n,l}\Big]\chi(\boldsymbol{r})=0,\ \ \Delta=\nabla^2, \quad (1)$$

where $\chi(\boldsymbol{r})=R_{nl}(r)\mathrm{Y}_{lm}(\theta,\phi)$. In the case of heavy interacting particles, we can write[17,18]

$$\Big[\frac{-1}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{l(l+1)}{2\mu r^2}+W_{nl}(r)-\frac{W_{nl}^2(r)}{2\tilde{m}}\Big]\psi_{nl}(r)=0, \quad (2)$$

where

$$W_{nl}(r)=V(r)-E_{nl}, \quad (3a)$$

$$\mu=\frac{m_1m_2}{m_1+m_2}, \quad (3b)$$

$$\eta=\mu\Big(\frac{m_1m_2}{m_1m_2-3\mu^2}\Big)^{1/3}, \quad (3c)$$

$$\tilde{m}=\eta^3/\mu^2=(m_1m_2\mu)/(m_1m_2-3\mu^2). \quad (3d)$$

Here, we consider the Yukawa potential[13]

$$V(r)=-V_0\frac{\mathrm{e}^{-\alpha r}}{r}, \quad (4)$$

where α is the screening parameter and V_0 is the strength of the potential. Substitution of the latter into Eq. (2), besides the approximations[19–21]

$$\frac{1}{r^2}\approx\frac{4\alpha^2\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2}, \quad (5)$$

†Corresponding author. E-mail: h.hasanabadi@shahroodut.ac.ir

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and

$$\frac{1}{r} \approx \frac{2\alpha \mathrm{e}^{-\alpha r}}{(1-\mathrm{e}^{-2\alpha r})}, \tag{6}$$

yields

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_{\mathrm{eff}}(r)\right]\psi_{n,l}(r) = \tilde{E}_{n,l}\psi_{n,l}(r), \tag{7}$$

where

$$V_{\mathrm{eff}}(r) = A\frac{\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2} + B\frac{\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})}, \tag{8a}$$

$$\tilde{E}_{n,l} = 2\mu E_{n,l} + \frac{\mu E_{n,l}^2}{\tilde{m}}, \tag{8b}$$

and

$$A = 4\alpha^2 l(l+1) - \frac{4\mu V_0^2\alpha^2}{\tilde{m}}, \tag{9a}$$

$$B = -4\mu V_0\alpha - \frac{4\alpha\mu V_0 E_{n,l}}{\tilde{m}} + \frac{4\mu V_0^2\alpha^2}{\tilde{m}}. \tag{9b}$$

In order to obtain the solution of Eq. (7), we use the powerful supersymmetry (SUSY) method. Bearing in mind Eq. (A1) in Appendix A, we search for the Riccati equation

$$\phi^2 - \phi' = V_{\mathrm{eff}} - \tilde{E}_{0,l}, \tag{10}$$

which is

$$\phi(r) = a - \frac{b}{1-\mathrm{e}^{-2\alpha r}}. \tag{11}$$

Substituting Eq. (11) into Eq. (10) and comparing equal powers we find

$$a = \frac{b^2 - B}{2b}, \tag{12a}$$

$$b = \alpha \pm \sqrt{\alpha^2 + A}, \tag{12b}$$

$$\tilde{E}_{0,l} = -(a^2 + B). \tag{12c}$$

Therefore, our partner potentials are

$$V_{\mathrm{eff}+}(r) = \phi^2 + \frac{\mathrm{d}\phi}{\mathrm{d}r} = \left(\frac{b^2-B}{2b}\right)^2 + \frac{b(b+2\alpha)\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2} + \frac{B}{1-\mathrm{e}^{-2\alpha r}}, \tag{13a}$$

$$V_{\mathrm{eff}-}(r) = \phi^2 - \frac{\mathrm{d}\phi}{\mathrm{d}r} = \left(\frac{b^2-B}{2b}\right)^2 + \frac{b(b-2\alpha)\mathrm{e}^{-2\alpha r}}{(1-\mathrm{e}^{-2\alpha r})^2} + \frac{B}{1-\mathrm{e}^{-2\alpha r}}, \tag{13b}$$

which are shape-invariant via a mapping of the $b \to b+2\alpha$. Thus, from Eq. (A2), the residuals are

$$R(a_1) = \left(\frac{a_0^2-B}{2a_0}\right)^2 - \left(\frac{a_1^2-B}{2a_1}\right)^2,$$

$$\cdots$$

$$R(a_n) = \left(\frac{a_{n-1}^2-B}{2a_{n-1}}\right)^2 - \left(\frac{a_n^2-B}{2a_n}\right)^2, \tag{14a}$$

and

$$\tilde{E}_{n,l}^{-} = \sum_{k=1}^{n} R(a_k) = \left(\frac{a_0^2-B}{2a_0}\right)^2 - \left(\frac{a_n^2-B}{2a_n}\right)^2, \tag{14b}$$

where $n = 0,1,2,\ldots$ and

$$a_n = a_0 + 2n\alpha,\ a_0 = b. \tag{15}$$

From Eqs. (12c) and (14), we have

$$\tilde{E}_{n,l} = \tilde{E}_{n,l}^{-} + \tilde{E}_{0,l} = -\left(\frac{a_n^2-B}{2a_n}\right)^2 - B, \tag{16}$$

which, from Eq. (8b), gives the energy relation as

$$2\mu E_{n,l} + \frac{\mu E_{n,l}^2}{\tilde{m}} + \left(\frac{a_n^2-B}{2a_n}\right)^2 + B = 0. \tag{17}$$

Energy spectra of the system for typical potential parameters are reported in Table 1. In Figs. 1 and 2 we plot the curves of $E_{n,l}$ versus α and $E_{n,l}$ versus V_0 for some values of n and l, respectively.

Table 1. Spectra of the system for $\alpha = 0.001$, $V_0 = 0.7$, $m_1 = m_2 = 1$.

$\|n,l\rangle$	$E_{0,l}$	$\|n,l\rangle$	$E_{1,l}$	$\|n,l\rangle$	$E_{2,l}$
$\|0,0\rangle$	−0.1477377549	$\|1,0\rangle$	−0.0339048814	$\|2,0\rangle$	−0.01414840172
$\|0,1\rangle$	−0.03049239863	$\|1,1\rangle$	−0.0131583619	$\|2,1\rangle$	−0.00708744432
$\|0,2\rangle$	−0.01300425360	$\|1,2\rangle$	−0.0070224716	$\|2,2\rangle$	−0.00425514097
$\|0,3\rangle$	−0.00699531871	$\|1,3\rangle$	−0.0042412665	$\|2,3\rangle$	−0.00274985460
$\|0,4\rangle$	−0.00423362231	$\|1,4\rangle$	−0.0027454520	$\|2,4\rangle$	−0.001853922307

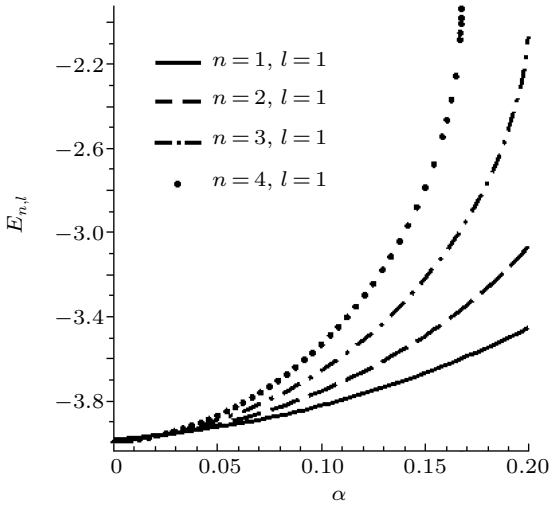


Fig. 1. $E_{n,l}$ versus α for $V_0 = 0.7$, $m_1 = m_2 = 1$.

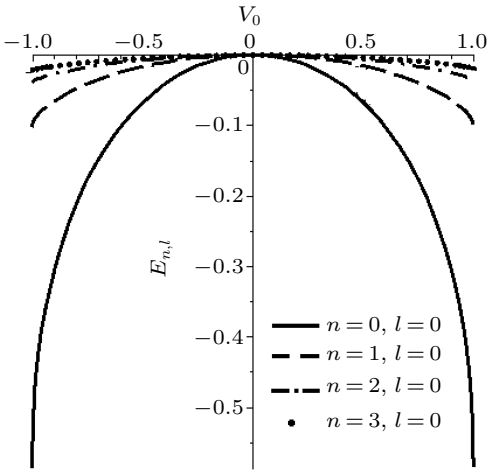


Fig. 2. $E_{n,l}$ versus V_0 for $\alpha = 0.001$, $m_1 = m_2 = 1$.

To obtain the wave function of the system we start from Eq. (7). By a change of variable of the form

$$z = \frac{1}{1-\exp(-2\alpha r)}, \tag{18}$$

we arrive at

$$\Big\{z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + (1-2z)\frac{\mathrm{d}}{\mathrm{d}z} + \frac{A}{4\alpha^2} + \frac{B}{4\alpha^2}\frac{1}{z} + \frac{\tilde{E}_{n,l}}{4\alpha^2}\frac{1}{z(1-z)}\Big\}\psi_{n,l}(z) = 0. \tag{19}$$

Taking the form of the wave function

$$\psi_{n,l}(z) = z^{\lambda_1}(1-z)^{\lambda_2}\mathrm{f}_{n,l}(z), \tag{20}$$

and inserting it into Eq. (19), one obtains a hypergeometric-type equation

$$\Big\{z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + [c' - (1+a'+b')z]\frac{\mathrm{d}}{\mathrm{d}z} - \Big[\lambda_1(\lambda_1+1) + \lambda_2(\lambda_2+1) + 2\lambda_1\lambda_2 - \frac{A}{4\alpha^2}\Big]\Big\}\mathrm{f}_{n,l}(z) = 0, \tag{21}$$

where

$$\begin{aligned} a' &= \lambda_1 + \lambda_2 + \frac{1}{2}\Big[1 + \sqrt{1 + \frac{A}{\alpha^2}}\Big], \\ b' &= \lambda_1 + \lambda_2 + \frac{1}{2}\Big[1 - \sqrt{1 + \frac{A}{\alpha^2}}\Big], \\ c' &= 1 + 2\lambda_1, \end{aligned} \tag{22}$$

with

$$\lambda_1^2 = -\frac{B+\tilde{E}_{n,l}}{4\alpha^2}, \quad \lambda_2^2 = -\frac{\tilde{E}_{n,l}}{4\alpha^2}, \tag{23}$$

and $\mathrm{f}_{n,l}(z)$ is a hypergeometric function

$$\mathrm{f}_{n,l}(z) = {}_2\mathrm{F}_1(a', b', c'; z). \tag{24}$$

From Eq. (20), we obtain the total wave function

$$\begin{aligned} \psi_{n,l}(z) &= z^{\lambda_1}(1-z)^{\lambda_2}\mathrm{f}_{n,l}(z) \\ &= z^{\lambda_1}(1-z)^{\lambda_2}{}_2\mathrm{F}_1(a', b', c'; z), \end{aligned} \tag{25}$$

or equivalently

$$\psi_{n,l}(r) = \Big(\frac{1}{1-\exp(-2\alpha r)}\Big)^{\lambda_1}\Big(1 - \frac{1}{1-\exp(-2\alpha r)}\Big)^{\lambda_2} \times {}_2\mathrm{F}_1\Big(a', b', c'; \frac{1}{1-\exp(-2\alpha r)}\Big). \tag{26}$$

3. Conclusion

In this paper it is shown that choosing a proper Pekeris-type approximation can bring the two-body spinless-Salpeter equation into a form which can be solved by the old but powerful SUSYQM technique without having to deal with the time-consuming and cumbersome numerical methodologies. Due to the semirelativistic nature of the equation, its two-body formulation, and the choice of the Yukawa potential, the results can be used in various fields where two-body systems are investigated.

Appendix A: Supersymmetry quantum mechanics

Within this appendix, we give a quite brief introduction to the SUSYQM. We quote these lines from Refs. [22]–[24]. Our first goal in SUSYQM is to find the solution of the Riccati equation

$$V_\pm = \Phi^2 \pm \Phi', \tag{A1}$$

where V is the potential of the Schrödinger equation and Φ is called the superpotential in the jargon. If

$$V_+(a_0, x) = V_-(a_1, x) + R(a_1), \tag{A2}$$

where a_1 is a new set of parameters uniquely determined from the old set a_0 via the mapping $F : a_0 \mapsto a_1 = F(a_0)$ and the residual term $R(a_1)$ does not include x, the partner potentials are shape-invariant and the necessary information of the system is obtained via[22–24]

$$E_n = \sum_{s=1}^{n} R(a_s), \tag{A3}$$

$$\phi_n^-(a_0, x) = \prod_{s=0}^{n-1}\Big(\frac{A^\dagger(a_s)}{[E_n - E_s]^{1/2}}\Big)\phi_n^-(a_0, x), \tag{A4}$$

$$\phi_n^-(a_0, x) = C\exp\Big\{-\int_0^x \mathrm{d}z\Phi(a_0, z)\Big\}, \tag{A5}$$

$$A_s^\dagger = -\frac{\partial}{\partial x} + \Phi(a_0, x). \tag{A6}$$

In other words, shape invariancy gives the complete and exact information about the spectrum of the bound states of the following Hamiltonians

$$H_s = -\frac{\partial^2}{\partial x^2} + V_-(a_s, x) + E_s. \tag{A7}$$

It should be noted that the energy eigenfunctions

$$H_s\phi_{n-s}^-(a_s, x) = E_n\phi_{n-s}^-(a_s, x), \ \ n \geq s \tag{A8}$$

of this family of Hamiltonians are related to each other by[3,4]

$$\phi_{n-s}^-(a_s, x) = \frac{A^\dagger}{[E_n - E_s]^{1/2}}\phi_{n-(s+1)}^-(a_{s+1}, x). \tag{A9}$$

As a final point, it should be mentioned that at level of our work, supersymmetry is in fact the factorization of Infeld and Hall[22–24] which was introduced many years ago and was based on the factorization of a linear second-order differential equation to two first-order ones.

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