New decomposition of the Fresnel operator corresponding to the optical transformation in $ABCD$-systems*

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By virtue of the coherent state representation of the newly introduced Fresnel operator and its group product property, we obtain new decomposition of the Fresnel operator as the product of the quadratic phase operator, the squeezing operator, and the fractional Fourier transformation operator, which in turn sheds light on the matrix optics design of *ABCD*-systems. The new decomposition for the two-mode Fresnel operator is also obtained by the use of entangled state representation.

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1. Introduction

In Ref. [1], by using the coherent state representation[2,3] $|z\rangle = \exp\{-|z|^2/2 + za^\dagger\}|0\rangle$ and the technique of integration within an ordered product (IWOP) of operators we have proposed a so-called Fresnel operator $U_1(r,s)$ in quantum optics, which is given as[4]

$$\begin{aligned} U_1(r,s) &= \sqrt{s}\int \frac{\mathrm{d}^2 z}{\pi}\,|sz - rz^*\rangle\langle z| \\ &= \exp\left(-\frac{r}{2s^*}a^{\dagger 2}\right)\exp\left[\left(a^\dagger a + \frac{1}{2}\right)\ln\frac{1}{s^*}\right] \\ &\quad\times \exp\left(\frac{r^*}{2s^*}a^2\right). \end{aligned} \tag{1}$$

This corresponds to optical Fresnel transformation characteristic of ray transfer matrix elements (A,B,C,D), $AD-BC=1$, connecting the input light field $f(x)$ and output light field $g(x)$ by the Fresnel integration[5–7]

$$\begin{aligned} g(x') &= \frac{1}{\sqrt{2\pi \mathrm{i}B}}\int_{-\infty}^{\infty} \\ &\quad\times \exp\left[\frac{\mathrm{i}}{2B}\left(Ax^2 - 2x'x + Dx'^2\right)\right] f(x)\,\mathrm{d}x. \end{aligned} \tag{2}$$

The r and s in Eq. (1) are related to $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ transfer matrix by

$$\begin{aligned} s &= \frac{1}{2}\left[A + D - \mathrm{i}\left(B - C\right)\right], \\ r &= -\frac{1}{2}\left[A - D + \mathrm{i}\left(B + C\right)\right], \end{aligned} \tag{3}$$

and the unimodularity condition $AD-BC=1$ is equivalent to $|s|^2 - |r|^2 = 1$. In addition, we have proved that the matrix element of the Fresnel operator in the coordinate $|x\rangle$ representation just coincides with the Fresnel integration kernel[8,9]

$$\begin{aligned} &\langle x'|U_1(r,s)|x\rangle \\ &= \frac{1}{\sqrt{2\pi \mathrm{i}B}}\exp\left[\frac{\mathrm{i}}{2B}\left(Ax^2 - 2x'x + Dx'^2\right)\right]. \end{aligned} \tag{4}$$

Moreover, using the coherent state overlap

$$\langle z|z'\rangle = \exp\left[-\frac{1}{2}\left(|z|^2 + |z'|^2\right) + z^*z'\right]$$

and the IWOP technique we can obtain the multiplication rule of the Fresnel operator, that is,

$$\begin{aligned} &U_1(r,s)\,U_1(r',s') \\ &= \sqrt{ss'}\int \frac{\mathrm{d}^2 z\mathrm{d}^2 z'}{\pi^2}\,|sz - rz^*\rangle\langle z|\, s'z' - r'z'^*\rangle\langle z'| \\ &= U(r'',s''), \end{aligned} \tag{5}$$

where

$$\begin{aligned} &\begin{pmatrix} s & -r \\ -r^* & s^* \end{pmatrix}\begin{pmatrix} s' & -r' \\ -r'^* & s'^* \end{pmatrix} = \begin{pmatrix} s'' & -r'' \\ -r''^* & s''^* \end{pmatrix}, \\ &|s''|^2 - |r''|^2 = 1, \end{aligned} \tag{6}$$

or $s'' = ss' + rr'^*$, $r'' = rs'^* + r's$. Equation (5) indicates that the product of two Fresnel operator is still a Fresnel operator, conversely, a Fresnel operator $U_1(r'',s'')$ can be decomposed as $U_1(r,s)\,U_1(r',s')$ abided by condition (6).

Alternately, make the identity $z = (x + \mathrm{i}p)/\sqrt{2}$ and recast $|z\rangle$ in the canonical coherent state form

$$|z\rangle = \left|\begin{pmatrix} x \\ p \end{pmatrix}\right\rangle = \exp\left[\mathrm{i}\left(pX - xP\right)|0\rangle\right],$$

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$$X=\frac{a+a^{\dagger}}{\sqrt{2}},\ \ P=\frac{a-a^{\dagger}}{\sqrt{2}\mathrm{i}}, \tag{7}$$

then we will be able to express

$$\begin{aligned}
&U_1(r,s)\\
&=F(A,B,C)\\
&=\frac{\sqrt{A+D-\mathrm{i}(B-C)}}{\sqrt{2}}\\
&\quad\times\int\frac{\mathrm{d}x\mathrm{d}p}{2\pi}\left|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}x\\ p\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}x\\ p\end{pmatrix}\right|\\
&=\sqrt{\frac{2}{A+D+\mathrm{i}(B-C)}}:\exp\left\{\frac{A-D+\mathrm{i}(B+C)}{2[A+D+\mathrm{i}(B-C)]}a^{\dagger 2}\right.\\
&\quad+\left[\frac{2}{A+D+\mathrm{i}(B-C)}-1\right]a^{\dagger}a\\
&\quad\left.-\frac{A-D+\mathrm{i}(B+C)}{2[A+D+\mathrm{i}(B-C)]}a^{2}\right\}:,
\end{aligned} \tag{8}$$

in which we see that

$$\begin{pmatrix}x\\ p\end{pmatrix}\to\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}x\\ p\end{pmatrix}$$

in the coherent state basis, which is mapped onto $F(A,B,C)$. The multiplication rule for F is $F(A',B',C')F(A,B,C)=F(A'',B'',C'')$, where

$$\begin{aligned}
\begin{pmatrix}A'' & B''\\ C'' & D''\end{pmatrix}&=\begin{pmatrix}A' & B'\\ C' & D'\end{pmatrix}\begin{pmatrix}A & B\\ C & D\end{pmatrix}\\
&=\begin{pmatrix}A'A+B'C & A'B+B'D\\ C'A+D'C & C'B+D'D\end{pmatrix}.
\end{aligned} \tag{9}$$

Thus the Fresnel operator $F(A,B,C)$ is decomposed as

$$\begin{aligned}
F(A,B,C)=&\exp\left(\frac{\mathrm{i}C}{2A}X^2\right)\exp\left[-\frac{\mathrm{i}}{2}(XP+PX)\ln A\right]\\
&\times\exp\left(-\frac{\mathrm{i}B}{2A}P^2\right),
\end{aligned} \tag{10}$$

which is the mapping of the matrix decomposition

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}1 & 0\\ C/A & 1\end{pmatrix}\begin{pmatrix}A & 0\\ 0 & A^{-1}\end{pmatrix}\begin{pmatrix}1 & B/A\\ 0 & 1\end{pmatrix}. \tag{11}$$

In this work we shall show a new decomposition of the Fresnel operator. By decomposing the *ABCD* matrix as an ortho-phase transformation, a dilation transformation, and a rotation transformation, we decompose the Fresnel operator as the product of a quadratic phase operator, a squeezing operator, and a fractional Fourier transformation operator, which helps to directly derive the transformation matrix element $\langle x'|F(A,B,C)|n\rangle$ in the context of quantum optics, where $|n\rangle$ is the number state, and in turn sheds light on the design of *ABCD*-systems in matrix optics. The new decomposition for the two-mode Fresnel operator is also obtained with the use of entangled state representation.

2. New decomposition of the single-mode Fresnel operator $F(A,B,C)$

We can decompose $\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ as ortho-phase transformation, dilation transformation, and rotation transformation via the following processes

$$\begin{aligned}
&\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\\
&=\begin{pmatrix}A\cos\theta-B\sin\theta & A\sin\theta+B\cos\theta\\ C\cos\theta-D\sin\theta & C\sin\theta+D\cos\theta\end{pmatrix},
\end{aligned} \tag{12}$$

such that the upper-right matrix element is zero, $A\sin\theta+B\cos\theta=0$, then θ is determined by

$$\cos\theta=\frac{A}{\sqrt{A^2+B^2}},\ \ \sin\theta=-\frac{B}{\sqrt{A^2+B^2}}. \tag{13}$$

Therefore it follows that

$$\begin{aligned}
&\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\\
&=\begin{pmatrix}\sqrt{A^2+B^2} & 0\\ \dfrac{CA+DB}{\sqrt{A^2+B^2}} & \dfrac{1}{\sqrt{A^2+B^2}}\end{pmatrix}.
\end{aligned} \tag{14}$$

Further decomposing

$$\begin{aligned}
&\begin{pmatrix}\sqrt{A^2+B^2} & 0\\ \dfrac{CA+DB}{\sqrt{A^2+B^2}} & \dfrac{1}{\sqrt{A^2+B^2}}\end{pmatrix}\\
&=\begin{pmatrix}1 & 0\\ \dfrac{CA+DB}{A^2+B^2} & 1\end{pmatrix}\begin{pmatrix}\sqrt{A^2+B^2} & 0\\ 0 & \dfrac{1}{\sqrt{A^2+B^2}}\end{pmatrix},
\end{aligned} \tag{15}$$

we finally obtain

$$\begin{aligned}
\begin{pmatrix}A & B\\ C & D\end{pmatrix}=&\begin{pmatrix}1 & 0\\ \dfrac{CA+DB}{A^2+B^2} & 1\end{pmatrix}\begin{pmatrix}\sqrt{A^2+B^2} & 0\\ 0 & \dfrac{1}{\sqrt{A^2+B^2}}\end{pmatrix}\\
&\times\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}.
\end{aligned} \tag{16}$$

We now analyze the Fresnel operator responsible for the matrix decomposition in Eq. (16) and derive F's canonical operator form correspondingly. We show

$$\begin{aligned}
F(A,B,C)=&F\left(1,0,\frac{CA+DB}{A^2+B^2}\right)F\left(\sqrt{A^2+B^2},0,0\right)\\
&\times F(\cos\theta,-\sin\theta,\sin\theta),
\end{aligned} \tag{17}$$

where $F\left(1,0,(CA+DB)/(A^2+B^2)\right)$ is the quadratic phase operator,

$$F\left(1,0,\frac{CA+DB}{A^2+B^2}\right)$$

$$= \sqrt{\frac{2\left(A^2+B^2\right)}{2\left(A^2+B^2\right)-\mathrm{i}\left(CA+DB\right)}}$$
$$\times :\exp\left[\frac{i\left(CA+DB\right)}{2\left(A^2+B^2\right)-\mathrm{i}\left(CA+DB\right)}\frac{a^{\dagger 2}+2a^{\dagger}a+a^2}{2}\right]:$$
$$= \exp\left[\frac{\mathrm{i}\left(CA+DB\right)}{2\left(A^2+B^2\right)}X^2\right], \tag{18}$$

where in the last step we have used the operator identity[3]

$$\exp\left[\lambda X^2\right] = \frac{1}{\sqrt{1-\lambda}} :\exp\left[\frac{\lambda}{1-\lambda}X^2\right]: . \tag{19}$$

$F(\sqrt{A^2+B^2},0,0)$ is the squeezing operator,

$$F\left(\sqrt{A^2+B^2},0,0\right)$$
$$= \operatorname{sech}^{1/2}\sigma :\exp\left[\frac{1}{2}a^{\dagger 2}\tanh\sigma+\left(\operatorname{sech}\sigma-1\right)a^{\dagger}a\right.$$
$$\left.-\frac{1}{2}a^2\tanh\sigma\right]:$$
$$= \exp\left[\frac{-\mathrm{i}}{2}\left(XP+PX\right)\ln\sqrt{A^2+B^2}\right], \tag{20}$$

where $\sqrt{A^2+B^2}=\mathrm{e}^{\sigma}, (1-A^2+B^2)/(1+A^2+B^2)=\tanh\sigma$, while $F\left(\cos\theta,-\sin\theta,\sin\theta\right)$ is the fractional Fourier transformation operator

$$F\left(\frac{A}{\sqrt{A^2+B^2}},\frac{B}{\sqrt{A^2+B^2}},\frac{-B}{\sqrt{A^2+B^2}}\right)$$
$$= \sqrt{\frac{\sqrt{A^2+B^2}}{A+\mathrm{i}B}} :\exp\left\{\left(\frac{\sqrt{A^2+B^2}}{A+\mathrm{i}B}-1\right)a^{\dagger}a\right\}:$$
$$= \mathrm{e}^{\mathrm{i}\theta/2}\mathrm{e}^{\mathrm{i}\theta a^{\dagger}a}. \tag{21}$$

Since

$$\exp\left[\frac{\mathrm{i}\left(x^2+p^2\right)}{2\tan\theta}-\frac{\mathrm{i}xp}{\sin\theta}\right]$$
$$= \sqrt{2\pi i\sin\theta\,\mathrm{e}^{-\mathrm{i}\theta}}\left\langle p\right|\exp\left[\mathrm{i}\left(\frac{x}{2}-\theta\right)a^{\dagger}a\right]\left|x\right\rangle \tag{22}$$

is a kernel of the fractional Fourier transformation, ($|p\rangle$ is conjugate state of $|x\rangle$), $F\left(A,B,C\right)$ has its new decomposition

$$F\left(A,B,C\right)$$
$$= \mathrm{e}^{\mathrm{i}\theta/2}\exp\left[\frac{\mathrm{i}\left(CA+DB\right)}{2\left(A^2+B^2\right)}X^2\right]$$
$$\times\exp\left[-\frac{\mathrm{i}}{2}\left(XP+PX\right)\ln\sqrt{A^2+B^2}\right]\mathrm{e}^{\mathrm{i}\theta a^{\dagger}a}, \tag{23}$$

where $\theta=\arg\tan\left(-B/A\right)$. Using Eq. (23) and

$$\exp\left[\mathrm{i}\frac{\lambda}{2}\left(XP+PX\right)\right] = \int\frac{\mathrm{d}x}{\sqrt{\mu}}\left|\frac{x}{\mu}\right\rangle\left\langle x\right|, \quad \mu=\mathrm{e}^{\lambda}, \tag{24}$$

we can directly derive matrix element

$$\left\langle x'\right|F\left(A,B,C\right)\left|n\right\rangle$$
$$= \mathrm{e}^{\mathrm{i}\theta/2}\exp\left[\frac{\mathrm{i}\left(CA+DB\right)}{2\left(A^2+B^2\right)}x'^2\right]$$
$$\times\mathrm{e}^{\mathrm{i}\theta n}\left\langle x'\right|\exp\left[-\frac{i}{2}\left(XP+PX\right)\ln\sqrt{A^2+B^2}\right]\left|n\right\rangle$$
$$= \frac{1}{\left(A^2+B^2\right)^{1/4}}\mathrm{e}^{\mathrm{i}\theta/2}\exp\left[\frac{\mathrm{i}\left(CA+DB\right)}{2\left(A^2+B^2\right)}x'^2\right]$$
$$\times\mathrm{e}^{\mathrm{i}\theta n}\left\langle x'/\sqrt{A^2+B^2}\right|\left.n\right\rangle$$
$$= \left[\pi\left(A^2+B^2\right)\right]^{1/4}\mathrm{e}^{\mathrm{i}\theta/2}\exp\left[\frac{\mathrm{i}\left(CA+DB\right)}{2\left(A^2+B^2\right)}x'^2\right]$$
$$\times\mathrm{e}^{\mathrm{i}\theta n}\exp\left[\frac{-x'^2}{2\left(A^2+B^2\right)}\right]$$
$$\times\mathrm{H}_n\left(x'/\sqrt{A^2+B^2}\right)\frac{1}{\sqrt{2^n n!}}, \tag{25}$$

where $|n\rangle$ is the number state. Then using the formula

$$\sum_{n=0}^{\infty}\frac{t^n}{2^n n!}\mathrm{H}_n\left(x_1\right)\mathrm{H}_n\left(x_2\right)$$
$$= \frac{1}{\sqrt{1-t^2}}\exp\left[\frac{t^2\left(x_1^2+x_2^2\right)-2tx_1x_2}{t^2-1}\right], \tag{26}$$

and Eq. (25), we obtain

$$\left\langle x'\right|F\left(A,B,C\right)\left|x\right\rangle$$
$$= \sum_n\left\langle x'\right|F\left(A,B,C\right)\left|n\right\rangle\left\langle n\right|\left.x\right\rangle$$
$$= \left[\pi^2\left(A^2+B^2\right)\right]^{-1/4}\mathrm{e}^{\mathrm{i}\theta/2}\exp\left[\frac{\mathrm{i}\left(CA+DB\right)}{2\left(A^2+B^2\right)}x'^2\right]$$
$$\times\exp\left[-\frac{x^2}{2}-\frac{-x'^2}{2\left(A^2+B^2\right)}\right]$$
$$\times\sum_n\frac{1}{\sqrt{2^n n!}}\mathrm{e}^{\mathrm{i}\theta n}\mathrm{H}_n\left(x'/\sqrt{A^2+B^2}\right)\mathrm{H}_n\left(x\right)$$
$$= \left[\pi^2\left(A^2+B^2\right)\right]^{-1/4}\mathrm{e}^{\mathrm{i}\theta/2}\exp\left[\frac{\mathrm{i}\left(CA+DB\right)}{2\left(A^2+B^2\right)}x'^2\right]$$
$$\times\exp\left[-\frac{x^2}{2}-\frac{-x'^2}{2\left(A^2+B^2\right)}\right]$$
$$\times\frac{1}{\sqrt{1-\mathrm{e}^{2\mathrm{i}\theta}}}$$
$$\times\exp\left\{\frac{\mathrm{e}^{2\mathrm{i}\theta}\left[x^2+\dfrac{x'^2}{\left(A^2+B^2\right)}-\dfrac{2\mathrm{e}^{2\mathrm{i}\theta}x'x}{\left(A^2+B^2\right)}\right]}{\mathrm{e}^{2\mathrm{i}\theta}-1}\right\}$$
$$= \frac{1}{\sqrt{2\pi\mathrm{i}B}}\exp\left[\frac{\mathrm{i}}{2B}\left(Ax^2-2x'x+Dx'^2\right)\right], \tag{27}$$

which is just the Fresnel integration kernel as expected.

3. New decomposition of two-mode Fresnel operator $F_2(A,B,C)$

Similar in spirit to the single-mode case (1), we introduce the two-mode Fresnel operator $U_2(r,s)$ through the following 2-mode coherent state representation[10–12]

$$U_2(r,s) = s\int \frac{\mathrm{d}^2 z_1 \mathrm{d}^2 z_2}{\pi^2} \left|sz_1 + rz_2^*, rz_1^* + sz_2\right\rangle \langle z_1, z_2|, \quad (28)$$

which indicates that $U_2(r,s)$ is a mapping of classical symplectic transform $(z_1,z_2) \to (sz_1 + rz_2^*, rz_1^* + sz_2)$ in phase space, where

$$|z_1,z_2\rangle = \exp\left\{-\frac{1}{2}|z_1|^2 - \frac{1}{2}|z_2|^2 + z_1 a_1^\dagger + z_2 a_2^\dagger\right\}|00\rangle$$

is a usual two-mode coherent state. Concretely, the ket in Eq. (28) is

$$|sz_1 + rz_2^*, rz_1^* + sz_2\rangle = |sz_1 + rz_2^*\rangle_1 \otimes |rz_1^* + sz_2\rangle_2, \quad (29)$$

s and r still satisfy the unimodularity condition $|s|^2 - |r|^2 = 1$. Using the IWOP technique and the normal ordering of the vacuum projector $|00\rangle\langle 00| = :\exp\left(-a_1^\dagger a_1 - a_2^\dagger a_2\right):$, we perform the integral in Eq. (28) and obtain

$$\begin{aligned} & U_2(r,s) \\ &= \frac{1}{s^*}\exp\left(\frac{r}{s^*}a_1^\dagger a_2^\dagger\right) : \exp\left[\left(\frac{1}{s^*} - 1\right)\left(a_1^\dagger a_1 + a_2^\dagger a_2\right)\right] : \\ &\quad \times \exp\left(-\frac{r^*}{s^*}a_1 a_2\right), \end{aligned} \quad (30)$$

or

$$\begin{aligned} & U_2(r,s) \\ &= F_2(A,B,C) \\ &= \frac{2}{A+D+\mathrm{i}(B-C)}\exp\left(\frac{D-A-\mathrm{i}(B+C)}{A+D+\mathrm{i}(B-C)}a_1^\dagger a_2^\dagger\right) \\ &\quad \times :\exp\left[\left(\frac{2}{A+D+\mathrm{i}(B-C)} - 1\right)\left(a_1^\dagger a_1 + a_2^\dagger a_2\right)\right] : \\ &\quad \times \exp\left(\frac{A-D-\mathrm{i}(B+C)}{A+D+\mathrm{i}(B-C)}a_1 a_2\right). \end{aligned} \quad (31)$$

We now analyze the two-mode Fresnel operator responsible for the classical decomposition in Eq. (31). It is remarkable that two-mode Fresnel operators also possess the group multiplication rule $F_2(A',B',C')F_2(A,B,C) = F_2(A'',B'',C'')$ (see Eq. (16)), thus according to Eq. (17), $F_2(A,B,C)$ also has the decomposition

$$\begin{aligned} F_2(A,B,C) &= F_2\left(1,0,\frac{CA+DB}{A^2+B^2}\right)F_2\left(\sqrt{A^2+B^2},0,0\right) \\ &\quad \times F_2(\cos\theta, -\sin\theta, \sin\theta), \end{aligned} \quad (32)$$

where

$$F_2\left(1,0,\frac{CA+DB}{A^2+B^2}\right)$$

$$= \exp\left\{\frac{\mathrm{i}(CA+DB)}{A^2+B^2}\left[(Q_1-Q_2)^2 + (P_1+P_2)^2\right]\right\} \quad (33)$$

is the two-mode quadratic phase operator. Note that in the last step we have used the operator identity

$$\begin{aligned} & \exp\left\{\frac{\lambda}{2}\left[(Q_1-Q_2)^2 + (P_1+P_2)^2\right]\right\} \\ &= \int \frac{\mathrm{d}^2\eta}{\pi}\exp\left[\frac{\lambda}{2}\left[(Q_1-Q_2)^2 + (P_1+P_2)^2\right]\right]|\eta\rangle\langle\eta| \\ &= \int \frac{\mathrm{d}^2\eta}{\pi} : \exp\left\{(\lambda-1)|\eta|^2 + a_1^\dagger\eta - a_2^\dagger\eta^* + a_1\eta^* - a_2\eta \right. \\ &\quad \left. + \left(a_1 - a_2^\dagger\right)\left(a_2 - a_1^\dagger\right)\right\} : \\ &= \frac{1}{1-\lambda} : \exp\left[\frac{\lambda}{2(1-\lambda)}\left((Q_1-Q_2)^2 + (P_1+P_2)^2\right)\right] :, \end{aligned} \quad (34)$$

where $|\eta\rangle$ is the two-mode entangled state[13–15]

$$\begin{aligned} |\eta\rangle &= \exp\left(-\frac{|\eta|^2}{2} + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right)|00\rangle, \\ \eta &= \eta_1 + \mathrm{i}\eta_2, \end{aligned} \quad (35)$$

which obeys

$$\left(a_1 - a_2^\dagger\right)|\eta\rangle = \eta|\eta\rangle, \quad \left(a_2 - a_1^\dagger\right)|\eta\rangle = -\eta^*|\eta\rangle, \quad (36)$$

and

$$\int \frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta| = 1. \quad (37)$$

$F_2\left(\sqrt{A^2+B^2},0,0\right)$ is the two-mode squeezing operator

$$\begin{aligned} & F_2\left(\sqrt{A^2+B^2},0,0\right) \\ &= \exp\left[\mathrm{i}(Q_1P_2 + Q_2P_1)\ln\sqrt{A^2+B^2}\right], \end{aligned} \quad (38)$$

while

$$\begin{aligned} & F_2\left(\frac{A}{\sqrt{A^2+B^2}},\frac{B}{\sqrt{A^2+B^2}},\frac{-B}{\sqrt{A^2+B^2}}\right) \\ &= \frac{\sqrt{A^2+B^2}}{A+iB} : \exp\left\{\left(\frac{\sqrt{A^2+B^2}}{A+\mathrm{i}B} - 1\right)\left(a_1^\dagger a_1 + a_2^\dagger a_2\right)\right\} : \\ &= \mathrm{e}^{\mathrm{i}\theta}\,\mathrm{e}^{\mathrm{i}\theta\left(a_1^\dagger a_1 + a_2^\dagger a_2\right)} \end{aligned} \quad (39)$$

is the two-mode fractional Fourier transformation operator.

4. Summary

In this paper, by virtue of the coherent state representation of the newly introduced Fresnel operator and its group product property we obtained a new decomposition of the Fresnel operator as the product of the quadratic phase operator, the

squeezing operator, and the fractional Fourier transformation operator, which in turn sheds light on the matrix optics design of *ABCD*-systems. The new decomposition for the two-mode Fresnel operator was also obtained by making use of the entangled state representation.

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