New operator identities with regard to the two-variable Hermite polynomial by virtue of entangled state representation*

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By virtue of the entangled state representation we concisely derive some new operator identities with regard to the two-variable Hermite polynomial (TVHP). By them and the technique of integration within an ordered product (IWOP) of operators we further derive new generating function formulas of the TVHP. They are useful in quantum optical theoretical calculations. It is seen from this work that by combining the IWOP technique and quantum mechanical representations one can derive some new integration formulas even without really performing the integration.

Keywords: two-variable Hermite polynomial, entangled state representation, operator identities

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1. Introduction

It is well known that Hermite polynomial (HP) is an important special function, which forms a class of orthogonal and complete set in function space.[1–3] Among them, the two-variable Hermite polynomial (TVHP) $\mathrm{H}_{m,n}(\xi,\xi^*)$

$$\mathrm{H}_{m,n}(\xi,\xi^*)=\sum_{l=0}^{\min[m,n]}\frac{(-1)^l m!n!\xi^{m-l}\xi^{*n-l}}{l!(m-l)!(n-l)!} \tag{1}$$

has become more and more useful (note that $\mathrm{H}_{m,n}(\xi,\xi^*)$ is not a direct product of a two independent single-variable HP). For example, in quantum optics theory $\mathrm{H}_{m,n}(\xi,\xi^*)$ can be considered as the basis of generalized Bargmann function space corresponding to the two-mode Fock state,[4] and is very useful for discussing the quasi-probability distribution of some quantum states.[5] In Fourier optics theory, $\mathrm{H}_{m,n}(\xi,\xi^*)$ is proved to be the eigenfunction of the complex fractional Fourier transform, the corresponding phenomenon may be observed in the light propagation in a graded index (GRIN) medium,[6] this eigenmode also exists in the two-dimensional Talbot effect demonstrated in a GRIN medium.[7] In Refs. [8]–[10], it is pointed out that the two-mode squeezed number state is just a TVHP excitation on the two-mode squeezed vacuum state. Very recently, Ma's group has investigated the nonclassicality of a TVHP state, considered as a two-mode photon subtracted squeezed vacuum state.[11]

Recalling that the entangled state representation $|\xi\rangle$ is expressed as[12–16]

$$|\xi\rangle=\exp\left(-\frac{1}{2}|\xi|^2+\xi a^\dagger+\xi^* b^\dagger-a^\dagger b^\dagger\right)|00\rangle, \tag{2}$$

which obeys

$$\left(a+b^\dagger\right)|\xi\rangle=\xi|\xi\rangle,\quad \left(a^\dagger+b\right)|\xi\rangle=\xi^*|\xi\rangle, \tag{3}$$

where $\xi=\xi_1+\mathrm{i}\xi_2$, $|00\rangle$ is the two-mode vacuum state, $a^\dagger$ and $b^\dagger$ are the Bose creation operators with $\left[a,a^\dagger\right]=\left[b,b^\dagger\right]=1$. $|\xi\rangle$ is orthonormal and complete,

$$\int\frac{\mathrm{d}^2\xi}{\pi}|\xi\rangle\langle\xi|=\int\frac{\mathrm{d}^2\xi}{\pi}:\mathrm{e}^{-\left(a^\dagger+b-\xi^*\right)\left(a+b^\dagger-\xi\right)}:=1, \tag{4}$$

where : : stands for normal ordering, and we have used the technique of integration within an ordered product (IWOP) of operators[17–20] as well as $|00\rangle\langle 00|=:\exp[-a^\dagger a-b^\dagger b]:$. According to the generating function of $\mathrm{H}_{m,n}(\xi,\xi^*)$

$$\sum_{m,n=0}^{\infty}\frac{t^m t'^n}{m!n!}\mathrm{H}_{m,n}(\xi,\xi^*)=\mathrm{e}^{-tt'+t\xi+t'\xi^*}, \tag{5}$$

we can expand

$$|\xi\rangle=\exp\left(-\frac{|\xi|^2}{2}\right)\sum_{m,n=0}^{\infty}\frac{\mathrm{H}_{m,n}(\xi,\xi^*)a^{\dagger m}b^{\dagger n}}{m!n!}|00\rangle, \tag{6}$$

then the overlap between $\langle\xi|$ and the two-mode Fock state is

$$\langle\xi|\,m,n\rangle=\frac{\exp\left[-\frac{1}{2}|\xi|^2\right]}{\sqrt{m!n!}}\mathrm{H}^*_{m,n}(\xi,\xi^*), \tag{7}$$

where

$$|m,n\rangle=\frac{a^{\dagger m}b^{\dagger n}}{\sqrt{m!n!}}|00\rangle.$$

Equation (6) exhibits quantum entanglement.[21] In this work we shall employ the entangled state representation for deriving some new operator identities regarding to $\mathrm{H}_{m,n}(\xi,\xi^*)$ and

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then present their applications. By virtue of them, we can also derive some new generating function formulas of $\mathrm{H}_{m,n}(\xi,\xi^*)$, which is quite useful in the calculating normalization of some quantum states. We conclude that by combining the IWOP technique and quantum mechanical representations one can derive some new integration formulas even without really performing the integration.

2. New operator identities regarding the two-variable Hermite polynomial

In order to obtain the normally ordered expansion of $\mathrm{H}_{m,n}\left(a+b^\dagger,a^\dagger+b\right)$, by noticing $\left[a+b^\dagger,a^\dagger+b\right]=0$, we have

$$\begin{aligned}&\mathrm{e}^{-tt'+t\left(a+b^\dagger\right)+t'\left(a^\dagger+b\right)}\\ =&:\mathrm{e}^{t\left(a+b^\dagger\right)+t'\left(a^\dagger+b\right)}:\\ =&\sum_{m,n=0}^{\infty}\frac{t^m t'^n}{m!n!}:\left(a+b^\dagger\right)^m\left(a^\dagger+b\right)^n:,\end{aligned}\tag{8}$$

where we have used the Baker–Hausdorff formula

$$\begin{aligned}&\mathrm{e}^A\mathrm{e}^B=\mathrm{e}^B\mathrm{e}^A\mathrm{e}^{[A,B]},\\ &[A,[A,B]]=[B,[A,B]]=0.\end{aligned}\tag{9}$$

Comparing with Eq. (5) we see the identity

$$\mathrm{H}_{m,n}\left(a+b^\dagger,a^\dagger+b\right)=:\left(a+b^\dagger\right)^m\left(a^\dagger+b\right)^n:.\tag{10}$$

Using Eqs. (3) and (4) as well as the IWOP technique, we have

$$\begin{aligned}&\mathrm{H}_{m,n}\left(a+b^\dagger,a^\dagger+b\right)\\ =&\int\frac{\mathrm{d}^2\xi}{\pi}\mathrm{H}_{m,n}(\xi,\xi^*)|\xi\rangle\langle\xi|\\ =&\int\frac{\mathrm{d}^2\xi}{\pi}\mathrm{H}_{m,n}(\xi,\xi^*):\mathrm{e}^{-\left(a^\dagger+b-\xi^*\right)\left(a+b^\dagger-\xi\right)}:.\end{aligned}\tag{11}$$

Equations (10) and (11) imply an integration formula

$$\int\frac{\mathrm{d}^2\xi}{\pi}\mathrm{H}_{m,n}(\xi,\xi^*)\exp\left[-\left(\alpha^*-\xi^*\right)(\alpha-\xi)\right]=\alpha^m\alpha^{*n}.\tag{12}$$

Thus we see that by combining the IWOP technique and quantum mechanical representations one can derive some new integration formulas even without really performing the integration. Moreover, from the antinormally ordered expansion (denoted by $\vdots\ \vdots$)

$$\begin{aligned}&\mathrm{e}^{-tt'+t'\left(a^\dagger+b\right)+t\left(a+b^\dagger\right)}\\ =&\vdots\mathrm{e}^{-2tt'+t'\left(a^\dagger+b\right)+t\left(a+b^\dagger\right)}\vdots\\ =&\sum_{m,n=0}^{\infty}\frac{2^{(m+n)/2}t^m t'^n}{m!n!}\vdots\mathrm{H}_{m,n}\left(\frac{a+b^\dagger}{\sqrt{2}},\frac{a^\dagger+b}{\sqrt{2}}\right)\vdots\end{aligned}\tag{13}$$

and comparing Eq. (5) with Eq. (13) we see

$$\mathrm{H}_{m,n}\left(a+b^\dagger,a^\dagger+b\right)=2^{(m+n)/2}\vdots\mathrm{H}_{m,n}\left(\frac{a+b^\dagger}{\sqrt{2}},\frac{a^\dagger+b}{\sqrt{2}}\right)\vdots.\tag{14}$$

On the other hand, it is seen that

$$\begin{aligned}&\sum_{m,n=0}^{\infty}\frac{t^m t'^n}{m!n!}\left(a+b^\dagger\right)^m\left(a^\dagger+b\right)^n\\ =&:\mathrm{e}^{tt'+t'\left(b+a^\dagger\right)}\mathrm{e}^{t\left(a+b^\dagger\right)}:\\ =&\sum_{m,n=0}^{\infty}\frac{(\mathrm{i}t)^m(\mathrm{i}t')^n}{m!n!}:\mathrm{H}_{m,n}\left[-\mathrm{i}\left(a+b^\dagger\right),-\mathrm{i}\left(a^\dagger+b\right)\right]:,\end{aligned}\tag{15}$$

this leads to the new operator identity

$$\begin{aligned}&\left(a+b^\dagger\right)^m\left(a^\dagger+b\right)^n\\ =&\ \mathrm{i}^{m+n}:\mathrm{H}_{m,n}\left[-\mathrm{i}\left(a+b^\dagger\right),-\mathrm{i}\left(a^\dagger+b\right)\right]:.\end{aligned}\tag{16}$$

Similarly, using Eqs. (3) and (4), we also have

$$\begin{aligned}&\left(a+b^\dagger\right)^m\left(a^\dagger+b\right)^n\\ =&\int\frac{\mathrm{d}^2\xi}{\pi}\xi^m\xi^{*n}:\exp\left[-\left(a^\dagger+b-\xi^*\right)\left(a+b^\dagger-\xi\right)\right]:.\end{aligned}\tag{17}$$

It is seen from Eqs. (16) and (17) that another integration formula is

$$\begin{aligned}&\int\frac{\mathrm{d}^2\xi}{\pi}\xi^m\xi^{*n}\exp\left[-\left(\alpha^*-\xi^*\right)(\alpha-\xi)\right]\\ =&\ \mathrm{i}^{m+n}\mathrm{H}_{m,n}(-\mathrm{i}\alpha,-\mathrm{i}\alpha^*).\end{aligned}\tag{18}$$

This is the reciprocal relation of Eq. (12). It is clearly shown that Eqs. (12) and (18) are the mutual integration transformation between $\mathrm{H}_{m,n}(\xi,\xi^*)$ and the polynomial $\xi^m\xi^{*n}$.

To derive another more complicated generating function formula about TVHP, we introduce another two-mode entangled state involved in modes $c^\dagger,d^\dagger$

$$\begin{aligned}&|\zeta\rangle=\exp\left(-\frac{1}{2}|\zeta|^2+\zeta c^\dagger+\zeta^* d^\dagger-c^\dagger d^\dagger\right)|00\rangle,\\ &\zeta=\zeta_1+\mathrm{i}\zeta_2,\end{aligned}\tag{19}$$

where $\left[c,c^\dagger\right]=1$ and $\left[d,d^\dagger\right]=1$, obeying

$$\left(c+d^\dagger\right)|\zeta\rangle=\zeta|\zeta\rangle,\quad\left(c^\dagger+d\right)|\zeta\rangle=\zeta^*|\zeta\rangle,\tag{20}$$

and

$$\int\frac{\mathrm{d}^2\zeta}{\pi}|\zeta\rangle\langle\zeta|=\int\frac{\mathrm{d}^2\zeta}{\pi}:\mathrm{e}^{-\left(c^\dagger+d-\zeta^*\right)\left(c+d^\dagger-\zeta\right)}:=1.\tag{21}$$

As a result of Eq. (11) we deduce

$$\begin{aligned}G\equiv&\sum_{m,n=0}^{\infty}\frac{t^n s^m}{n!m!}\mathrm{H}_{m,n}\left(a+b^\dagger,a^\dagger+b\right)\mathrm{H}_{m,n}\left(c+d^\dagger,c^\dagger+d\right)\\ =&\sum_{m,n=0}^{\infty}\frac{t^n s^m}{n!m!}:\left(a+b^\dagger\right)^m\left(a^\dagger+b\right)^n\left(c+d^\dagger\right)^m\left(c^\dagger+d\right)^n:\\ =&:\exp\left[s\left(a+b^\dagger\right)\left(c+d^\dagger\right)+t\left(a^\dagger+b\right)\left(c^\dagger+d\right)\right]:.\end{aligned}\tag{22}$$

In reference to

$$\int\frac{\mathrm{d}^2z}{\pi}\mathrm{e}^{\eta|z|^2+fz+gz^*}=-\frac{1}{\eta}\mathrm{e}^{-fg/\eta}$$

with $\mathrm{Re}(\eta)<0$, the right-hand side of Eq. (22) can be the result of the following integration with use of the IWOP technique

$$G = \int \frac{\mathrm{d}^2\zeta}{\pi} : \mathrm{e}^{-|\zeta|^2+\zeta\left[\left(c^\dagger+d\right)+s\left(a+b^\dagger\right)\right]+\zeta^*\left[\left(c+d^\dagger\right)+t\left(a^\dagger+b\right)\right]-st\left(a^\dagger+b\right)\left(a+b^\dagger\right)-\left(c^\dagger+d\right)\left(c+d^\dagger\right)} :$$
$$= : \int \frac{\mathrm{d}^2\zeta}{\pi} |\zeta\rangle\langle\zeta| \, \mathrm{e}^{-st\left(a^\dagger+b\right)\left(a+b^\dagger\right)+s\zeta\left(a+b^\dagger\right)+t\zeta^*\left(a^\dagger+b\right)} : , \tag{23}$$

where, within the normal ordering $: \, :$, the terms involving $\left(a+b^\dagger\right)$ and $\left(a^\dagger+b\right)$ can be expressed as the following integration

$$\exp\left[-st\left(a^\dagger+b\right)\left(a+b^\dagger\right)+s\zeta\left(a+b^\dagger\right)+t\zeta^*\left(a^\dagger+b\right)\right]$$
$$= \frac{1}{1-ts}\int \frac{\mathrm{d}^2\xi}{\pi} \exp\left[\frac{-|\xi|^2}{1-ts}+\xi\left(a^\dagger+b+\frac{s\zeta}{1-ts}\right)+\xi^*\left(a+b^\dagger+\frac{t\zeta^*}{1-ts}\right)-\frac{ts|\zeta|^2}{1-ts}-\left(a^\dagger+b\right)\left(a+b^\dagger\right)\right]. \tag{24}$$

Substituting Eq. (24) into Eq. (23) and using Eqs. (3) and (4) as well as Eqs. (20) and (21) we see

$$G = \frac{1}{1-ts}\int \frac{\mathrm{d}^2\xi\,\mathrm{d}^2\zeta}{\pi^2} |\xi\rangle|\zeta\rangle\langle\zeta|\langle\xi| \exp\left[\frac{-ts\left(|\xi|^2+|\zeta|^2\right)}{1-ts}+\frac{s\zeta\xi}{1-ts}+\frac{t\zeta^*\xi^*}{1-ts}\right]$$
$$= \frac{1}{1-ts}\exp\left\{\frac{-ts}{1-ts}\left[\left(a^\dagger+b\right)\left(a+b^\dagger\right)+\left(c^\dagger+d\right)\left(c+d^\dagger\right)\right]+\frac{s\left(a+b^\dagger\right)\left(c+d^\dagger\right)}{1-ts}+\frac{t\left(c^\dagger+d\right)\left(a^\dagger+b\right)}{1-ts}\right\}. \tag{25}$$

Since $\left(a^\dagger+b\right)$, $\left(a+b^\dagger\right)$, $\left(c^\dagger+d\right)$, and $\left(c+d^\dagger\right)$ are all commutative among themselves, we can make replacement $\left(a^\dagger+b\right)\to y$, $\left(a+b^\dagger\right)\to x$, $\left(c^\dagger+d\right)\to y'$, $\left(c+d^\dagger\right)\to x'$, then by comparing Eq. (25) with Eq. (22) we obtain the complicated generating function formula about TVHP

$$\sum_{m,n=0}^{\infty}\frac{s^m t^n}{m!n!}\mathrm{H}_{m,n}(x,y)\,\mathrm{H}_{m,n}\left(x',y'\right)$$
$$= \frac{1}{1-ts}\exp\left[\frac{sxx'+tyy'-ts\left(xy+x'y'\right)}{1-ts}\right]. \tag{26}$$

3. Applications

We now present some applications of Eq. (26). To begin with, we point out that Eq. (26) can be used for deriving another generating function formula. In fact, using the generating function of Laguerre polynomial[3,22]

$$\sum_{m=0}^{\infty}\mathrm{L}_m(x)\,s^m = (1-s)^{-1}\exp\left(\frac{-xs}{1-s}\right), \tag{27}$$

and its relation to the TVHP,[14,23] $\mathrm{H}_{m,m}(x,y) = (-1)^m m!\mathrm{L}_m(xy)$, we can reexpress Eq. (26) as

$$\sum_{m,n=0}^{\infty}\frac{s^m t^n \mathrm{H}_{m,n}(x,y)\,\mathrm{H}_{m,n}\left(x',y'\right)}{m!n!}$$
$$= \frac{\mathrm{e}^{tyy'}}{1-st}\exp\left[\frac{-\left(xy+x'y'-tyy'-\frac{1}{t}xx'\right)ts}{1-st}\right]$$
$$= \mathrm{e}^{tyy'}\sum_{m=0}^{\infty}\frac{(-st)^m}{m!}\mathrm{H}_{m,m}\left[\mathrm{i}\left(\sqrt{t}y'-\frac{x}{\sqrt{t}}\right),\right.$$
$$\left.\mathrm{i}\left(\sqrt{t}y-\frac{x'}{\sqrt{t}}\right)\right]. \tag{28}$$

Comparing the same power of s on the above two sides yields

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\mathrm{H}_{m,n}(x,y)\,\mathrm{H}_{m,n}\left(x',y'\right)$$
$$= (-t)^m\,\mathrm{e}^{tyy'}\mathrm{H}_{m,m}\left[\mathrm{i}\left(\sqrt{t}y'-\frac{x}{\sqrt{t}}\right),\mathrm{i}\left(\sqrt{t}y-\frac{x'}{\sqrt{t}}\right)\right]. \tag{29}$$

Further, by noticing

$$\mathrm{e}^{t'a}\mathrm{e}^{ta^\dagger} = : \mathrm{e}^{\left(-\mathrm{i}t'\right)\mathrm{i}a+(-\mathrm{i}t)\mathrm{i}a^\dagger-(-\mathrm{i}t)\left(-\mathrm{i}t'\right)} :$$
$$= \sum_{m,n=0}^{\infty}\frac{(-\mathrm{i}t)^m\left(-\mathrm{i}t'\right)^n}{m!n!} : \mathrm{H}_{m,n}\left(\mathrm{i}a^\dagger,\mathrm{i}a\right) : \tag{30}$$

and comparing it with

$$\mathrm{e}^{t'a}\mathrm{e}^{ta^\dagger} = \sum_{m,n=0}^{\infty}\frac{t'^n t^m}{n!m!}a^n a^{\dagger m},$$

we have the compact operator identity

$$a^n a^{\dagger m} = (-\mathrm{i})^{m+n} : \mathrm{H}_{m,n}\left(\mathrm{i}a^\dagger,\mathrm{i}a\right) : . \tag{31}$$

It then follows from Eq. (26) that

$$\mathrm{e}^{sab}\mathrm{e}^{ta^\dagger b^\dagger}$$
$$= \sum_{m,n=0}^{\infty}\frac{(-1)^{m+n}s^m t^n}{m!n!} : \mathrm{H}_{m,n}\left(\mathrm{i}a^\dagger,\mathrm{i}a\right)\mathrm{H}_{m,n}\left(\mathrm{i}b^\dagger,ib\right) :$$
$$= \frac{1}{1-ts} : \exp\left[\frac{ts\left(a^\dagger a+b^\dagger b\right)+sa^\dagger b^\dagger+tab}{1-ts}\right] :$$
$$= \frac{1}{1-ts}\exp\left(\frac{s}{1-ts}a^\dagger b^\dagger\right)\exp\left(-\left(a^\dagger a+b^\dagger b\right)\ln(1-ts)\right)$$
$$\times\exp\left(\frac{t}{1-ts}ab\right), \tag{32}$$

where at the last step we have used the formula $\mathrm{e}^{\lambda a^\dagger a} = : \exp[\left(\mathrm{e}^\lambda-1\right)a^\dagger a] : .$[18,24] This result of Eq. (32) agrees with that of Ref. [25]. On the other hand, using the antinormally ordered operator $\vdots \mathrm{H}_{m,n}\left(a^\dagger,a\right)\vdots = a^{\dagger m}a^n$, where using its P-representation as well as Eq. (12) we easily prove that

$$\vdots \mathrm{H}_{m,n}\left(a^\dagger,a\right)\vdots = \int \frac{\mathrm{d}^2 z}{\pi}\mathrm{H}_{m,n}\left(z^*,z\right)|z\rangle\langle z|$$

$$= \int \frac{\mathrm{d}^2 z}{\pi} \mathrm{H}_{m,n}(z^*, z) : \mathrm{e}^{-\left(a^\dagger - z^*\right)(a-z)} :$$
$$= a^{\dagger m} a^n, \tag{33}$$

we can obtain

$$\begin{aligned}
&\mathrm{e}^{t a^\dagger b^\dagger} \mathrm{e}^{sab} \\
&= \sum_{m,n=0}^{\infty} \frac{s^m t^n}{m!n!} :\mathrm{H}_{m,n}\left(a, a^\dagger\right) \mathrm{H}_{m,n}\left(b, b^\dagger\right): \\
&= \frac{1}{1-ts} :\exp\left[\frac{-ts\left(a^\dagger a + b^\dagger b\right) + t a^\dagger b^\dagger + sab}{1-ts}\right]: \\
&= \frac{1}{1-ts} \exp\left(\frac{s}{1-ts} ab\right) :\exp\left(\frac{-ts}{1-ts}\left(a^\dagger a + b^\dagger b\right)\right): \\
&\quad \times \exp\left(\frac{t}{1-ts} a^\dagger b^\dagger\right).
\end{aligned} \tag{34}$$

Finally, according to Eqs. (29) and (31) and using

$$\mathrm{H}_{m,m}(x, y) = (-1)^m m! \mathrm{L}_m(xy),$$

we also have

$$\begin{aligned}
&a^m b^m \mathrm{e}^{a^\dagger b^\dagger \tanh\lambda} \\
&= \sum_{n=0}^{\infty} \frac{\tanh^n \lambda}{n!} a^m a^{\dagger n} b^m b^{\dagger n} \\
&= (-1)^m \sum_{n=0}^{\infty} \frac{(-\tanh\lambda)^n}{n!} : \mathrm{H}_{m,n}\left(\mathrm{i}a, \mathrm{i}a^\dagger\right) \mathrm{H}_{m,n}\left(\mathrm{i}b, \mathrm{i}b^\dagger\right) : \\
&= m! \tanh^m \lambda \, \mathrm{e}^{a^\dagger b^\dagger \tanh\lambda} \\
&\quad \times : \mathrm{L}_m\left(-a^\dagger a - b^\dagger b - ab\coth\lambda - a^\dagger b^\dagger \tanh\lambda\right) : ,
\end{aligned} \tag{35}$$

which relates to Laguerre polynomial. Based on this, we obtain that the two-mode photon-subtracted squeezed vacuum state is expressed as[26]

$$\begin{aligned}
|\lambda\rangle_m &\equiv a^m b^m S_2(\lambda) |00\rangle = a^m b^m \mathrm{e}^{a^\dagger b^\dagger \tanh\lambda} |00\rangle \\
&= m! \tanh^m \lambda \, \mathrm{e}^{a^\dagger b^\dagger \tanh\lambda} \mathrm{L}_m\left(-a^\dagger b^\dagger \tanh\lambda\right) |00\rangle \\
&= m! \tanh^m \lambda \mathrm{L}_m\left(-a^\dagger b^\dagger \tanh\lambda\right) S_2(\lambda) |00\rangle ,
\end{aligned} \tag{36}$$

where $S_2(\lambda) = \exp\left[\lambda\left(a^\dagger b^\dagger - ab\right)\right]$ is the two-mode squeezing operator with λ being a real squeezing parameter. From Eq. (36), $|\lambda\rangle_m$ can be equivalent to Laguerre polynomial excitation on squeezed vacuum state. Recall the resut of Ref. [27], we have calculated its normalization factor

$$\begin{aligned}
&\langle 00| \mathrm{e}^{ab\tanh\lambda} a^{\dagger m} b^{\dagger m} a^m b^m \mathrm{e}^{a^\dagger b^\dagger \tanh\lambda} |00\rangle \\
&= (m!)^2 \sinh^{2m}\lambda \mathrm{P}_m(\cosh 2\lambda),
\end{aligned} \tag{37}$$

where $\mathrm{P}_m(x)$ is Legendre polynomial[1,28]

$$\mathrm{P}_m(x) = \sum_{l=0}^{[m/2]} \frac{(-1)^l (2m-2l)! x^{m-2l}}{2^m l!(m-l)!(m-2l)!}. \tag{38}$$

Using Eqs. (36) and (37) as well as the coherent state's completeness relation

$$\int \frac{\mathrm{d}^2\alpha \mathrm{d}^2\beta}{\pi^2} |\alpha, \beta\rangle \langle\alpha, \beta| = 1,$$

it follows that

$$\begin{aligned}
&\int \frac{\mathrm{d}^2\alpha \mathrm{d}^2\beta}{\pi^2} \mathrm{L}_m(-\alpha\beta\tanh\lambda) \mathrm{L}_m(-\alpha^*\beta^*\tanh\lambda) \\
&\quad \times \mathrm{e}^{-|\alpha|^2 - |\beta|^2 + (\alpha\beta + \alpha^*\beta^*)\tanh\lambda} \\
&= \cosh^{2m}\lambda \mathrm{P}_m(\cosh 2\lambda),
\end{aligned} \tag{39}$$

which is a new integration formula.

4. Summary

In summary, by virtue of the entangled state representation we concisely derived some new operator identities with regard to two-variable Hermite polynomials. They are useful in quantum optical theoretical calculations. By combining the IWOP technique and quantum mechanical representations one can derive some new integration formulas even without really performing the integration.

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