Analysis of the generalized Camassa and Holm equation with the improved element-free Galerkin method*

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In this paper, we analyze the generalized Camassa and Holm (CH) equation by the improved element-free Galerkin (IEFG) method. By employing the improved moving least-square (IMLS) approximation, we derive the formulas for the generalized CH equation with the IEFG method. A variational method is used to obtain the discrete equations, and the essential boundary conditions are enforced by the penalty method. Because there are fewer coefficients in the IMLS approximation than in the MLS approximation, and in the IEFG method, fewer nodes are selected in the entire domain than in the conventional EFG method, the IEFG method should result in a higher computing speed. The effectiveness of the IEFG method for the generalized CH equation is investigated by numerical examples in this paper.

Keywords: meshless method, improved moving least-square (IMLS) approximation, improved element-free Galerkin (IEFG) method, generalized Camassa and Holm (CH) equation

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1. Introduction

The Camassa and Holm (CH) equation is a nonlinear partial differential equation which was initially formulated by Camassa and Holm.[1] The CH equation was also derived by Dai[2] as a model equation in hyperelastic rods. Boyd [3] concluded that the CH equation is of interest for two reasons. First, it is a model for small amplitude, shallow water waves—just as consistent in this limit as the KdV equation, which has been intensively studied for a century. Second, the equation is of great intrinsic interest in the study of solitary waves because (i) it is exactly integrable, (ii) its solutions, including multiple solitons are very simple, and (iii) it is novel in that its solitary wave has a discontinuous first derivative, in contrast to the great smoothness of most previously known species of solitary waves. These novel solitary waves are called peakons. Recently the CH equation and its generalized forms have aroused great research interest.[4–12]

The element-free Galerkin (EFG) method is one of the promising meshless methods that have attracted the great attention of researchers.[13–46] This method does not require any element connectivity data, and does not suffer much degradation in accuracy when nodal arrangements are very irregular. The method has been successfully applied to elasticity, heat conduction, and fatigue crack growth modeling. The moving least-square (MLS) is employed for the construction of the shape function in the EFG method. The MLS approximation was developed from the conventional least-square method, and in a practical numerical process, it essentially involves the application of the conventional method to each selected point. A disadvantage of the conventional method is that the final algebraic equation system is sometimes ill-conditioned. Hence, in the MLS approximation, the ill-conditioned algebraic equation system should be considered. However, it is difficult to determine which algebraic equation system is ill-conditioned before the equation is solved. Thus, sometimes a poor solution or even an incorrect numerical solution will be obtained. To solve this problem, the improved moving least-square (IMLS) approximation has been developed to obtain the approximation function.[47–49] In the IMLS, the orthogonal function system with a weight function is used as the basis function. The algebraic equation system in the IMLS approximation is not ill-conditioned, and it can be solved without involving matrix inversion.

Based on the IMLS approximation and the EFG method, an improved element-free Galerkin (IEFG) method has been developed.[50–57] In the IEFG method, less nodes are selected in the entire domain than in the conventional EFG method. Hence, the IEFG method should result in a higher computing speed. Furthermore, the IEFG method has greater computational precision than the EFG method when the same nodes are selected in the domain.

In this paper, the improved element-free Galerkin (IEFG) method is used to obtain the numerical solution of the generalized CH equation. The variational method is used to obtain the corresponding discrete equations and the essential boundary conditions are enforced by penalty method. The present results are compared with existing results available in extant literature.

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2. Improved moving least-square approximation

The moving least-square (MLS) method, first used by mathematicians for data fitting and surface construction, is now a widely used alternative for constructing EFG shape functions for approximation.

2.1. MLS approximation

In the moving least-square approximation,[13] Lancaster and Salkauskas defined a local approximation by

$$u^h(\boldsymbol{x})\bar{\boldsymbol{x}}) = \sum_{j=1}^{m} p_j(\bar{\boldsymbol{x}})a_j(\boldsymbol{x}) \equiv \boldsymbol{p}^{\mathrm{T}}(\bar{\boldsymbol{x}})\boldsymbol{a}(\boldsymbol{x}), \tag{1}$$

where m is the number of terms in the basis, $p_j(\boldsymbol{x})$ are the monomial basis functions, and $\boldsymbol{a}_j(\boldsymbol{x})$ are their coefficients, which as indicated are functions of the spatial coordinates $\boldsymbol{x}$. This defines the quadratic form

$$J = \sum_{i=1}^{n} w(\boldsymbol{x}-\boldsymbol{x}_i)\left[u^h(\boldsymbol{x},\boldsymbol{x}_i) - u(\boldsymbol{x}_i)\right]^2, \tag{2}$$

where $w(\boldsymbol{x}-\boldsymbol{x}_i)$ are weight functions with compact support, $\boldsymbol{x}_i$ are the nodes in the compacted support domain of $\boldsymbol{x}$. Equation (2) can be rewritten into the vector form

$$J = (\boldsymbol{pa}-\boldsymbol{u})^{\mathrm{T}}\boldsymbol{W}(\boldsymbol{x})(\boldsymbol{pa}-\boldsymbol{u}), \tag{3}$$

where

$$\boldsymbol{u}^{\mathrm{T}} = (u_1, u_2, \ldots, u_n), \tag{4}$$

$$\boldsymbol{p} = \begin{bmatrix} p_1(x_1) & p_2(x_1) & \cdots & p_m(x_1) \\ p_1(x_2) & p_2(x_2) & \cdots & p_m(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ p_1(x_n) & p_2(x_n) & \cdots & p_m(x_n) \end{bmatrix}, \tag{5}$$

and

$$\boldsymbol{W}(\boldsymbol{x}) = \begin{bmatrix} w(\boldsymbol{x}-\boldsymbol{x}_1) & 0 & \cdots & 0 \\ 0 & w(\boldsymbol{x}-\boldsymbol{x}_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w(\boldsymbol{x}-\boldsymbol{x}_n) \end{bmatrix}. \tag{6}$$

To find the coefficients $\boldsymbol{a}(\boldsymbol{x})$, we obtain the extremum of J by

$$\frac{\partial J}{\partial \boldsymbol{a}} = \boldsymbol{A}(\boldsymbol{x})\boldsymbol{a}(\boldsymbol{x}) - \boldsymbol{B}(\boldsymbol{x})\boldsymbol{u} = 0, \tag{7}$$

which results in the equation system

$$\boldsymbol{A}(\boldsymbol{x})\boldsymbol{a}(\boldsymbol{x}) = \boldsymbol{B}(\boldsymbol{x})\boldsymbol{u}, \tag{8}$$

where A is called the moment matrix and is given by

$$\boldsymbol{A} = \boldsymbol{p}^{\mathrm{T}}\boldsymbol{W}(\boldsymbol{x})\boldsymbol{p}, \tag{9}$$

$$\boldsymbol{B} = p^{\mathrm{T}}\boldsymbol{W}(\boldsymbol{x}). \tag{10}$$

Then we have

$$\boldsymbol{a}(\boldsymbol{x}) = \boldsymbol{A}^{-1}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})\boldsymbol{u}. \tag{11}$$

The approximation $u^h(\boldsymbol{x})$ can be expressed as

$$u(\boldsymbol{x}) \approx u^h(\boldsymbol{x}) = \sum_{i=1}^{n} \boldsymbol{\Phi}_i^k(\boldsymbol{x})u_i = \boldsymbol{\Phi}^k(\boldsymbol{x})\boldsymbol{u}, \tag{12}$$

where the shape functions are given by

$$\boldsymbol{\Phi}^k(\boldsymbol{x}) = (\boldsymbol{\Phi}_1^k(\boldsymbol{x}), \ldots, \boldsymbol{\Phi}_n^k(\boldsymbol{x})) = \boldsymbol{p}^{\mathrm{T}}(\boldsymbol{x})\boldsymbol{A}^{-1}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x}), \tag{13}$$

with the superscript k being the order of the polynomial basis.

2.2. Improved moving least-square (IMLS) procedure

In an MLS approximation, Eq. (8) is sometimes ill-conditioned, even in the presence of a singular phenomenon. Moreover, it is difficult to obtain the correct numerical solution. Using the weighted orthogonal basis functions, the IMLS approximation was presented in Refs. [47]–[49].

For $\forall f(\boldsymbol{x}), g(\boldsymbol{x}) \in \mathrm{span}(p)$, define

$$(f,g) = \sum_{I=1}^{n} w(\boldsymbol{x}-\boldsymbol{x}_I)f(\boldsymbol{x}_I)g(\boldsymbol{x}_I), \tag{14}$$

then $(f,\ g)$ will be an inner product, and $\mathrm{span}(p)$ is a Hilbert space.

In the Hilbert space $\mathrm{span}(p)$, for the set of points $\{x_i\}$ and the weight functions $\{w_i\}$, if the functions $p_1(\boldsymbol{x}), p_2(\boldsymbol{x}), \ldots, p_m(\boldsymbol{x})$ satisfy the conditions

$$\begin{aligned}(p_k, p_j) &= \sum_{i=1}^{n} w_i p_k(x_i) p_j(x_i) \\ &= \begin{cases} 0, & k \neq j, \\ A_k, & k = j, \end{cases} \quad (k, j = 1, 2, \ldots, m),\end{aligned} \tag{15}$$

then the function set $p_1(\boldsymbol{x}), p_2(\boldsymbol{x}), \ldots, p_m(\boldsymbol{x})$ is called a weighted orthogonal function set with a weight function $\{w_i\}$ about points $\{x_i\}$.

Equation (8) can be written as

$$\begin{bmatrix} (p_1,p_1) & (p_1,p_2) & \cdots & (p_1,p_m) \\ (p_2,p_1) & (p_2,p_2) & \cdots & (p_2,p_m) \\ \vdots & \vdots & \ddots & \vdots \\ (p_m,p_1) & (p_m,p_2) & \cdots & (p_m,p_m) \end{bmatrix} \begin{bmatrix} a_1(\boldsymbol{x}) \\ a_2(\boldsymbol{x}) \\ \vdots \\ a_m(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} (p_1,u_I) \\ (p_2,u_I) \\ \vdots \\ (p_m,u_I) \end{bmatrix}. \tag{16}$$

If the basis function set $p_i(\boldsymbol{x}) \in \mathrm{span}(\boldsymbol{p})$ $(i = 1, 2, \ldots, m,)$ is a weighted orthogonal function set about points $\{\boldsymbol{x}_i\}$, i.e., if

$$(p_i, p_j) = 0, \quad (i \neq j), \tag{17}$$

then equation (16) becomes

$$\begin{bmatrix} (p_1,p_1) & 0 & \cdots & 0 \\ 0 & (p_2,p_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (p_m,p_m) \end{bmatrix} \begin{bmatrix} a_1(\boldsymbol{x}) \\ a_2(\boldsymbol{x}) \\ \vdots \\ a_m(\boldsymbol{x}) \end{bmatrix}$$

$$= \begin{bmatrix} (p_1, u_I) \\ (p_2, u_I) \\ \vdots \\ (p_m, u_I) \end{bmatrix}. \tag{18}$$

We can then directly obtain the coefficients $a_i(\boldsymbol{x})$ as follows:

$$a_i(\boldsymbol{x}) = \frac{(p_i, u_I)}{(p_i, p_i)}, \quad (i = 1, 2, \ldots, m), \tag{19}$$

i.e.,

$$\boldsymbol{a}(\boldsymbol{x}) = \bar{A}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})\boldsymbol{u}, \tag{20}$$

where

$$\bar{A}(\boldsymbol{x}) = \begin{bmatrix} \dfrac{1}{(p_1, p_1)} & 0 & \ldots & 0 \\ 0 & \dfrac{1}{(p_2, p_2)} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \dfrac{1}{(p_m, p_m)} \end{bmatrix}. \tag{21}$$

From Eqs. (19) and (12), the expression of the approximation function $u^{\mathrm{h}}(\boldsymbol{x})$ is obtained as

$$u^{\mathrm{h}}(\boldsymbol{x}) = \bar{\boldsymbol{\Phi}}(\boldsymbol{x})\boldsymbol{u} = \sum_{I=1}^{n} \bar{\Phi}_I(\boldsymbol{x}) u_I, \tag{22}$$

where $\bar{\boldsymbol{\Phi}}(\boldsymbol{x})$ is the shape function and

$$\begin{aligned} \bar{\boldsymbol{\Phi}}(\boldsymbol{x}) &= (\bar{\Phi}_1(\boldsymbol{x}), \bar{\Phi}_2(\boldsymbol{x}), \ldots, \bar{\Phi}_n(\boldsymbol{x})) \\ &= \boldsymbol{p}^{\mathrm{T}}(\boldsymbol{x})\bar{\boldsymbol{A}}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x}). \end{aligned} \tag{23}$$

This is an IMLS approximation in which the coefficients $a_i(\boldsymbol{x})$ are obtained simply and directly. It is therefore impossible to form an ill-conditioned or singular equations system, and we can thus obtain the correct solution.

From Eq. (23), we have

$$\bar{\Phi}_I(\boldsymbol{x}) = \sum_{j=1}^{m} p_j(\boldsymbol{x})[\bar{A}(\boldsymbol{x})\boldsymbol{B}(\boldsymbol{x})]_{jI}, \tag{24}$$

which represents the shape function of the IMLS approximation corresponding to node I. Then, the partial derivatives of $\bar{\Phi}_I(\boldsymbol{x})$ can be obtained as

$$\bar{\Phi}_{I,i}(\boldsymbol{x}) = \sum_{j=1}^{m} [p_{j,i}(\bar{\boldsymbol{A}}\boldsymbol{B})_{jI} + p_j(\bar{\boldsymbol{A}}_{,i}\boldsymbol{B} + \bar{\boldsymbol{A}}\boldsymbol{B}_{,i})_{jI}]. \tag{25}$$

The weighted orthogonal basis function set $\boldsymbol{p} = (p_i)$ can be formed with the Schmidt method,

$$\begin{aligned} &p_1 = 1, \\ &p_i = r^{i-1} - \sum_{k=1}^{i-1} \frac{(r^{i-1}, p_k)}{(p_k, p_k)} p_k, \quad i = 2, 3, \ldots. \end{aligned} \tag{26}$$

When the weighted orthogonal basis functions (26) are used, there are fewer coefficients in the trial function. In an IMLS approximation, fewer nodes are needed in a domain of influence than in an MLS approximation. Thus, with the meshless method that is formed with an IMLS approximation, fewer nodes are selected than those with the conventional meshless methods in the whole domain.

3. Improved element-free Galerkin (IEFG) method for the generalized Camassa and Holm (CH) equation

The generalized Camassa and Holm (CH) equation is given as

$$\begin{aligned} &u_t + 2ku_x - u_{xxt} + (\beta + 1)\boldsymbol{u}^m u_x - \beta u_x u_{xx} - u u_{xxx} = 0, \\ &(a < x < b), \end{aligned} \tag{27a}$$

the boundary conditions are

$$u(a, t) = \alpha_1, \quad u(b, t) = \beta_1, \tag{27b}$$

the initial condition is

$$u(x, 0) = f(\boldsymbol{x}). \tag{27c}$$

In Eq. (27a) subscripts x and t denote differentiation.

The weak form of Eq. (27) is obtained as follows:

$$\begin{aligned} &\int_\Gamma \delta u^{\mathrm{T}}(u_t + 2ku_x - u_{xxt} + (\beta + 1)\boldsymbol{u}^m u_x \\ &- \beta u_x u_{xx} - u u_{xxx})\mathrm{d}\Gamma + \alpha \delta u^{\mathrm{T}}(u - \bar{u})|_{\Gamma_u} = 0, \end{aligned} \tag{28}$$

where Γ denotes the domain of x, Γ_u denotes the boundaries, $\bar{u}$ are the values of x at the boundaries, and α is a penalty parameter.

By the subsection integration, we can obtain

$$\begin{aligned} &\int_\Gamma \delta u^{\mathrm{T}} u_t \mathrm{d}\Gamma + 2k \int_\Gamma \delta u^{\mathrm{T}} u_x \mathrm{d}\Gamma + \int_\Gamma \delta u_x^{\mathrm{T}} u_{xt} \mathrm{d}\Gamma \\ &+ (\beta + 1) \int_\Gamma \delta u^{\mathrm{T}}(u^m u_x)\mathrm{d}\Gamma - \beta \int_\Gamma \delta u^{\mathrm{T}}(u_x u_{xx})\mathrm{d}\Gamma \\ &- \int_\Gamma \delta u^{\mathrm{T}}(u u_{xxx})\mathrm{d}\Gamma + \alpha \delta u^{\mathrm{T}}(u - \bar{u})|_{\Gamma_u} = 0. \end{aligned} \tag{29}$$

With Eq. (24), we can obtain the approximation of the test function

$$u^{\mathrm{h}}(x, t) = \sum_{I=1}^{n} \Phi_I(x) \cdot u_I(t) = \boldsymbol{\Phi}(x) \cdot U(t), \tag{30}$$

$$\begin{aligned} \frac{\partial u^{\mathrm{h}}(x, t)}{\partial t} &= \frac{\partial}{\partial t} \sum_{I=1}^{n} \Phi_I(x) \cdot u_I(t) \\ &= \sum_{I=1}^{n} \Phi_I(x) \cdot \frac{\partial U_I(t)}{\partial t} = \boldsymbol{\Phi}(x)\dot{U}(t), \end{aligned} \tag{31}$$

$$\begin{aligned} \frac{\partial u^{\mathrm{h}}(x, t)}{\partial x} &= \frac{\partial}{\partial x} \sum_{I=1}^{n} \Phi_I(x) \cdot u_I(t) \\ &= \sum_{I=1}^{n} \Phi_{I,x}(x) \cdot u_I(t) = \boldsymbol{\Phi}_x(x) U(t), \end{aligned} \tag{32}$$

$$\frac{\partial^2 u^{\mathrm{h}}(x, t)}{\partial x \partial t} = \frac{\partial}{\partial x \partial t} \sum_{I=1}^{n} \Phi_I(x) \cdot u_I(t)$$

$$= \sum_{I=1}^{n} \Phi_{I,x}(x) \cdot \frac{\partial u_I(t)}{\partial t} = \boldsymbol{\Phi}_x(x)\dot{U}(t), \quad (33)$$

$$\frac{\partial^2 u^{\mathrm{h}}(x,t)}{\partial x^2} = \frac{\partial^2}{\partial x \partial x} \sum_{I=1}^{n} \Phi_I(x) \cdot u_I(t)$$

$$= \sum_{I=1}^{n} \Phi_{I,xx}(x) \cdot u_I(t) = \boldsymbol{\Phi}_{xx}(x)U(t), \quad (34)$$

$$\frac{\partial^3 u^{\mathrm{h}}(x,t)}{\partial x^3} = \frac{\partial^3}{\partial x \partial x \partial x} \sum_{I=1}^{n} \Phi_I(x) \cdot u_I(t)$$

$$= \sum_{I=1}^{n} \Phi_{I,xxx}(x) \cdot u_I(t) = \Phi_{xxx}(x)U(t), \quad (35)$$

where

$$\boldsymbol{\Phi}(x) = (\Phi_1(x), \Phi_2(x), \ldots, \Phi_n(x)), \quad (36)$$

$$\boldsymbol{U}(t) = (u_1(t), u_2(t), \ldots, u_n(t))^{\mathrm{T}}, \quad (37)$$

$$\dot{\boldsymbol{U}}(t) = \left(\frac{\partial u_1(t)}{\partial t}, \frac{\partial u_2(t)}{\partial t}, \ldots, \frac{\partial u_n(t)}{\partial t}\right)^{\mathrm{T}}, \quad (38)$$

$$\boldsymbol{\Phi}_x(x) = (\Phi_{1,x}(x), \Phi_{2,x}(x), \ldots, \Phi_{n,x}(x)), \quad (39)$$

$$\boldsymbol{\Phi}_{xx}(x) = (\Phi_{1,xx}(x), \Phi_{2,xx}(x), \ldots, \Phi_{n,xx}(x)), \quad (40)$$

$$\boldsymbol{\Phi}_{xxx}(x) = (\Phi_{1,xxx}(x), \Phi_{2,xxx}(x), \ldots, \Phi_{n,xxx}(x)). \quad (41)$$

Substituting Eqs. (30)–(41) into Eq. (29) yields

$$\begin{aligned}
&\delta U^{\mathrm{T}}(t)K\dot{U}(t) + \delta U^{\mathrm{T}}(t)TU(t) \\
&+ \delta U^{\mathrm{T}}(t)G\dot{U}(t) + \delta U^{\mathrm{T}}(t)\int_{\Gamma} \beta\Phi(u^m u_x)\,\mathrm{d}\Gamma \\
&+ \delta U^{\mathrm{T}}(t)\int_{\Gamma} 2\Phi(u_x u_{xx})\,\mathrm{d}\Gamma - \delta U^{\mathrm{T}}(t)\int_{\Gamma} \Phi(u u_{xxx})\,\mathrm{d}\Gamma \\
&+ \delta U^{\mathrm{T}}(t)HU(t) - \delta U^{\mathrm{T}}(t)Q = 0. \quad (42)
\end{aligned}$$

Since $\delta U^{\mathrm{T}}(t)$ are arbitrary in Eq. (42), the following equations can be obtained

$$\begin{aligned}
&(\boldsymbol{K}+\boldsymbol{G})\cdot\dot{\boldsymbol{U}}(t) + (\boldsymbol{H}+\boldsymbol{T})\cdot\boldsymbol{U}(t) + \int_{\Gamma} (\beta+1)\Phi(u^m u_x)\,\mathrm{d}\Gamma \\
&- \int_{\Gamma} \beta\Phi(u_x u_{xx})\,\mathrm{d}\Gamma - \int_{\Gamma} \Phi(u u_{xxx})\,\mathrm{d}\Gamma = \boldsymbol{Q}(t), \quad (43)
\end{aligned}$$

where

$$K_{IJ} = \int_{\Gamma} \Phi_I(x)\Phi_J(x)\,\mathrm{d}\Gamma, \quad (44)$$

$$G_{IJ} = \int_{\Gamma} \Phi_{I,x}(x)\Phi_{J,x}(x)\,\mathrm{d}\Gamma, \quad (45)$$

$$T_{IJ} = 2k\int_{\Gamma} \Phi_I(x)\Phi_{J,x}(x)\,\mathrm{d}\Gamma, \quad (46)$$

$$H_{IJ} = \alpha(\Phi_I\Phi_J|_{x=a} + \Phi_I\Phi_J|_{x=b}), \quad (47)$$

$$Q_I = \alpha(\Phi_I\bar{u}|_{x=a} + \Phi_I\bar{u}|_{x=b}). \quad (48)$$

Forming the time discretization of Eq. (43) with the central difference method, we obtain

$$\begin{aligned}
&(\boldsymbol{K}+\boldsymbol{G})\frac{\boldsymbol{U}^{n+1}-\boldsymbol{U}^n}{\Delta t} + (\boldsymbol{H}+\boldsymbol{T})\frac{\boldsymbol{U}^{n+1}+\boldsymbol{U}^n}{2} + \\
&+ \int_{\Gamma} (\beta+1)\Phi^m \frac{(u^m u_x)^{n+1} + (u^m u_x)^n}{2}\,\mathrm{d}\Gamma \\
&- \int_{\Gamma} \beta\Phi \frac{(u_x u_{xx})^{n+1} + (u_x u_{xx})^n}{2}\,\mathrm{d}\Gamma \\
&- \int_{\Gamma} \Phi \frac{(u u_{xxx})^{n+1} + (u u_{xxx})^n}{2}\,\mathrm{d}\Gamma = \frac{\boldsymbol{Q}^{n+1}+\boldsymbol{Q}^n}{2}, \quad (49)
\end{aligned}$$

where is the time of step and

$$\boldsymbol{U}^n = \boldsymbol{U}(n\Delta t) = (u_1(n\Delta t), u_2(n\Delta t), \ldots, u_n(n\Delta t)). \quad (50)$$

The nonlinear term $(uu_x)^{n+1}$ and $(u^2u_x)^{n+1}$ in the above Eq. (49) are linearized by using the following term

$$(uv)^{n+1} = u^{n+1}v^n + u^n v^{n+1} - u^n v^n, \quad (51)$$

then we obtain

$$(u_x u_{xx})^{n+1} = u_x^{n+1}u_{xx}^n + u_x^n u_{xx}^{n+1} - u_x^n u_{xx}^n, \quad (52)$$

$$(u^2 u_x)^{n+1} = 2u^{n+1}u^n u_x^n + u_x^{n+1}u^n u^n - 2u^n u^n u_x^n, \quad (53)$$

$$(uu_{xxx})^{n+1} = u^{n+1}u_{xxx}^n + u^n u_{xxx}^{n+1} - u^n u_{xxx}^n. \quad (54)$$

Substituting Eqs. (50)–(54) into Eq. (49) yields

$$\begin{aligned}
&(\boldsymbol{K}+\boldsymbol{G})\frac{\boldsymbol{U}^{n+1}-\boldsymbol{U}^n}{\Delta t} + (\boldsymbol{H}+\boldsymbol{T})\frac{\boldsymbol{U}^{n+1}+\boldsymbol{U}^n}{2} \\
&+ \int_{\Gamma} (\beta+1)\Phi^m \frac{(\boldsymbol{U}^n)^m(2\Phi_x\Phi^{\mathrm{T}} + \Phi\Phi_x^{\mathrm{T}})}{2}\,\mathrm{d}\Gamma\cdot\boldsymbol{U}^{n+1} \\
&- \int_{\Gamma} \beta\Phi \frac{U^n(\Phi_{xx}\Phi_x^{\mathrm{T}} + \Phi_x\Phi_{xx}^{\mathrm{T}})}{2}\,\mathrm{d}\Gamma\cdot U^{n+1} \\
&- \int_{\Gamma} \Phi \frac{\boldsymbol{U}^n(\Phi_{xxx}\Phi^{\mathrm{T}} + \Phi\Phi_{xxx}^{\mathrm{T}})}{2}\,\mathrm{d}\Gamma\cdot\boldsymbol{U}^{n+1} \\
&= \frac{\boldsymbol{Q}^{n+1}+\boldsymbol{Q}^n}{2}. \quad (55)
\end{aligned}$$

By solving the above iteration equation, we can obtain the numerical solution of the generalized CH equation.

4. Numerical examples

The application of the IEFG method to the numerical solution of the generalized CH equation is demonstrated in this section by several examples, which are chosen because their exact solutions are known. A regular arrangement of nodes and the background meshes of cells are used for numerical integrations to calculate the system equation. The linear basis and the cubic spline weight function are used in the IEFG method.

4.1. Example 1: modified Degaperis–Procesi (DP) equation

When $k = 0$, $\beta = 3$, and $m = 2$ in Eq. (27), the generalized CH equation reduces into the modified DP equation in the following form

$$u_t - u_{xxt} + 4u^2u_x - 3u_x u_{xx} - uu_{xxx} = 0, \quad (a < x < b) \quad (56)$$

with initial condition

$$u(x,0) = -\frac{15}{8}\,\mathrm{sech}^2\left(\frac{x}{2}\right) \quad (57)$$

and boundary conditions

$$u(a,t) = -\frac{15}{8}\operatorname{sech}^2\left(\frac{a}{2} - \frac{5}{4}t\right), \tag{58}$$

$$u(b,t) = -\frac{15}{8}\operatorname{sech}^2\left(\frac{b}{2} - \frac{5}{4}t\right). \tag{59}$$

The exact solution is

$$u(x,t) = -\frac{15}{8}\operatorname{sech}^2\left(\frac{x}{2} - \frac{5}{4}t\right). \tag{60}$$

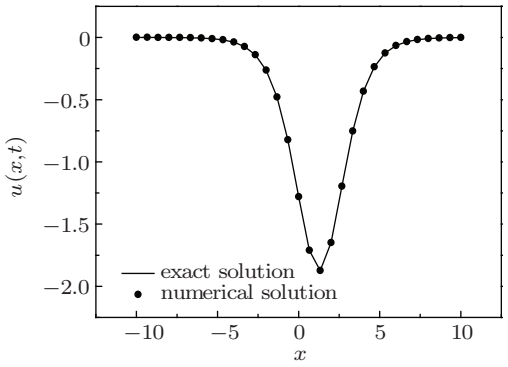


Fig. 1. Numerical solution and exact solution of $u(x,t)$ at $t = 0.5$ (Example 1).

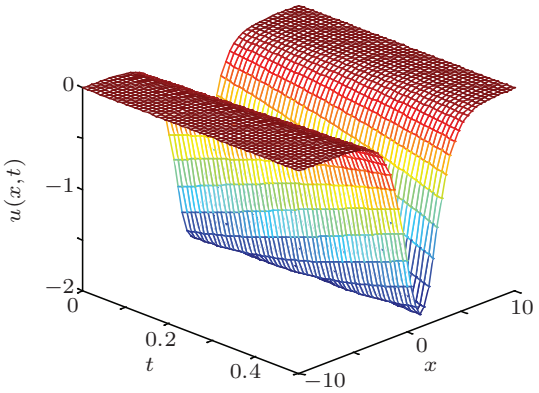


Fig. 2. (color online) Surface of the exact solution with the IEFG method (Example 1).

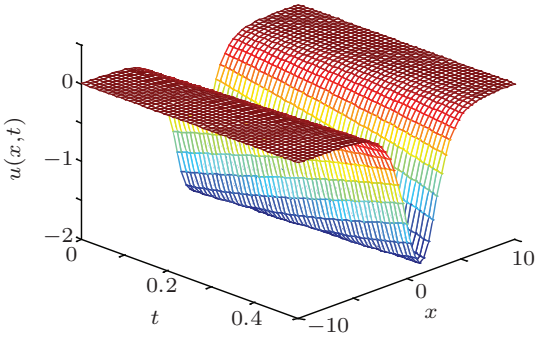


Fig. 3. (color online) Surface of the numerical solution with the IEFG method (Example 1).

The IEFG method is used to solve the above equation in steps of $\Delta t = 0.001$ and the penalty factor $\alpha = 10^5$, and $d_{\max} = 2.5$, $a = -10$, $b = 10$. In Fig. 1, the numerical solution and exact solution at time $t = 0.5$ are plotted. The surfaces of the numerical solution with the IEFG method and exact solution are plotted in Figs. 2 and 3. It shows that the IEFG method works well.

4.2. Example 2: CH equation

When $k = 0$, $\beta = 2$, and $m = 1$ in Eq. (27), the generalized CH equation reduces into the CH equation in the following form

$$u_t - u_{xxt} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0, \quad (a < x < b) \tag{61}$$

with initial condition

$$u(x,t) = c\,\mathrm{e}^{-|x|} \tag{62}$$

and boundary conditions

$$u(a,t) = c\,\mathrm{e}^{-|a-ct|}, \tag{63}$$

$$u(b,t) = c\,\mathrm{e}^{-|b-ct|}. \tag{64}$$

The exact solution is

$$u(x,t) = c\,\mathrm{e}^{-|x-ct|}. \tag{65}$$

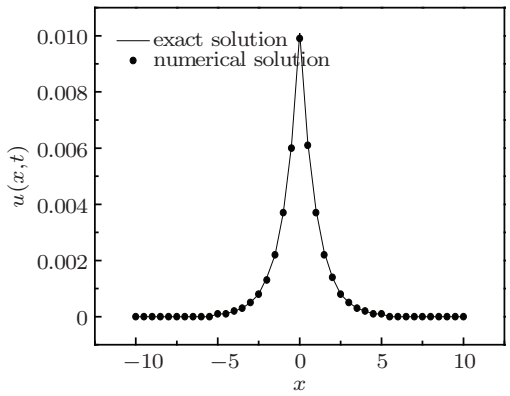


Fig. 4. Numerical solution and exact solution of $u(x,t)$ at $t = 0.5$ (Example 2).

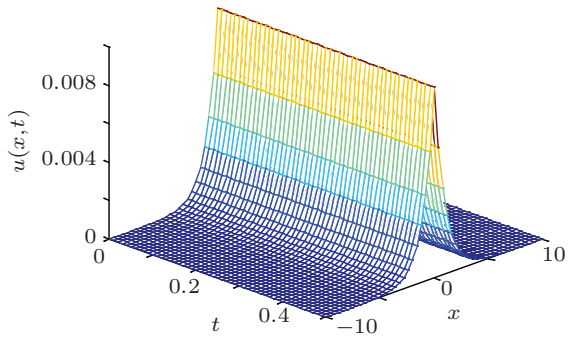


Fig. 5. (color online) Surface of the exact solution with the IEFG method (Example 2).

The IEFG method is used to solve the above equation in steps of $\Delta t = 0.001$ and the penalty factor $\alpha = 10^5$ and

$d_{\max} = 2.7$, and $a = -10$, $b = 10$. In Fig. 4, the numerical solution and exact solution at time $t = 0.5$ are plotted. The surfaces of the numerical solution with the IEFG method and exact solution are plotted in Figs. 5 and 6. It shows that the IEFG method works well.

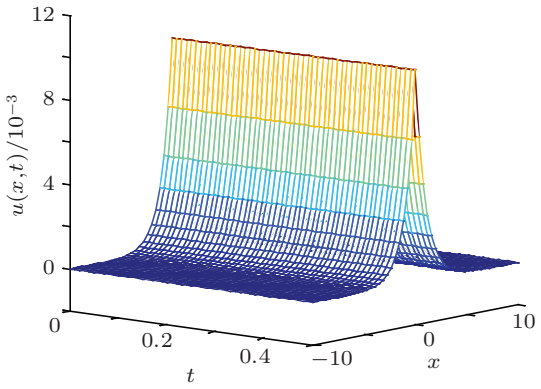


Fig. 6. (color online) Surface of the numerical solution with the IEFG method (Example 2).

These figures show that numerical results obtained by using the IEFG to solve the generalized CH equation are in good agreement with those from the exact solutions.

5. Conclusion

In this paper, the generalized CH equation is investigated using the IEFG method. A system of nonlinear discrete equations is obtained through the variational method. The numerical results which are in good agreement with the exact solutions show that the technique is accurate and efficient. A mesh-free method does not require a mesh to discretize the domain of the problem under consideration, and the approximate solution is constructed entirely based on a set of scattered nodes. This method does not require mesh generation, which makes them advantageous for problems that require frequent re-meshing, such as those arising in nonlinear analysis. Due to the simplicity of its implementation, the proposed IEFG method promises to be a potential alternative to the difference method and the finite element method for solving the non-linear generalized CH equation. The IEFG method should be extended to solve the problems of nonlinear partial differential equations which originates from the theory of solitons and other areas.

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