A double optical solitary wave in a nonlinear Schrödinger-type equation*

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A qualitative analysis method to efficiently solve the shallow wave equations is improved, so that a more complicated nonlinear Schrödinger equation can be considered. By using the detailed study, some quite strange optical solitary waves are obtained in which the bright and dark optical solitary waves are allowed to coexist.

Keywords: nonlinear Schrödinger equation, optical wave, exotic wave

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1. Introduction

Optical solitary waves, temporal and spatial solitons, have been the subject of intense theoretical and experimental studies in recent years. In nonlinear optics, the main nonlinear equation governing the pulse evolution is the famous nonlinear Schrödinger (NLS) equation which, depending on the sign of the group-velocity dispersion, has two distinct types of localized solutions, bright or dark solitons.

Optical solitary waves have been the subject of intense current research, which is motivated by their important applications in the areas of high-capacity fiber telecommunications and all-optical switches due to their capability of propagating over a long distance without attenuation and changing their shapes. Much interest is aroused in exotic optical solitary waves for their transmitions of signals. Some optical solitary waves have been found in many models.[1–4]

It is reported in Ref. [5] that one type of the Schrödinger equation

$$\mathrm{i}\partial\psi + \frac{\partial_x^2}{1-\mu\partial_x^2}\psi + \psi|\psi|^2 - \frac{\beta}{2+\alpha}(2\psi\partial_x^2|\psi|^2 + \alpha\psi^*\partial_x^2\psi^2) = 0 \qquad (1)$$

admits abundant exotic optical solitary waves such as compactons, peakons, cuspons, kinkons, and tipons. Here ψ is the complex amplitude of the propagating field in the waveguide and the asterisk $(*)$ denotes complex conjugation. The variable t represents the distance along the direction of propagation. μ, α, and β are rescaled coupling constants which depend on overlap integrals of the modes in the waveguides. Equation (1) can be regarded as a model for coupled optical waveguides. Some special cases of Eq. (1) have previously been investigated. The case $\mu = \alpha = 0$ appeared in the context of Heisenberg ferromagnets in Ref. [6] and the case β was studied as a model of the nonlocal dispersion effect on self-interaction in Ref. [7].

Owing to many strange waves and wide applications, it is reasonable to think that there should be other undiscovered exotic phenomena. We will improve the work in shallow wave equations[8–10] to study the structures of the optical solitary waves of Eq. (1).

The method in Refs. [8]–[10] can deal with the existence of solutions to a type of nonlinear partial differential equation. After integration, this equation has a term like $P(\varphi) = \varphi^2(b\varphi^2 + c\varphi + d)$. Solutions can be determined by analyzing the zeros of $P(\varphi)$. There are two different aspects in our method. One is the polynomial form $P(\varphi) = \varphi^2(a\varphi^3 + b\varphi^2 + c\varphi + d)$ and the more complicated situation of zeros, and the other is two singular points existing in the integral form while only one singular point appears in those models.[8–10] We know that a singular point may correspond to a nonsmooth point in the solution, while two singular points can lead to a solution with two nonsmooth points. So the previous method in Refs. [8]–[10] does not work well. We solve the problem by checking the signs of some function values at selected points. By the improved method, we obtain a very detailed classification of travelling waves in Eq. (1).

2. Qualitative behavior

In this section, in order to better solve Eq. (1), we study the first-order integral form $\phi_x^2 = F(\phi)$ and the qualitative behavior near the zeros or poles.

To find approximate solutions to Eq. (1) by analytic methods, one can multiply with $1-\mu\partial_x^2$ and neglect all fourth-order nonlinear terms. Note that the fourth-order terms vanish identically when $\mu = 0$ or $\beta = 0$ and thus the results give exact solutions to Eq. (1) for these limiting cases.[1]

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If making an ansatz of real stationary solutions in a rotating frame of reference ($\Lambda > 0$) of the form ($\beta \geq 0$)

$$\psi(x,t) = \left(\frac{\Lambda\mu+1}{3\mu+2\beta}\right)^{1/2} \phi(\xi)\exp(\mathrm{i}\Lambda t), \quad \xi = \frac{x}{\sqrt{6\mu+2\beta}}.$$

Equation (1) becomes an ordinary differential equation for the envelope function $\phi(\xi)$ as expressed below,

$$-\gamma\phi + \phi^3 + \frac{1}{\varepsilon-1}(1-\phi^2)\phi_{\xi\xi} - \phi\phi_\xi^2 = 0, \tag{2}$$

where γ is the effective spectral parameter and $(\varepsilon-1)^{-1}$ is the effective strength of the nonlinear dispersion with

$$\gamma = \frac{(3\mu+2\beta)\Lambda}{\Lambda\mu+1}, \quad \varepsilon = \frac{9\mu+4\beta}{3\mu+2\beta}.$$

In this section we consider the case $\mu = 0$, and give the classification of travelling wave solutions of Eq. (1).

By integrating Eq. (2) with respect to ξ, we obtain

$$\phi_\xi^2 = \frac{\phi^2(\phi^2-\gamma)/2 - 2h}{\phi^2-1}. \tag{3}$$

For convenience, we define $P(\phi)$ as $P(\phi) = \phi^2(\phi^2-\gamma)/2 + h$ and $P'(\phi) = \phi(2\phi^2-\gamma)$.

We point out that the following facts have a close relationship with the formation of solutions to Eq. (2).

(i) If $F(\phi)$ has a simple zero at $\phi = \phi_0$, the solution ϕ to Eq. (3) satisfies $\phi(\xi) \sim m + (\xi-\xi_0)^2$, where $\phi(\xi_0) = \phi_0$. It is easy to find that periodic smooth solutions exist if $F(\phi)$ has two simple zeros.

(ii) If $F(\phi)$ has a double zero at $\phi = \phi_0$, the solution ϕ of Eq. (3) satisfies $\phi(\xi) \sim m + \exp(-|\xi|)$ as $\xi \to \infty$. It is easy to find that smooth solitary wave solutions exist if $F(\phi)$ has a simple zero and a double zero.

(iii) If $F(\phi)$ has a double pole at $\phi = \phi_0$, we obtain

$$\varphi_x(\xi) \sim \begin{cases} |\xi-\xi_0|^{-1/3}, & \xi \downarrow \xi_0, \\ -|\xi-\xi_0|^{-1/3}, & \xi \uparrow \xi_0, \end{cases}$$

where $\phi(\xi_0) = c$ and η is a constant. It is easy to find that looped wave solutions exist if $F(\phi)$ has a pole lying between the maximum value and the minimum value of the wave. Cusped wave solutions exist if this pole is just the maximum value or the minimum value of the wave.

(iv) Peaked waves appear when the solution ϕ according to Eq. (3) suddenly changes direction at $\phi = c$ where $F(0) \neq 0$ and $\varphi(\xi_0) = 0$, that is, $\lim\limits_{\xi\uparrow\xi_0} \phi_\xi(\xi) = -\lim\limits_{\xi\downarrow\xi_0} \phi_\xi(\xi) \neq \pm\infty$.

From the above facts, it can be found that formation of solutions to Eq. (3) is related to the distributions of the singular points and zeros of $F(\phi)$ which are given in the different parameter regions (see Figs. 1–6).

Case 1 $\gamma \leq 0$

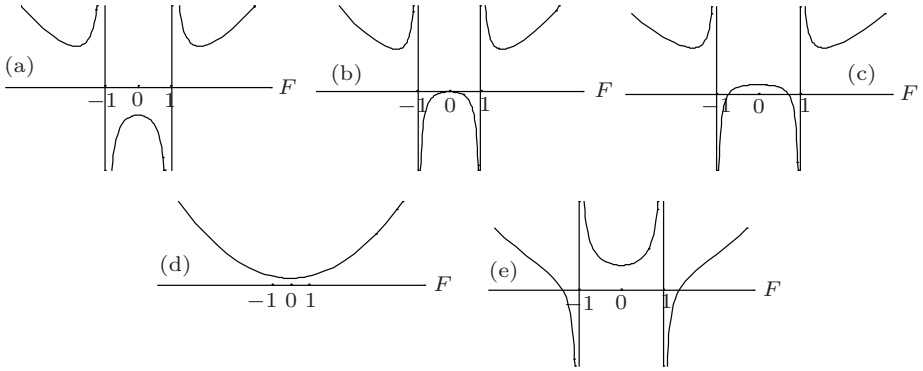


Fig. 1. Graphs of the polynomial $F(\phi)$ as $\gamma \leq 0$: (a) $h < 0$, (b) $h < 0$, (c) $0 < h < (1-\gamma)/4$, (d) $h = (1-\gamma)/4$, (e) $h > (1-\gamma)/4$.

Case 2 $0 < \gamma < 1$

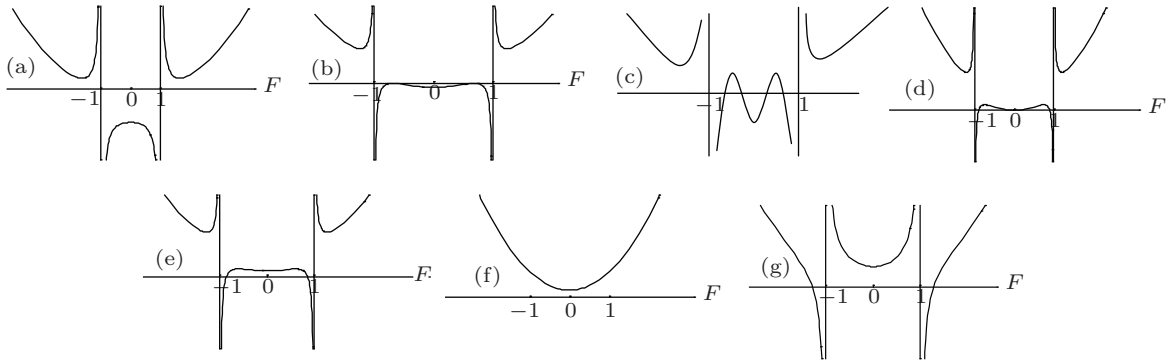


Fig. 2. Graphs of the polynomial $F(\phi)$ as $0 < \gamma < 1$: (a) $h < -\gamma^2/16$, (b) $h = -\gamma^2/16$, (c) $-\gamma^2/16 < h < 0$, (d) $h = 0$, (e) $0 < h < (1-\gamma)/4$, (f) $h = (1-\gamma)/4$, (g) $h > (1-\gamma)/4$.

Case 3 $\gamma = 1$

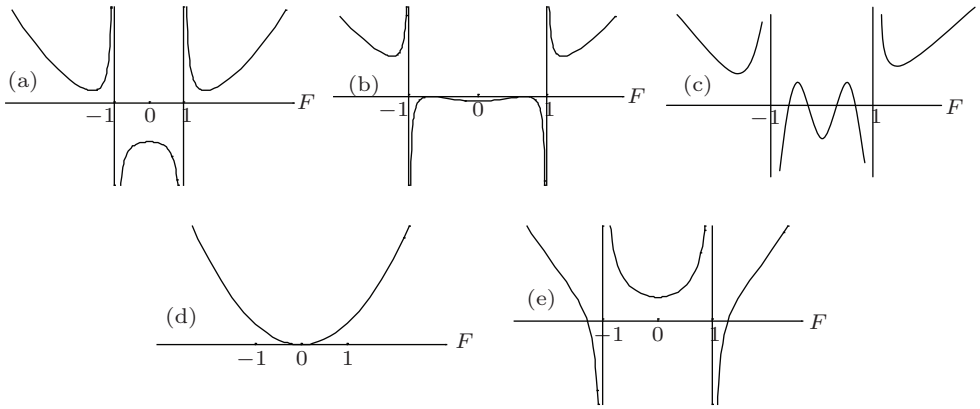


Fig. 3. Graphs of the polynomial $F(\phi)$ as $\gamma = 1$: (a) $h < -\gamma^2/16$, (b) $h = -\gamma^2/16$, (c) $-\gamma^2/16 < h < 0$, (d) $h = 0$, (e) $h > 0$.

Case 4 $1 < \gamma < 2$

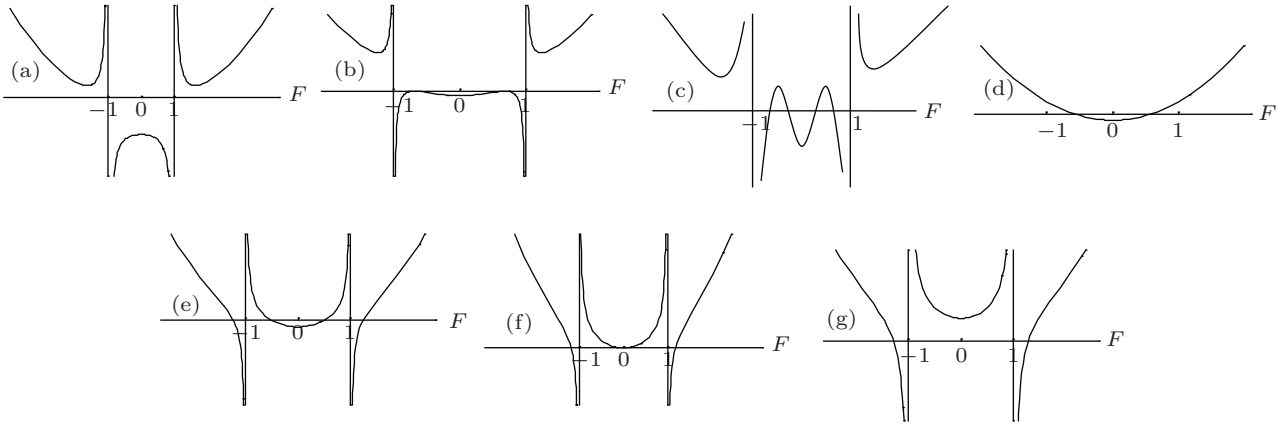


Fig. 4. Graphs of the polynomial $F(\phi)$ as $1 < \gamma < 2$: (a) $h < -\gamma^2/16$, (b) $h = -\gamma^2/16$, (c) $-\gamma^2/16 < h < (1-\gamma)/4$, (d) $h = (1-\gamma)/4$, (e) $(1-\gamma)/4 < h < 0$, (f) $h = 0$. (g) $h > 0$.

Case 5 $\gamma = 2$

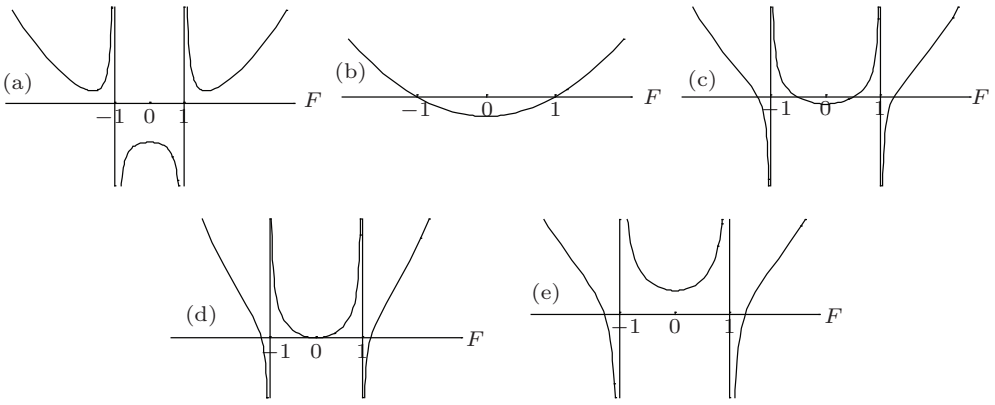


Fig. 5. Graphs of the polynomial $F(\phi)$ as $\gamma = 2$: (a) $h < -\gamma^2/16$, (b) $h = -\gamma^2/16$, (c) $-\gamma^2/16 < h < 0$, (d) $h = 0$, (e) $h > 0$.

Case 6 $\gamma > 2$

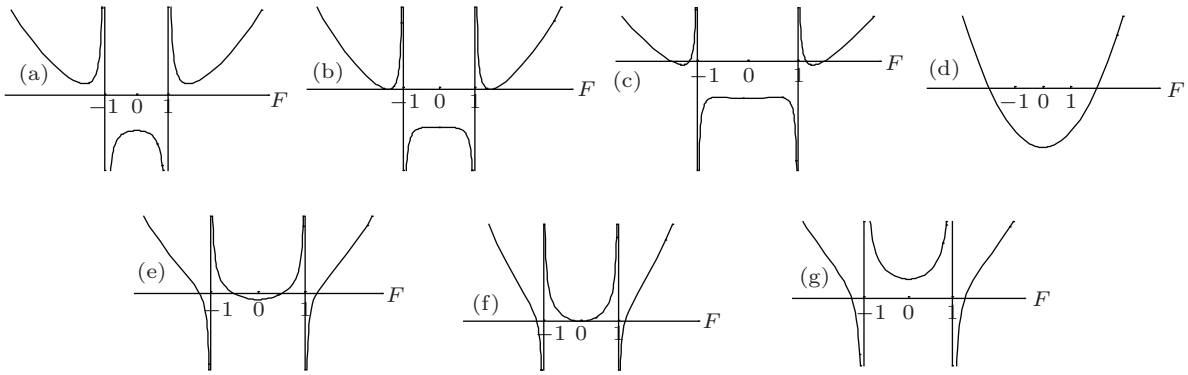


Fig. 6. Graph of the polynomial $F(\phi)$ as $\gamma > 2$: (a) $h < -\gamma^2/16$, (b) $h = -\gamma^2/16$, (c) $-\gamma^2/16 < h < (1-\gamma)/4$, (d) $h = (1-\gamma)/4$, (e) $(1-\gamma)/4 < h < 0$, (f) $h = 0$, (g) $h > 0$.

3. Double optical solitary wave

Using the results in Section 2, we can obtain abundant optical solitary waves (see Fig. 7) and the corresponding parameter regions. Due to the existence of two singular points, our equation has many exotic optical waves with two nonsmooth points at $\phi = \pm 1$.

(I) If $\gamma \leq 0$, we obtain that: for $h \leq 0$, there are no boundary wave solutions; for $0 < h < (1-\gamma)/4$, there is a smooth solitary wave solution; for $h = (1-\gamma)/4$, there is a double peaked periodic wave solution; for $h > (1-\gamma)/4$, there is a double cusped periodic wave solution.

(II) If $0 < \gamma < 1$, we obtain that: for $h \leq -\gamma^2/16$, there are no boundary wave solutions; for $-\gamma^2/16 < h < 0$, there are smooth periodic wave solutions; for $h = 0$, there is a double smooth solitary wave solution; for $0 < h < (1-\gamma)/4$, there is a smooth periodic wave solution; for $h = (1-\gamma)/4$, there is a double peaked wave solution; for $h > (1-\gamma)/4$, there is a double cusped periodic wave solution.

(III) If $\gamma = 1$, we obtain that: for $h \leq -\gamma^2/16$, there are no boundary wave solutions; for $-\gamma^2/16 < h < 0$, there are smooth periodic wave solutions; for $h = 0$, there is a double peaked solitary wave solution; for $h > 0$, there is a double cusped periodic wave solution.

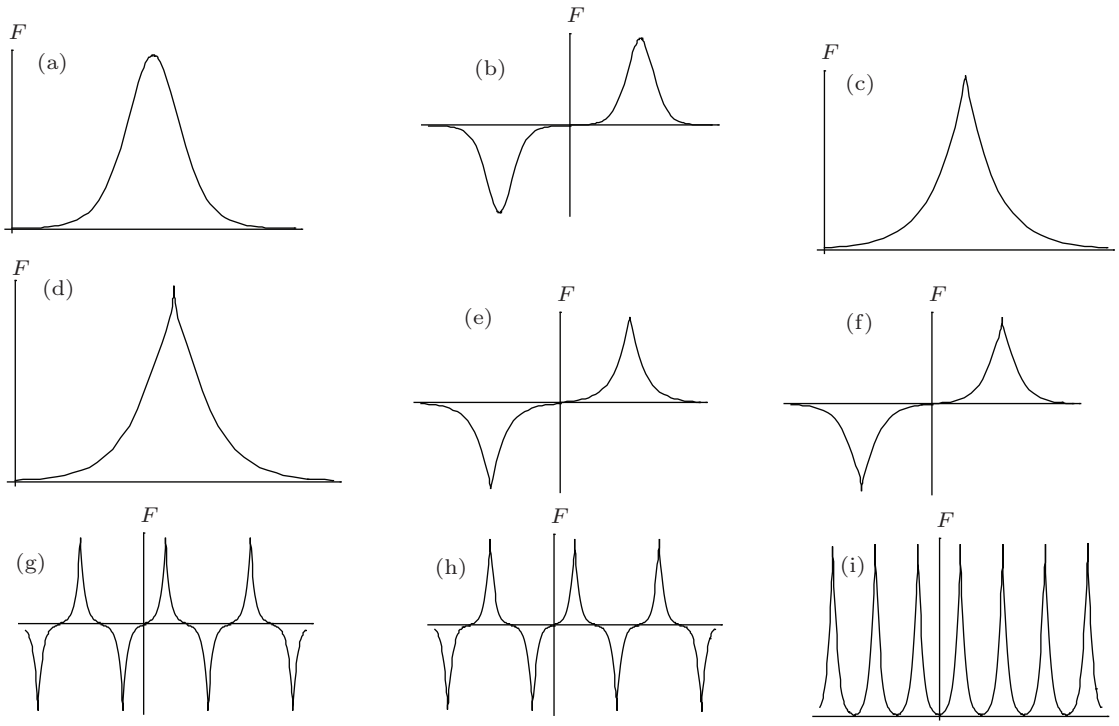


Fig. 7. Travelling waves in Eq. (1): (a) smooth solitary waves, (b) double smooth solitary waves, (c) peaked solitary waves, (d) cusped solitary waves, (e) double peaked solitary waves, (f) double cusped solitary waves, (g) double peaked periodic waves, (h) double cusped periodic waves, (i) cusped periodic waves.

(IV) If $1 < \gamma < 2$, we obtain that for $h \leq -\gamma^2/16$, there are no boundary wave solutions; for $-\gamma^2/16 < h < (1-\gamma)/4$, there are smooth periodic wave solutions; for $h = (1-\gamma)/4$, there are smooth periodic wave solutions; $(1-\gamma)/4 < h < 0$, there are cusped periodic wave solutions; for $h = 0$, there is a double cusped solitary wave solution; for $h > 0$, there is a double peaked periodic wave solution.

(V) If $\gamma = 2$, we obtain that: for $h \leq -\gamma^2/16$, there are no boundary wave solutions; for $-\gamma^2/16 < h < 0$, there are peaked solitary wave solutions; for $h = 0$, there is a double cusped solitary wave solution; for $h > 0$, there is a double cusped periodic wave solution.

(VI) If $\gamma > 2$, we obtain that: for $h < -\gamma^2/16$, there are no boundary wave solutions; for $h = -\gamma^2/16$, there are cusped solitary wave solutions; for $-\gamma^2/16 < h < (1-\gamma)/4$, there are cusped solitary wave solutions; for $h = (1-\gamma)/4$, there are no boundary wave solutions; for $(1-\gamma)/4 < h < 0$, there are peaked solitary wave solutions; for $h = 0$, there is a double cusped solitary wave solution; for $h > 0$, there is a double cusped periodic wave solution.

4. Conclusions

By an improved method, some strange double optical waves in a nonlinear Schrödinger-type equation are obtained. Those strange solutions include double smooth solitary waves, double cusped waves, double peaked waves, double peaked periodic waves, and double cusped periodic waves. Under different parametric conditions, various sufficient conditions to guarantee the existence of the above solutions are given. It can be widely applied to other nonlinear wave equations. Our study may be useful for further understanding the role that the nonlinearly dispersive terms play in optical wave solutions.

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