Singular solitons and other solutions to a couple of nonlinear wave equations

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This paper addresses the extended (G'/G)-expansion method and applies it to a couple of nonlinear wave equations. These equations are modified the Benjamin–Bona–Mahoney equation and the Boussinesq equation. This extended method reveals several solutions to these equations. Additionally, the singular soliton solutions are revealed, for these two equations, with the aid of the ansatz method.

Keywords: (G'/G)-expansion method, travelling wave solutions, singular soliton, mBBM and Boussinesq equations

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1. Introduction

The study of nonlinear evolution equations (NLEEs) is very important in various areas of applied mathematics and theoretical physics. These NLEEs form the fabric of research in these areas of study. There are various issues to address in the area of NLEEs. They are the integrability issues, conservation laws, stochasticity aspects, Bäcklund transform, and several others. This paper is going to focus on one of the important issue of NLEEs. It is the integrability aspect. There is a plethora of integration tools available in order to retrieve solutions to these NLEEs. However, one must exercise caution before the application of these tools. These multiple forms of integration architecture lead to a profound solution spectrum. This is a true blessing to various NLEEs.

The investigation of exact and numerical solutions, in particular travelling wave solutions, of NLEEs plays an important role in soliton theory.[1,2] Soliton phenomena are explained with nonlinear equations. These equations are mathematical models of complex physical phenomena that appear in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, and geochemistry. Exact solutions of NLEEs may provide much physical information and help one to understand the mechanism that governs these physical models.

Recently interest has increased in the travelling wave solutions of nonlinear evolution equations. There are many methods to solve these equations in literature. Some of them are: inverse scattering transform,[3] thehomogeneous balance method,[4–6] the Jacobi elliptic function method,[7] Hirota's bilinear method,[8–10] the sine–cosine method,[11,12] Cole Hopf transformation,[13] the F-expansion method,[14–21] and the auxiliary equation method.[22–27]

More recently, an extended (G'/G)-expansion method has been suggested to look for the travelling wave solutions of NLEEs. By using the extended (G'/G)-expansion method, some nonlinear equations[28–31] have been succesfully solved.

Some of the integration tools that are very commonly applied to extract these several solutions structures are the Adomian decomposition method, the variational iteration method, the (G'/G)-expansion method, the exp-function method, the first integral method, the simplest equation method, and several others. This paper is going to address an extended version of the (G'/G)-expansion method that will be applied to a couple of NLEEs. They are the modified Benjamin–Bona–Mahoney (mBBM) equation and the Boussinesq equation (BE). This integration architecture will be first spelled out in this paper and then applied to these two NLEEs. Several solutions will be revealed. Subsequently the ansatz method will be applied to the same two equations and singular 1-soliton solutions will be revealed along with constraint conditions that will be deemed necessary in order for the singular soliton solutions to exist.[32–35]

2. Extended (G'/G)-expansion method

This section will address this new version of the (G'/G)-expansion method that is also known as the extended (G'/G)-expansion method. The study will be split into two subsections. The first subsection will describe this method in a succinct form. Subsequently, it will be applied to a couple of NLEEs.

2.1. Brief description of the method

Suppose that a nonlinear equation, say in two independent variables x and t, is given by

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0, \tag{1}$$

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where $u = u(x,t)$ is an unknown complex-valued function, P is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.

The main steps of the extended (G'/G)-expansion method are the following:

Step 1 By taking $u(x,t) = u(\xi)$, $\xi = x - vt$, equation (1) can be converted to an ordinary differential equation (ODE)

$$P(u, -vu', u', v^2u'', -vu'', u'', \ldots) = 0. \tag{2}$$

Step 2 Suppose that the solution of ODE (2) can be expressed by a polynomial in (G'/G) as follows:

$$u(\xi) = a_0 + \sum_{i=1}^{m} a_i \left(\frac{G'}{G}\right)^i + b_i \left(\frac{G'}{G}\right)^{-i}, \tag{3}$$

where $G = G(\xi)$ satisfies the second-order linear differential equation in the form

$$G'' + \lambda G' + \mu G = 0, \tag{4}$$

where a_0, a_i, b_i, v, λ, and μ are constants to be determined later. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in ODE (2).

Step 3 Substituting Eq. (3) into Eq. (2) and using Eq. (4), collecting all terms with the same power of (G'/G) together and then equating the coefficients of each power of (G'/G) to zero gives a system of algebraic equations for a_0, a_i, b_i, v, λ, and μ.

Step 4 These constants are written in their places in Eq. (3) and if the solutions of the second-order linear ordinary differential equation (4) are used, the soliton solutions of nonlinear partial differential equation (1) are obtained.

2.2. Application to a couple of NLEEs

This subsection will apply the extended version of the (G'/G)-expansion to a couple of NLEEs. They are the modified BBM equation and the BE. These will be studied in detail in the following subsections.

2.2.1. Modified BBM equation

We consider the mBBM equation in the form

$$u_t + \alpha u_x + \beta u^2 u_x - \gamma u_{xxt} = 0. \tag{5}$$

To look for the travelling wave solution of Eq. (5), we make the transformation $u(x,t) = u(\xi)$, $\xi = x - vt$, where v is the wave speed to be determined later, and integrating it with respect to ξ once yields

$$c - vu + \alpha u + \frac{\beta}{3}u^3 + v\gamma u'' = 0, \tag{6}$$

where c is an integration constant that is to be determined later.

Balancing the terms u^3 and u'' in Eq. (2) yields the leading order $m = 1$. Therefore, we can write the solution of Eq. (6) in the form

$$u(\xi) = a_0 + a_1\left(\frac{G'}{G}\right) + b_1\left(\frac{G'}{G}\right)^{-1}, \tag{7}$$

Substituting Eqs. (4) and (7) into Eq. (6), collecting all terms with the same power of (G'/G) and setting each coefficient to zero, we obtain the following system of algebraic equations

$$\begin{aligned}
&\left(\frac{G'}{G}\right)^{-3}: \ \frac{1}{3}b_1^3\beta + 2b_1v\gamma\mu^2 = 0,\\
&\left(\frac{G'}{G}\right)^{-2}: \ a_0b_1^2\beta + 3b_1v\gamma\lambda\mu = 0,\\
&\left(\frac{G'}{G}\right)^{-1}: \ -b_1v + b_1\alpha + a_0^2b_1\beta + a_1b_1^2\beta + b_1v\gamma\lambda^2\\
&\qquad + 2b_1v\gamma\mu = 0,\\
&\left(\frac{G'}{G}\right)^{0}: \ c - a_0v + a_0\lambda + \frac{1}{3}a_0^3\beta + 2a_0a_1b_1\beta + b_1v\gamma\lambda\\
&\qquad + a_1v\gamma\lambda\mu = 0,\\
&\left(\frac{G'}{G}\right)^{1}: \ -a_1v + a_1\alpha + a_0^2a_1\beta + a_1^2b_1\beta + a_1v\gamma\lambda^2\\
&\qquad + 2a_1v\gamma\mu = 0,\\
&\left(\frac{G'}{G}\right)^{2}: \ a_0a_1^2\beta + 3a_1v\gamma\lambda = 0,\\
&\left(\frac{G'}{G}\right)^{3}: \ \frac{1}{3}a_1^3\beta + 2a_1v\gamma = 0.
\end{aligned}$$

From the solution system, we obtain the following cases with the aid of Mathematica.

Case 1

$$\begin{aligned}
a_0 &= \frac{\lambda\sqrt{3\alpha\gamma}}{\sqrt{-2\beta - \beta\gamma\lambda^2 + 4\beta\gamma\mu}},\\
a_1 &= \frac{3(\alpha\gamma\lambda + a_0^2\beta\gamma\lambda)}{a_0\beta(-1 + \gamma\lambda^2 + 2\gamma\mu)}, \ b_1 = 0,\\
v &= \frac{1}{3}(3\alpha + 3a_0^2\beta - a_0a_1\beta\lambda - a_1^2\beta\mu),\\
c &= \frac{1}{3}(2a_0^3\beta + a_0\alpha\gamma\lambda^2 + a_0^3\beta\gamma\lambda^2 + a_1\alpha\gamma\lambda^3 - 4a_0\alpha\gamma\mu\\
&\quad - 4a_0^3\beta\gamma\mu + 2a_1\alpha\gamma\lambda\mu).
\end{aligned}$$

Case 2

$$\begin{aligned}
a_0 &= -i\sqrt{\frac{\alpha}{\beta}}, \quad a_1 = 0, \quad b_1 = \frac{3(\alpha\gamma\lambda\mu + a_0^2\beta\gamma\lambda\mu)}{a_0\beta(-1 + \gamma\lambda^2 + 2\gamma\mu)},\\
v &= \frac{-b_1^2\beta - a_0b_1\beta\lambda + 3\alpha\mu + 3a_0^2\beta\mu}{3\mu},\\
c &= \frac{1}{3}(3a_0v - 3a_0\alpha - a_0^3\beta - 3b_1v\gamma\lambda).
\end{aligned}$$

Case 3

$$\begin{aligned}
&a_0 \neq 0, \quad a_1 = \frac{2a_0}{\lambda}, \quad b_1 = a_1\mu,\\
&v = \frac{1}{3}(-a_0a_1\beta\lambda + 2a_1^2\beta\mu),\\
&c = \frac{1}{3}(-2a_0\alpha + a_1\alpha\lambda - 2a_0a_1^2\beta\mu).
\end{aligned}$$

For Case 1, we have the following solutions.

(i) When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function travelling wave solution

$$u(\xi) = \frac{\lambda\sqrt{3\alpha\gamma}}{\sqrt{-\beta(2+\gamma(\lambda^2-4\mu))}} + \frac{3(\alpha+a_0^2\beta)\gamma\lambda\left\{\left[-\lambda+\frac{\sqrt{\lambda^2-4\mu}(A\cosh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]+B\sinh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)])}{B\cosh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]+A\sinh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]}\right]\right\}}{2a_0\beta(-1+\gamma(\lambda^2+2\mu))}, \quad (8)$$

where $\xi = x-(3\alpha+3a_0^2\beta-a_0a_1\beta\lambda-a_1^2\beta\mu)t/3$, A and B are arbitrary constants.

(ii) When $\lambda^2-4\mu<0$, we obtain the trigonometric function travelling wave solution

$$u(\xi) = \frac{\lambda\sqrt{3\alpha\gamma}}{\sqrt{-\beta(2+\gamma(\lambda^2-4\mu))}} + \frac{3(\alpha+a_0^2\beta)\gamma\lambda\left\{\left[-\lambda+\frac{\sqrt{4\mu-\lambda^2}(B\cos[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]-A\sinh[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)])}{A\cos[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]+B\sin[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]}\right]\right\}}{2a_0\beta(-1+\gamma(\lambda^2+2\mu))}, \quad (9)$$

where $\xi = x-\frac{1}{3}(3\alpha+3a_0^2\beta-a_0a_1\beta\lambda-a_1^2\beta\mu)t$, A and B are arbitrary constants.

In particular, if we take $A=0$, $B\neq 0$, then the solutions (8) and (9) become

$$u(\xi) = \frac{\lambda\sqrt{3\alpha\gamma}}{\sqrt{-\beta(2+\gamma(\lambda^2-4\mu))}} - \frac{3(\alpha+a_0^2\beta)\gamma\lambda[\lambda-\sqrt{\lambda^2-4\mu}\tanh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]]}{2a_0\beta(-1+\gamma(\lambda^2+2\mu))}, \quad (10)$$

$$u(\xi) = \frac{\lambda\sqrt{3\alpha\gamma}}{\sqrt{-\beta(2+\gamma(\lambda^2-4\mu))}} - \frac{3(\alpha+a_0^2\beta)\gamma\lambda[\lambda-\sqrt{4\mu-\lambda^2}\cot[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]]}{2a_0\beta(-1+\gamma(\lambda^2+2\mu))}. \quad (11)$$

On the other hand, if we take $A\neq 0$, $B=0$, then solutions (8) and (9) become

$$u(\xi) = \frac{\lambda\sqrt{3\alpha\gamma}}{\sqrt{-\beta(2+\gamma(\lambda^2-4\mu))}} - \frac{3(\alpha+a_0^2\beta)\gamma\lambda[\lambda-\sqrt{\lambda^2-4\mu}\coth[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]]}{2a_0\beta(-1+\gamma(\lambda^2+2\mu))}, \quad (12)$$

$$u(\xi) = \frac{\lambda\sqrt{3\alpha\gamma}}{\sqrt{-\beta(2+\gamma(\lambda^2-4\mu))}} - \frac{3(\alpha+a_0^2\beta)\gamma\lambda[\lambda-\sqrt{4\mu-\lambda^2}\tan[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]]}{2a_0\beta(-1+\gamma(\lambda^2+2\mu))}. \quad (13)$$

For Case 2, we obtain the following solutions.

i) When $\lambda^2-4\mu>0$, we obtain the hyperbolic function travelling wave solution

$$u(\xi)=-\mathrm{i}\sqrt{\frac{\alpha}{\beta}}+\frac{6(\alpha+a_0^2\beta)\gamma\lambda\mu}{a_0\beta(\lambda-p\sqrt{\lambda^2-4\mu})(-1+\gamma(\lambda^2+2\mu))}, \quad (14)$$

where

$$p=\frac{A\cosh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]+B\sinh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]}{A\sinh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]+B\cosh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]},$$

$$\xi=x-\Big(\frac{-b_1^2\beta-a_0b_1\beta\lambda+3\alpha\mu+3a_0^2\beta\mu}{3\mu}\Big)t,$$

A and B are arbitrary constants.

ii) When $\lambda^2-4\mu<0$, we obtain the trigonometric function travelling wave solution

$$u(\xi)=-\mathrm{i}\sqrt{\frac{\alpha}{\beta}}-\frac{6(\alpha+a_0^2\beta)\gamma\lambda\mu}{a_0\beta(-1+\gamma(\lambda^2+2\mu))(-\lambda+q\sqrt{4\mu-\lambda^2})}, \quad (15)$$

where

$$q=\frac{-A\sin[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]+B\cos[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]}{A\cos[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]+B\sin[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]},$$

$$\xi=x-\Big(\frac{-b_1^2\beta-a_0b_1\beta\lambda+3\alpha\mu+3a_0^2\beta\mu}{3\mu}\Big)t,$$

A and B are arbitrary constants.

In particular, if we take $A=0$, $B\neq 0$, then solutions (14) and (15) become

$$u(\xi) = -\mathrm{i}\sqrt{\frac{\alpha}{\beta}} + \frac{6(\alpha+a_0^2\beta)\gamma\lambda\mu}{a_0\beta(-1+\gamma(\lambda^2+2\mu))(\lambda-\sqrt{\lambda^2-4\mu}\tanh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)])}, \quad (16)$$

$$u(\xi) = -\mathrm{i}\sqrt{\frac{\alpha}{\beta}} - \frac{6(\alpha+a_0^2\beta)\gamma\lambda\mu}{a_0\beta(-1+\gamma(\lambda^2+2\mu))(\lambda-\sqrt{4\mu-\lambda^2}\cot[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)])}. \quad (17)$$

On the other hand, if we take $A\neq 0$, $B=0$, then solutions (14) and (15) become

$$u(\xi) = -\mathrm{i}\sqrt{\frac{\alpha}{\beta}} - \frac{6(\alpha+a_0^2\beta)\gamma\lambda\mu}{a_0\beta(-1+\gamma(\lambda^2+2\mu))(\lambda-\sqrt{\lambda^2-4\mu}\coth[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)])}, \quad (18)$$

$$u(\xi) = -\mathrm{i}\sqrt{\frac{\alpha}{\beta}} - \frac{6(\alpha+a_0^2\beta)\gamma\lambda\mu}{a_0\beta(-1+\gamma(\lambda^2+2\mu))(\lambda+\sqrt{4\mu-\lambda^2}\tan[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)])}. \quad (19)$$

For Case 3, we gin the following solutions.

(I) When $\lambda^2-4\mu>0$, we obtain the hyperbolic function travelling wave solution

$$u(\xi)=\frac{-2a_1\lambda\mu+a_0p(-p\lambda^2+\lambda\sqrt{\lambda^2-4\mu}+4p\mu)}{\lambda(\lambda-p\sqrt{\lambda^2-4\mu})}, \quad (20)$$

where

$$p=\frac{A\cosh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]+B\sinh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]}{A\sinh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]+B\cosh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]},$$

$$\xi=x-\frac{1}{3}(-a_0a_1\beta\lambda+2a_1^2\beta\mu)t,$$

A and B are arbitrary constants.

(II) When $\lambda^2-4\mu<0$, we obtain the trigonometric function travelling wave solution

$$u(\xi)=\frac{-2a_1\lambda\mu+a_0q(q(\lambda^2-4\mu)+\lambda\sqrt{4\mu-\lambda^2})}{\lambda(\lambda-q\sqrt{4\mu-\lambda^2})}, \quad (21)$$

where

$$q=\frac{-A\sin[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]+B\cos[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]}{A\cos[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]+B\sin[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]},$$

$$\xi=x-\frac{1}{3}(-a_0a_1\beta\lambda+2a_1^2\beta\mu)t,$$

A and B are arbitrary constants.

In particular, if we take $A=0$, $B\neq0$, then the solutions (20) and (21) become

$$u(\xi)=\frac{a_0}{\lambda}\sqrt{\lambda^2-4\mu}\tanh\left[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)\right]-\frac{2a_1\mu}{\lambda-\sqrt{\lambda^2-4\mu}\coth\left[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)\right]}, \quad (22)$$

$$u(\xi)=\frac{1}{\lambda}\left\{a_0\sqrt{4\mu-\lambda^2}\cot\left[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)\right]\right\}-\frac{2a_1\mu}{\lambda-\sqrt{4\mu-\lambda^2}\cot\left[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)\right]}. \quad (23)$$

On the other hand, if we take $A\neq0$, $B=0$, then solutions (20) and (21) become

$$u(\xi)=\frac{a_0}{\lambda}\sqrt{\lambda^2-4\mu}\coth\left[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)\right]-\frac{2a_1\mu}{\lambda-\sqrt{\lambda^2-4\mu}\coth\left[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)\right]}, \quad (24)$$

$$u(\xi)=-\frac{1}{\lambda}(a_0\sqrt{4\mu-\lambda^2}\tan\left[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)\right])-\frac{2a_1\mu}{\lambda+\sqrt{4\mu-\lambda^2}\tan\left[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)\right]}. \quad (25)$$

Remark 1 Solutions (10)–(13), (16)–(19), and (22)–(25) are similar to Yusufoğlu and Bekir's[36] solutions obtained by using sine–cosine, tanh methods. However we observe that solutions (8), (9), (14), (15), (20), and (21) are presented here for the first time.

2.2.2. Boussinesq equation

We consider the Boussinesq equation in the form

$$u_{tt}-u_{xx}+(u^2)_{xx}+u_{xxxx}=0. \quad (26)$$

To look for the travelling wave solution of Eq. (26), we make the transformation $u(x,t)=u(\xi)$, $\xi=x-vt$, where v is the wave speed to be determined later, and integrating it with respect to ξ twice yields

$$c+v^2u-u+u^2+u''=0, \quad (27)$$

where c is an integration constant that is to be determined later.

Balancing the terms u^2 and u'' in Eq. (6) yields the leading order $m=2$. Therefore, we can write the solution of Eq. (27) in the form

$$u(\xi)=a_0+a_1\left(\frac{G'}{G}\right)+a_2\left(\frac{G'}{G}\right)^2+b_1\left(\frac{G'}{G}\right)^{-1}+b_2\left(\frac{G'}{G}\right)^{-2}. \quad (28)$$

Substituting Eqs. (4) and (28) into Eq. (27), collecting all terms with the same power of (G'/G) and setting each coefficient to zero, we obtain the following system of algebraic equations

$$\left(\frac{G'}{G}\right)^{-4}:\ b_2^2+6b_2\mu^2=0,$$

$$\left(\frac{G'}{G}\right)^{-3}:\ 2b_1b_2+10b_2\lambda\mu+2b_1\mu^2=0,$$

$$\left(\frac{G'}{G}\right)^{-2}:\ b_1^2-b_2+2a_0b_2+b_2v^2+4b_2\lambda^2+8b_2\mu+3b_1\lambda\mu=0,$$

$$\left(\frac{G'}{G}\right)^{-1}:\ -b_1+2a_0b_1+2a_1b_2+b_1v^2+6b_2\lambda+b_1\lambda^2+2b_1\mu=0,$$

$$\left(\frac{G'}{G}\right)^{0}:\ -a_0+a_0^2+2a_1b_1+2b_2+2a_2b_2+c+a_0v^2+b_1\lambda+a_1\lambda\mu+2a_2\mu^2=0,$$

$$\left(\frac{G'}{G}\right)^{1}:\ -a_1+2a_0a_1+2a_2b_1+a_1v^2+a_1\lambda^2+2a_1\mu+6a_2\lambda\mu=0,$$

$$\left(\frac{G'}{G}\right)^{2}:\ a_1^2-a_2+2a_0a_2+a_2v^2+3a_1\lambda+4a_2\lambda^2+8a_2\mu=0,$$

$$\left(\frac{G'}{G}\right)^{3}:\ 2a_1+2a_1a_2+10a_2\lambda=0,$$

$$\left(\frac{G'}{G}\right)^{4}:\ 6a_2+a_2^2=0.$$

From the solution system, we obtain the following cases with the aid of Mathematica.

Case A

$$a_1=0,\quad a_2=0,\quad \mu\neq0,\quad b_1=-6\lambda\mu,\quad b_2=-6\mu^2,$$

$$v=\sqrt{\frac{b_1-2a_0b_1-b_1\lambda^2-2b_1\mu+36\lambda\mu^2}{b_1}},$$

$$c=a_0-a_0^2-a_0v^2-b_1\lambda+12\mu^2.$$

Case B

$$a_1=-6\lambda,\quad a_2=-6,\quad b_1=0,\quad b_2=0,$$
$$v=\sqrt{1-2a_0-\lambda^2-8\mu},$$
$$c=a_0^2+a_0\lambda^2+8a_0\mu+6\lambda^2\mu+12\mu^2.$$

For Case A, we have the following solutions.

(A) When $\lambda^2-4\mu>0$, we obtain the hyperbolic function travelling wave solution

$$u(\xi)=a_0+\frac{12(\lambda^2-p\lambda\sqrt{\lambda^2-4\mu}-2\mu)\mu}{(\lambda-p\sqrt{\lambda^2-4\mu})^2},\qquad(29)$$

where

$$p=\frac{A\cosh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]+B\sinh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]}{A\sinh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]+B\cosh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]},$$
$$\xi=x-\Big(\sqrt{\frac{b_1-2a_0b_1-b_1\lambda^2-2b_1\mu+36\lambda\mu^2}{b_1}}\Big)t,$$

A and B are arbitrary constants.

(B) When $\lambda^2-4\mu<0$, we obtain the trigonometric function travelling wave solution

$$u(\xi)=a_0+\frac{12\mu(\lambda^2-2\mu-q\lambda\sqrt{4\mu-\lambda^2})}{(\lambda-q\sqrt{4\mu-\lambda^2})^2},\qquad(30)$$

where

$$q=\frac{-A\sin[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]+B\cos[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]}{A\cos[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]+B\sin[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]},$$
$$\xi=x-\Big(\sqrt{\frac{b_1-2a_0b_1-b_1\lambda^2-2b_1\mu+36\lambda\mu^2}{b_1}}\Big)t,$$

A and B are arbitrary constants.

In particular, if we take $A=0$, $B\neq0$, then solutions (29) and (30) become

$$u(\xi)=a_0+\frac{12\mu(\lambda^2-2\mu-\lambda\sqrt{\lambda^2-4\mu}\tanh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)])}{(\lambda-\sqrt{\lambda^2-4\mu}\tanh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)])^2},\qquad(31)$$

$$u(\xi)=a_0+\frac{12\mu(\lambda^2-2\mu-\lambda\sqrt{4\mu-\lambda^2})\cot[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]}{(\lambda-\sqrt{4\mu-\lambda^2}\cot[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)])^2}.\qquad(32)$$

On the other hand, if we take $A\neq0$, $B=0$, then solutions (29) and (30) become

$$u(\xi)=a_0+\frac{12\mu(\lambda^2-2\mu-\lambda\sqrt{\lambda^2-4\mu}\coth[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)])}{(\lambda-\sqrt{\lambda^2-4\mu}\coth[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]^2},\qquad(33)$$

$$u(\xi)=a_0+\frac{12\mu(\lambda^2-2\mu+\lambda\sqrt{4\mu-\lambda^2})\tan[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]}{(\lambda+\sqrt{4\mu-\lambda^2}\tan[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)])^2}.\qquad(34)$$

For Case B, we gain the following solutions.

A) When $\lambda^2-4\mu>0$, we obtain the hyperbolic function travelling wave solution

$$u(\xi)=\frac{\{-(A^2-B^2)(a_0+3\lambda^2-6\mu)+(A^2+B^2)(a_0+6\mu)\cosh[\sqrt{\lambda^2-4\mu}(x-vt)]+2AB(a_0+6\mu)\sinh[\sqrt{\lambda^2-4\mu}(x-vt)]\}}{2\{B\cosh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]+A\sinh[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)]\}^2},\qquad(35)$$

where $\xi=x-(\sqrt{1-2a_0-\lambda^2-8\mu})t$, A and B are arbitrary constants.

B) When $\lambda^2-4\mu<0$, we obtain the trigonometric function travelling wave solution

$$u(\xi)=\frac{\{(A^2+B^2)(a_0+3\lambda^2-6\mu)+(A^2-B^2)(a_0+6\mu)\cos[\sqrt{4\mu-\lambda^2}(x-vt)]+2AB(a_0+6\mu)\sin[\sqrt{4\mu-\lambda^2}(x-vt)]\}}{2\{A\cos[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]+B\sin[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)]\}^2},\qquad(36)$$

where $\xi=x-(\sqrt{1-2a_0-\lambda^2-8\mu})t$, A and B are arbitrary constants.

In particular, if we take $A=0$, $B\neq0$, then solutions (35) and (36) become

$$u(\xi)=a_0+\frac{3\lambda^2}{2}-\frac{3}{2}(\lambda^2-4\mu)\tanh\Big[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)\Big]^2,\qquad(37)$$

$$u(\xi)=a_0+\frac{3\lambda^2}{2}+\frac{3}{2}(\lambda^2-4\mu)\cot\Big[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)\Big]^2.\qquad(38)$$

On the other hand, if we take $A\neq0$, $B=0$, then solutions (35) and (36) become

$$u(\xi)=a_0+\frac{3\lambda^2}{2}-\frac{3}{2}(\lambda^2-4\mu)\coth\Big[\frac{1}{2}\sqrt{\lambda^2-4\mu}(x-vt)\Big]^2,\qquad(39)$$

$$u(\xi)=a_0+\frac{3\lambda^2}{2}+\frac{3}{2}(\lambda^2-4\mu)\tan\Big[\frac{1}{2}\sqrt{4\mu-\lambda^2}(x-vt)\Big]^2.\qquad(40)$$

Remark 2 Solutions (31)–(34) and (37)–(40) are similar to Wazwaz's solutions obtained by the extended tanh method, the rational sinh method, the second rational cosh method, and the rational exponential functions method.[37] However solutions (29), (30), (35), and (36) are new solutions for the BE.

Remark 3 We have not given solutions of equations when $\lambda^2-4\mu=0$ because these solutions have not satisfied the original equations (5) and (6).

Remark 4 Solutions (8), (9), (14), (15), (20), (21), (29), (30), (35), and (36) cannot be obtained by the method in Ref. [38]. It shows that our method is more powerful in constructing exact solutions of NLEEs.

3. Singular solitons

This section will focus on obtaining the singular 1-soliton solution to the two NLEES by the ansatz method. The mathe-

matical technique is self-explanatory as seen in the following two subsections.

3.1. Modified BBM equation

In this subsection the following form of the mBBM equation will be solved

$$u_t + au_x + bu^2u_x + cu_{xxx} = 0, \tag{41}$$

where a, b, and c are arbitrary constants while $u(x,t)$ is the dependent variable that is dependent on the two independent variables x and t. These are designated as the spatial and temporal variables respectively. This subsection will focus on the ansatz method of solving Eq. (41) for a singular 1-soliton solution. To start off, the ansatz we select is given by

$$u(x,t) = A\,\mathrm{csch}^p\tau, \tag{42}$$

where

$$\tau = B(x - vt). \tag{43}$$

In Eqs. (42) and (43), A and B are free parameters while v is the velocity of the soliton. Substituting Eq. (42) into Eq. (41) leads to

$$(v - a + cp^2B^2)\mathrm{csch}^p\tau - bA^2\mathrm{csch}^{3p}\tau - c(p+1)(p+2)vB^2\,\mathrm{csch}^{p+2}\,\tau = 0. \tag{44}$$

By the aid of the balancing principle, equating the exponents $3p$ and $p+2$ gives

$$3p = p + 2, \tag{45}$$

that implies

$$p = 1. \tag{46}$$

Now, setting the coefficients of the linearly independent functions $\mathrm{csch}^{p+j}\tau$, for $j = 0, 2$ to zero gives

$$v = \frac{a}{1 + cB^2}, \tag{47}$$

and

$$A = B\sqrt{\frac{6ac}{b(1+cB^2)}}, \tag{48}$$

which poses the constraint condition

$$abc(1 + cB^2) > 0. \tag{49}$$

Hence finally the singular 1-soliton solution to Eq. (41) is given by

$$u(x,t) = A\,\mathrm{csch}[B(x - vt)], \tag{50}$$

where the velocity of the soliton is given by Eq. (47) and the free parameters A and B are connected as given in Eq. (48). The constraint condition given by Eq. (49) holds for the singular soliton to exist.

3.2. Boussinesq equation

This subsection will address the BE that is given by

$$u_{tt} - k^2u_{xx} + a(u^2)_{xx} + bu_{xxxx} = 0, \tag{51}$$

where, for Eq. (51), in addition to a and b, k is a real valued constant. The starting hypothesis stays the same as that in Eq. (42). Substituting the hypothesis into Eq. (51) leads to

$$\left[(v^2-k^2)pB + bp^3B^3\right]\mathrm{csch}^p\tau + 4apAB\,\mathrm{csch}^{2p}\tau + \{(v^2-k^2)(p+1)B + bB^3(p+1)[p^2+(p+2)^2]\}\mathrm{csch}^{p+2}\tau + 2a(2p+1)AB\mathrm{csch}^{2p+2}\tau + bB^3(p+1)(p+2)(p+3)\mathrm{csch}^{p+4}\tau = 0. \tag{52}$$

From Eq. (52), the balancing principle gives

$$2p = p + 2, \tag{53}$$

that shows

$$p = 2. \tag{54}$$

It needs to be noted that the same value of the exponent p is yielded when the exponents $2p+2$ and $p+4$ are equated. Again, as in the previous equation, setting the coefficients of the linearly independent functions $\mathrm{csch}^{p+j}\tau$, for $j = 0, 2, 4$ gives

$$B = \frac{1}{2}\sqrt{\frac{k^2 - v^2}{b}}, \tag{55}$$

and

$$A = \frac{3(v^2 - k^2)}{2a}. \tag{56}$$

The relation for the free parameter B introduces the constraint condition

$$b(v^2 - k^2) < 0. \tag{57}$$

Thus, finally, the singular 1-soliton solution to BE is given by

$$u(x,t) = A\,\mathrm{csch}^2[B(x - vt)], \tag{58}$$

where the free parameters A and B are related to the soliton velocity v as given by Eqs. (55) and (56) and the constraint condition (57) must remain valid in order for the singular 1-soliton solution to exist.

4. Conclusions

This paper studied a couple of NLEEs and they are the mBBM equation and the BE. These equations are integrated with the aid of two integration machineries. They are the extended version of the (G'/G)-expansion method and the ansatz method. While the first approach revealed a plethora of solutions to the governing NLEEs, the second approach only gave singular 1-soliton solution to the equations. The constraint conditions also naturally fell out of the results that must remain valid in order for the solutions to exist. These results are going to be very useful in fluid dynamics where these NLEEs are predominantly studied.

In our future studies, these results will be extended further. The perturbation terms are going to be inroduced and the soliton perturbation theory will be applied. Furthermore, the stochastic perturbation terms will be taken into consideration and the corresponding Langevin equation will give the mean free velocity of the soliton. Additionally, the perturbed BE and mBBM equations will be integrated by using these as well as several other integration tools. These results will be reported in future.

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