Solving the spin-weighted spheroidal wave equation*

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In this paper we solve spin-weighted spheroidal wave equations through super-symmetric quantum mechanics with a different expression of the super-potential. We use the shape invariance property to compute the "excited" eigenvalues and eigenfunctions. The results are beneficial to researchers for understanding the properties of the spin-weighted spheroidal wave more deeply, especially its integrability.

Keywords: spin-weighted spherical wave equation, supersymmetric quantum mechanics, shape invariance, recurrence relation

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1. Introduction

As an extension of the associated Legendre functions, spin-weighted spheroidal wave functions have extensive applications in many branches of physics and mathematics.[1–4] Their differential equations are expressed as follows:

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)+\left[E+s+\beta^2\omega^2\cos^2\theta-2s\beta\cos\theta-\frac{(m+s\cos\theta)^2}{\sin^2\theta}\right]\Theta=0,\quad 0<\theta<\pi, \tag{1}$$

with the boundary condition that Θ is finite at $\theta=0,\pi$.

It is well known that the special mathematical functions generally come from the differential equations with appropriate condition, the Sturm–Liouville problem. Here we use Eq. (1) with $\beta=s=0$ as an example. By the transformations $x=\cos\theta$ and $E=l(l+1)$, equation (1) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}\Theta}{\mathrm{d}x}\right]-\frac{m^2}{1-x^2}\Theta+(l+1)l\Theta=0,\quad -1<x<+1, \tag{2}$$

with the corresponding boundary condition that $\Theta\,|_{x=\pm1}$ is limited. The solutions to the Sturm–Liouville problem are associated Legendre's functions $\mathrm{P}_l^m(x)$. There are two ways to obtain the solutions of Eq. (2). One can directly solve it to obtain, the associated-Legendre functions

$$\mathrm{P}_l^m(x),\quad l=m,m+1,\ldots,\quad m=0,1,2,3,\ldots \tag{3}$$

Another way is to first obtain its "ground" state for $l=m$, that is, $\Theta_0=\mathrm{P}_m^m=(1-x^2)^{m/2}$, then try to get its recurrence relations[5]

$$\mathrm{P}_l^m(x)=[(l+m)(l-m)]^{-1/2}\left[-lx+(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\right]\mathrm{P}_{l-1}^m(x). \tag{4}$$

The "excited" states $\mathrm{P}_l^m, l>m$ are then achieved through the recurrence relation Eq. (4). Note that the meaning of "ground" and "excited" here is different from that in physics and here we use them only for getting analog of the results in suppersymmetric quantum mechanics (SUSYQM). We distinguish them by the value of l: the "state" is called "ground" when the condition $l=m$ is satisfied, while other "states" with $l>m$ are called "excited".

As a kind of special function, the spin-weighted spheroided wave function's recurrence relations are very important in both theoretical and practical backgrounds. Though there are some recurrence relations that have been given recently,[6–9] the kind of recurrence relations like that of Eq. (4) has not been given as yet.[10] So what we will try to do in this paper is to obtain the recurrence relations which are the extension of Eq. (4) for spin-weighted spherical wave (SWSW) functions.

The rest of this paper is arranged as follows. In Section 2, we will briefly introduce the SUSYQM method. Section 3 will give a summary of the study of the spin-weighted spheroidal wave function. Our main results will be given in Section 4.

2. A brief introduction of SUSYQM

The spin-weighted spheroidal wave functions were solved by the perturbation method in quantum physics,[1–3] but the recurrence relations were not touched by then. Recently, a new method of SUSYQM has been applied to study them and some new results were obtained. Here we give only a brief introduction of SUSYQM. In SUSYQM, the Schrödinger equation

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}+[V(x)-E]\psi=0 \tag{5}$$

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is solved by introducing the key concept of the super-potential W, and the "ground" function ψ_0 is obtained by[11]

$$\psi_0(x) = N\exp\left[-\int W\,\mathrm{d}x\right], \tag{6}$$

where W can be determined from the potential $V(x)$ by

$$W^2 - \frac{\mathrm{d}W}{\mathrm{d}x} = V(x) - E. \tag{7}$$

It is easy to see that solving Eq. (7) will give the "ground" function $\psi_0(x)$. However, other solutions, that is, the "excited" state functions cannot be available directly through the super-potential W. This requires a new property, i.e., shape-invariance, for the super-potential W. There are two partner potentials $V^{\pm}$ defined by the super-potential $W(x;a)$:

$$V^{\pm}(x;a) = W^2(x;a) \pm \frac{\mathrm{d}W(x;a)}{\mathrm{d}x}, \tag{8}$$

where we suppose that a may be some parameter or parameters. If the partner potentials $V^{\pm}$ satisfy the following relations

$$V^{+}(x;a) = V^{-}(x;b) + R(a), \tag{9}$$

where b are parameters determined by parameters a and $R(a)$ depends just only on parameters a, then the super-potential W is said to be of shape-invariance.

Actually the shape-invariance properties make the partner potentials $V^{\pm}(x;a)$ similar in shape. By the shape-invariance property of W, the "excited" state function can be obtained by

$$\psi_n^{-}(x;a) = \mathscr{A}^{\dagger}(x;a)\psi_{n-1}^{-}(x;b), \tag{10}$$

$$\mathscr{A}^{\dagger}(x;a) = -\frac{\mathrm{d}}{\mathrm{d}x} + W(x;a),$$

$$n = 1,2,3,\ldots \tag{11}$$

It is obvious that equation (10) is also the recurrence relations.[11]

3. Retrospection of spin-weighted spheroidal functions

In order to employ SUSYQM to Eq. (1), we first put it into the Schrödinger form[6–10]

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}\theta^2} + \left[\frac{1}{4} + s + \beta^2\cos^2\theta - 2s\beta\cos\theta - \frac{(m+s\cos\theta)^2 - 1/4}{\sin^2\theta} + E\right]\Psi = 0, \tag{12}$$

by the transformation

$$\Theta(\theta) = \frac{\Psi(\theta)}{\sqrt{\sin\theta}}. \tag{13}$$

In previous literatures, equation (12) has been studied, in Refs. [6]– [9]. In Ref. [9], the authors have solved the equation completely with the expression of W_n:

$$W_n(\theta) = \sum_{k=1}^{[n/2]} b_{n,k}\sin^{2k-1}\theta\cos\theta + \sum_{k=1}^{[(n+1)/2]} a_{n,k}\sin^{2k-1}\theta. \tag{14}$$

The method used in Ref. [9] is very useful to describe the characteristic of θ, especially when one needs to study the behavior of the function at $\theta = 0, \pi$. And the authors have already obtained the "ground" and "excited" solutions. Whenever one wants to study the spin-weighted spheroidal wave functions as the functions of independent variable $x = \cos\theta$, their results are not convenient. So in Ref. [10], the authors used another expression

$$W_n(\theta) = \sin\theta\sum_{j=0}^{n-1} a_{n,j}\cos^j\theta \tag{15}$$

and obtained the "ground" solutions of Eq. (1). It should be noted that the $a_{n,j}$ in Eq. (15) are different from those in Eq. (14). Readers are referred to Refs. [6]– [10] for details.

Compared with Eq. (14), equation (15) can be turned into a much more simple expression of independent variable x with the transformation $x = \cos\theta$. So the method used in Ref. [10] is very useful for processing the form of $\Psi(x)$ and is much simpler than the former. However in Ref. [10], the authors only obtained the "ground" solutions. In this paper we will continue their work to get the recurrence relations to complete the whole solutions.

4. The recurrence relations for the spin-weighted spheroidal functions

The super-potential W is expended as the series sum of the parameter β as

$$W = W_0 + \sum_{n=1}^{\infty}\beta^n W_n. \tag{16}$$

In Ref. [10], this series form of W has been obtained, here we just give some terms

$$W_0(\theta) = -\frac{s + (m+1/2)\cos\theta}{\sin\theta}, \tag{17}$$

and all $W_n(\theta)$ have the following form

$$W_n(\theta) = \sin\theta\sum_{j=0}^{n-1} a_{n,j}\cos^j\theta, \quad n \geq 1, \tag{18}$$

the first two expressions of $W_n(\theta)$ are as follows:

$$W_1(\theta) = -\frac{s}{m+1}\sin\theta, \tag{19}$$

$$W_2(\theta) = -\frac{(m+s+1)(m-s+1)s}{(m+1)^3(2m+3)}\sin\theta + \frac{(m+s+1)(m-s+1)}{(m+1)^2(2m+3)}\sin\theta\cos\theta. \tag{20}$$

For other $a_{n,l}$, see Ref. [10] for details.

The "ground" eigenvalue has also been presented in Ref. [10] as

$$E_0 = \sum_{n=0}^{\infty} E_{0,n}\beta^n, \tag{21}$$

$$E_{0,0} = (m-s)(m+s+1),\ E_{0,1} = -\frac{2s^2}{m+1}, \tag{22}$$

$$E_{0,2} = -\frac{s^2}{m+1} - \frac{(m+s+1)(m-s+1)(m+2s^2+1)}{(2m+3)(m+1)^3}. \tag{23}$$

We now rewrite the super-potential W in the form of

$$W(\theta;A_{0,0},C_{0,0},\mathscr{A}) = -\left(m+\frac{1}{2}\right)A_{0,0}\cot\theta - sC_{0,0}\csc\theta + \sum_{n=1}^{\infty}\beta^n W_n(\theta;A_n) \tag{24}$$

with the definition

$$W_n(\theta;A_n) = \sin\theta\sum_{j=0}^{n-1}\bar{a}_{n,j}\cos^j\theta, \tag{25}$$

where A_n is the finite set $\{A_{n,j}|j=0,\ldots,n-1\}$ with $A_{n,j}$ constant, $\mathscr{A}_n$ is the finite set $\{A_{k,j}\in A_k|k=1,\ldots,n,\ j=0,\ldots,k-1\}$ and $\mathscr{A}$ is the infinite set $\{A_n|n=1,\ldots,\infty\}$,

$$\bar{a}_{n,j} = A_{n,j}a_{n,j}. \tag{26}$$

We define $V^{\pm}(\theta;A_{0,0},\mathscr{A})$ as follows:

$$V^{\pm}(\theta;A_{0,0},\mathscr{A}) = W^2(\theta;A_{0,0},\mathscr{A}) \pm W'(\theta;A_{0,0},\mathscr{A}) = \sum_{n=0}^{\infty}\beta^n V_n^{\pm}(\theta;A_{0,0},\mathscr{A}_n). \tag{27}$$

Just as mentioned above, there are shape-invariance properties, similar in shape and different in parameters, with the pair of partner potentials $V^{\pm}(\theta;A_{0,0},\mathscr{A})$. We then need to find some quantities $B_{0,0},\mathscr{B}_n$ to make the property relations

$$V_n^+(\theta;A_{0,0},\mathscr{A}_n) = V_n^-(\theta;B_{0,0},\mathscr{B}_n) + R_n(A_{0,0},\mathscr{A}_n,B_{0,0},\mathscr{B}_n) \tag{28}$$

retain with $R_n(A_{0,0},\mathscr{A}_n,B_{0,0},\mathscr{B}_n)$ pure quantities, where $B_{0,0}$, $\mathscr{B}_n$ are counterparts of $A_{0,0}$ and $\mathscr{A}_n$, respectively. It is easy to know that the relation is right for the cases $n=0$

$$V_0^+(\theta;A_{0,0},C_{0,0}) = V_0^-(\theta;B_{0,0},D_{0,0}) + R_0(A_{0,0},B_{0,0}), \tag{29}$$

where

$$A_{0,0} = 1,\ C_{0,0} = 1; \tag{30}$$

$$B_{0,0} = A_{0,0} + \frac{2}{2m+1},\ D_{0,0} = C_{0,0} = 1; \tag{31}$$

$$R_0 = (2m+1)A_{0,0} + 1. \tag{32}$$

Due to the fact that $D_{0,0} = C_{0,0} = 1$, we will not include $C_{0,0}$ and $D_{0,0}$ in explicit forms of $W_0(\theta;A_{0,0},C_{0,0})$ and $W_0(\theta;B_{0,0},D_{0,0})$. We write them as

$$W_0(\theta;A_{0,0}) = -\left(m+\frac{1}{2}\right)A_{0,0}\cot\theta - \frac{s}{\sin\theta} \tag{33}$$

and

$$W_0(\theta;B_{0,0}) = -\left(m+\frac{1}{2}\right)B_{0,0}\cot\theta - \frac{s}{\sin\theta} \tag{34}$$

for convenience.

We can get the following results

$$V_n^-(\theta;A_{0,0},\mathscr{A}_n) = \sum_{p=0}^{n}(P_{n,p}^-(A_{0,0},A_n) + H_{n,p}(\mathscr{A}_{n-1})\cos^p\theta \tag{35}$$

and

$$V_n^+(\theta;A_{0,0},\mathscr{A}_n) = \sum_{p=0}^{n}(P_{n,p}^+(A_{0,0},A_n) + H_{n,p}(\mathscr{A}_{n-1})\cos^p\theta, \tag{36}$$

where $P_{n,p}^+(A_{0,0},A_n)$, $P_{n,p}^-(A_{0,0},A_n)$, and $H_{n,p}(\mathscr{A}_{n-1})$ are defined by Eqs. (A7), (A8), and (A10) in the Appendix A.

We rewrite the formula of $V_n^+(\theta;A_{0,0},\mathscr{A}_n)$ as follows:

$$\begin{aligned} &V_n^+(\theta;A_{0,0},\mathscr{A}_n)\\ &= V_n^-(\theta;B_{0,0},\mathscr{B}_n) + R_n(A_{0,0},\mathscr{A}_n,B_{0,0},\mathscr{B}_n)\\ &= \sum_{p=0}^{n}(P_{n,p}^-(B_{0,0},B_n) + H_{n,p}(\mathscr{B}_{n-1}))\cos^p\theta\\ &\quad + R_n(A_{0,0},\mathscr{A}_n,B_{0,0},\mathscr{B}_n). \end{aligned} \tag{37}$$

In the term $V_n^-(\theta;B_{0,0},\mathscr{B}_n)$, $A_{n,j}$ is substituted into the forms $\bar{a}_{n,j} = A_{n,j}a_{n,j}$ by the corresponding terms $B_{n,j}$.

It is easy to obtain

$$\begin{aligned} &R_n(A_{0,0},\mathscr{A}_n,B_{0,0},\mathscr{B}_n)\\ &= P_{n,0}^+(A_{0,0},A_n) + H_{n,0}(\mathscr{A}_{n-1})\\ &\quad - P_{n,0}^-(B_{0,0},B_n) - H_{n,0}(\mathscr{B}_{n-1}), \end{aligned} \tag{38}$$

where

$$P_{n,0}^+(A_{0,0},A_n) = -2sA_{n,0}a_{n,0} - A_{n,1}a_{n,1}, \tag{39}$$

$$P_{n,0}^-(B_{0,0},B_n) = -2sB_{n,0}a_{n,0} + B_{n,1}a_{n,1}, \tag{40}$$

$$H_{n,0}(\mathscr{A}_{n-1}) = \sum_{k=1}^{n-1}A_{k,0}a_{k,0}A_{n-k,0}a_{n-k,0}, \tag{41}$$

$$H_{n,0}(\mathscr{B}_{n-1}) = \sum_{k=1}^{n-1}B_{k,0}a_{k,0}B_{n-k,0}a_{n-k,0}. \tag{42}$$

From Eq. (37), the following equations must be satisfied to maintain the shape-invariance property

$$P_{n,p}^+(A_{0,0},A_n) + H_{n,p}(\mathscr{A}_{n-1}) = P_{n,p}^-(B_{0,0},B_n) + H_{n,p}(\mathscr{B}_{n-1}),\quad p=1,2,\ldots,n. \tag{43}$$

Define

$$P^{+}_{n,p}(A_{0,0},A_n)+H_{n,p}(\mathscr{A}_{n-1})-H_{n,p}(\mathscr{B}_{n-1}) \equiv U_{n,p}(A_{0,0},\mathscr{A}_n,B_{0,0},\mathscr{B}_{n-1}), \tag{44}$$

and from Eq. (A8) in Appendix A we can obtain

$$P^{-}_{n,p}(B_{0,0},B_n) = (-(2m+1)B_{0,0}-p)B_{n,p-1}a_{n,p-1} - 2sB_{n,p}a_{n,p}+(1+p)B_{n,p+1}a_{n,p+1}. \tag{45}$$

It is easy to see that in the above equation, the quantities are all known functions with the only unknown quantities $B_{n,j}, j=0,1,\ldots,n-1$. So, we obtain

$$B_{n,p-1} = \frac{-2sB_{n,p}a_{n,p}+(p+1)B_{n,p+1}a_{n,p+1}-U_{n,p}(B_{0,0},\mathscr{A}_n,B_{0,0},\mathscr{B}_{n-1})}{((2m+1)B_{0,0}+p)a_{n,p-1}}, \quad p = n,n-1,n-2,\ldots,2,1, \tag{46}$$

or

$$B_{n,p-1}a_{n,p-1} = \frac{-2sB_{n,p}a_{n,p}+(p+1)B_{n,p+1}a_{n,p+1}-U_{n,p}(B_{0,0},\mathscr{A}_n,B_{0,0},\mathscr{B}_{n-1})}{((2m+1)B_{0,0}+p)}, \quad p = n,n-1,n-2,\ldots,2,1. \tag{47}$$

In Eq. (47) we have used the fact that $a_{n,p}=B_{n,p}=0$ whenever $p>n-1$ or $p<0$, and we can obtain $B_{n,n-1}$, $B_{n,n-2}$, …, $B_{n,0}$ in turn. So all the elements in B will be determined by the elements in A.

Keeping the fact in mind, we rewrite the quantities $R_n(A_{0,0},\mathscr{A}_n,B_{0,0},\mathscr{B}_n)$ in Eq. (38) as $R_n(A_{0,0},\mathscr{A}_n)$. Then, we can obtain the "excited" eigenvalues E_N and eigenfunctions Ψ_N through the recurrence relation[10]

$$E^{-}_N = E_0+\sum_{k=1}^{N}R(a_k), \tag{48}$$

$$E_0 = (m-s)(m+s+1)+\sum_{n=1}^{\infty}E_{0,n}\beta^n, \tag{49}$$

$$R(a_k) = R_0+\sum_{n=1}^{\infty}\beta^n R_n(a_k), \tag{50}$$

$$a_1=(\mathscr{A}_n),\ a_2=(\mathscr{B}_n),\ldots, \tag{51}$$

$$\mathscr{A}^{\dagger}(\theta;A_{0,0},\mathscr{A}) = -\frac{\mathrm{d}}{\mathrm{d}\theta}+W(\theta,A_{0,0},\mathscr{A}), \tag{52}$$

$$\Psi^{-}_N(\theta;A_{0,0},\mathscr{A}) = \mathscr{A}^{\dagger}(\theta;A_{0,0},\mathscr{A})\Psi^{-}_{N-1}(\theta;B_{0,0},\mathscr{B}), \quad N=1,2,3,\ldots \tag{53}$$

Finally, we can get the recurrence relations between $\Theta_N(\theta)$ and $\Theta_{N-1}(\theta)$

$$\Theta_N(\theta;A_{0,0},\mathscr{A}) = \left[-\frac{\mathrm{d}}{\mathrm{d}\theta}-\frac{\cos\theta}{2\sin\theta}+W(\theta;A_{0,0},\mathscr{A})\right] \times \Theta_{N-1}(\theta;B_{0,0},\mathscr{B}). \tag{54}$$

With the transformation $\cos\theta=x$, we can get

$$\Theta_N(x;A_{0,0},\mathscr{A}) = \left[(1-x^2)^{1/2}\frac{\mathrm{d}}{\mathrm{d}x}-\frac{x}{2(1-x^2)^{1/2}}+W(x;A_{0,0},\mathscr{A})\right] \times \Theta_{N-1}(x;B_{0,0},\mathscr{B}) = \left[(1-x^2)^{1/2}\frac{\mathrm{d}}{\mathrm{d}x}-\frac{s+((m+1/2)A_{0,0}+1/2)x}{(1-x^2)^{1/2}} + \sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\beta^n A_{n,j}a_{n,j}(1-x^2)^{1/2}x^j\right] \times \Theta_{N-1}(x;B_{0,0},\mathscr{B}) \tag{55}$$

and

$$\Psi_0(x;A_{0,0},\mathscr{A}) = \overline{N}\left(\frac{1-x}{1+x}\right)^{s/2}(1-x^2)^{(m+1/2)A_{0,0}/2} \times \exp\left[\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\beta^n A_{n,j}a_{n,j}\frac{1}{j+1}x^{j+1}\right], \tag{56}$$

$$\Theta_0(x;A_{0,0},\mathscr{A}) = \overline{N}\left(\frac{1-x}{1+x}\right)^{s/2}(1-x^2)^{(m+1/2)A_{0,0}/2-1/4} \times \exp\left[\sum_{n=1}^{\infty}\sum_{j=0}^{n-1}\beta^n A_{n,j}a_{n,j}\frac{1}{j+1}x^{j+1}\right], \tag{57}$$

where $\overline{N}$ is a constant.

From the above discussion, we draw our the conclusions that the spin-weighted equations have shape-invariance properties and their recurrence relations, the excited eigenvalues and eigenfunctions can be sequentially obtained.

Appendix A

With the help of

$$W^2(\theta;A_{0,0},\mathscr{A}) = W_0^2(\theta;A_{0,0})+\sum_{n=1}^{\infty}\beta^n\left[\sum_{k=1}^{n}W_k(\theta;A_k)W_{n-k}(\theta;A_{n-k}) + 2W_0(\theta;A_{0,0})W_n(\theta;A_n)\right] \tag{A1}$$

and

$$W'(\theta;A_{0,0},\mathscr{A}) = W_0'(\theta;A_{0,0}) + \sum_{n=1}^{\infty}\beta^n W_n'(\theta;A_n), \tag{A2}$$

where

$$W_{n-k}(\theta;A_{n-k}) = \sin\theta\sum_{j=0}^{n-k-1}\bar{a}_{n-k,j}\cos^j\theta,$$
$$W_k(\theta;A_k) = \sin\theta\sum_{i=0}^{k-1}\bar{a}_{k,i}\cos^i\theta,\ n\geq 1, \tag{A3}$$

we rewrite the formulae of $V_n^{\pm}(\theta;A_{0,0},\mathscr{A}_n)$ from Eq. (27) in the case $n\geq 1$ as follows:

$$V_n^{\mp}(\theta;A_{0,0},\mathscr{A}_n) = \mathrm{P}_n^{\mp}(\theta;A_{0,0},A_n) + \bar{\mathrm{P}}_n(\theta;\mathscr{A}_{n-1}), \tag{A4}$$

where

$$\mathrm{P}_n^{\mp}(\theta;A_{0,0},A_n) = 2W_0(\theta;A_{0,0})W_n(\theta;A_n) \mp \frac{\mathrm{d}W_n(\theta;A_n)}{\mathrm{d}\theta}$$
$$= \sum_{p=0}^{n}\mathrm{P}_{n,p}^{\mp}(A_{0,0},A_n)\cos^p\theta, \tag{A5}$$
$$\bar{\mathrm{P}}_n(\theta;\mathscr{A}_{n-1}) = \sum_{k=1}^{n-1}W_k(\theta;A_k)W_{n-k}(\theta;A_{n-k}). \tag{A6}$$

Then, we have

$$\mathrm{P}_{n,p}^{+}(A_{0,0},A_n) = (-(2m+1)A_{0,0}+p)\bar{a}_{n,p-1}$$
$$-\ 2s\bar{a}_{n,p} - (1+p)\bar{a}_{n,p+1}, \tag{A7}$$
$$\mathrm{P}_{n,p}^{-}(A_{0,0},A_n) = (-(2m+1)A_{0,0}-p)\bar{a}_{n,p-1}$$
$$-\ 2s\bar{a}_{n,p} + (1+p)\bar{a}_{n,p+1}, \tag{A8}$$
$$\bar{\mathrm{P}}_n(\theta;\mathscr{A}_{n-1}) = \sum_{k=1}^{n-1}\sin^2\theta\sum_{i=0}^{k-1}\sum_{j=0}^{n-k-1}\bar{a}_{k,i}\bar{a}_{n-k,j}\cos^{i+j}\theta$$
$$= \sum_{k=1}^{n-1}\sum_{p=0}^{n-2}\sum_{j=0}^{p}\bar{a}_{k,p-j}\bar{a}_{n-k,j}\cos^p\theta(1-\cos^2\theta)$$
$$= \sum_{p=0}^{n-2}\sum_{k=1}^{n-1}\sum_{j=0}^{p}\bar{a}_{k,p-j}\bar{a}_{n-k,j}\cos^p\theta$$
$$-\sum_{p=2}^{n}\sum_{k=1}^{n-1}\sum_{j=0}^{p-2}\bar{a}_{k,p-2-j}\bar{a}_{n-k,j}\cos^p\theta$$
$$= \sum_{p=0}^{n}\sum_{k=1}^{n-1}\sum_{j=0}^{p}(\bar{a}_{k,p-j}\bar{a}_{n-k,j} - \bar{a}_{k,p-2-j}\bar{a}_{n-k,j})$$
$$\times\ \cos^p\theta, \tag{A9}$$

where we use the assumption that $\bar{a}_{n,j}=0$ whenever $j\geq n$ or $j<0$.

If we define

$$H_{n,p}(\mathscr{A}_{n-1}) = \sum_{k=1}^{n-1}\sum_{j=0}^{p}(\bar{a}_{k,p-j}\bar{a}_{n-k,j} - \bar{a}_{k,p-2-j}\bar{a}_{n-k,j})$$
$$= \sum_{k=1}^{n-1}\sum_{j=0}^{p}(\bar{a}_{k,p-j} - \bar{a}_{k,p-2-j})\bar{a}_{n-k,j}, \tag{A10}$$

then

$$\bar{\mathrm{P}}_n(\theta;\mathscr{A}_{n-1}) = \sum_{p=0}^{n}H_{n,p}(\mathscr{A}_{n-1})\cos^p\theta. \tag{A11}$$

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