A symmetry-preserving difference scheme for high dimensional nonlinear evolution equations*

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(Received 16 October 2012; revised manuscript received 15 December 2012)

In this paper, a procedure for constructing discrete models of the high dimensional nonlinear evolution equations is presented. In order to construct the difference model, with the aid of the potential system of the original equation and compatibility condition, the difference equations which preserve all Lie point symmetries can be obtained. As an example, invariant difference models of the (2+1)-dimensional Burgers equation are presented.

Keywords: symmetry-preserving, potential systems, difference equation, Lie point symmetry

PACS: 02.20.–a, 04.60.Nc **DOI:** 10.1088/1674-1056/22/6/060201

1. Introduction

Symmetries[1–10] are intrinsic and fundamental features of the differential equations of mathematical physics. Lie point symmetries yield a number of useful properties of differential equations: 1) a group action transforms the complete set of solutions into itself; 2) there exists a standard procedure to obtain the whole set of invariants and differential invariants for a symmetry group; 3) for ordinary differential equations (ODEs) the known symmetry yields the reduction of the order. The invariance of partial differential equations (PDEs) is a necessary condition for the application of Noether's theorem on variational problems to obtain conservation laws, etc.

In many cases, the symmetries of differential equations are more important and better known than the equations themselves since these symmetries reflect fundamental physical laws. Thus, when discretizing, or otherwise modifying dynamical equations, it is of interest to preserve the original symmetries.

The structure of the admitted group essentially affects the construction of equations and grids. Group transformations can break the geometric structure of the mesh that influences the approximation and other properties of a difference equation. Applications of Lie group theory to difference equations[11–27] are much more recent with important work by Hydon[28] on continuous symmetries of discrete systems. The main point in this area which distinguishes different approaches is the transformation of the independent variables. Dorodnitsyn is the first to start with symmetries of a differential equation and to introduce a difference equation together with a symmetry-adapted mesh in such a way that all the symmetries of the original differential equation are preserved.[15–21] Dorodnitsyn's approach can be subsumed in the following way: first, determine the maximal Lie invariance algebra of the model under consideration. Hereafter, a discretization stencil, i.e., a number of grid points which the difference scheme will be based on, has to be chosen. The generators of Lie symmetries are then prolonged to all points of the stencil. From these prolonged generators, the invariants of the extended group action on the stencil are determined. The final step is to assemble the obtained invariants together to a difference approximation of the original differential equation.

In general it is not possible to discretize a differential equation on a simple regular and orthogonal mesh while preserving all of the point symmetries. It is either necessary to give up their point character,[14,22] or to construct a symmetry-adapted mesh which can be nonregular and nonorthogonal; in this article we follow the second procedure.[23] Up to now, the reviewed technique has been applied to a physically rather simple model, usually only involving time and one space dimension. It is understandable that the multi-dimensional case is even more delicate, as there is an increasing number of possibilities for assembling the difference invariants to finite difference schemes. For Dorodnitsyn's approach, when the dimension increases, it is difficult to find appropriate invariants to structure difference equations, in the continous limit, the difference equations reduce to the original system and preserve all of its Lie point symmetries.

In this paper, based on Dorodnitsyn's work, we give a

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11075055 and 11275072), the Innovative Research Team Program of the National Natural Science Foundation of China (Grant No. 61021004), National High Technology Research and Development Program of China (Grant No. 2011AA010101), the Leading Academic Discipline Project of Shanghai (Grant No. B412), the Research Fund for the Doctoral Program of Higher Education of China (Grant No. 20120076110024), and the Shanghai Knowledge Service Platform Project (Grant No. ZF1213).

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method to construct the difference models of high dimensional nonlinear evolution equation which preserves all Lie point symmetries of the original equation. The difference of this method to others is that we construct the difference models of the potential system instead of the original equation. The specific process is as follows: (i) for a high dimensional nonlinear evolution equation, the potential system of the original equation can be constructed firstly. One can see that the order of the potential system is lower than the original equation, this procedure can provide beneficial help for constructing the invariant. (ii) Following Bluman's theory, each local symmetry of the potential system projects onto a local symmetry of the given system, we can construct the difference models of the potential system which preserve all Lie point symmetries. (iii) We give a theorem which shows that if one can construct the difference models of the potential system, then the difference equations of the original equation can be obtained. (iv) One can check that the difference models preserve all Lie point symmetries of original equation.

This paper is arranged as follows: in Section 2, the overview of the discretization procedure and the extended method to construct the difference models of high dimensional PDEs are introduced. In Section 3, using the method to the (2+1)-dimensional Burgers equation, the difference equations of preserving the symmetries of the original equation are obtained. Finally, some conclusions and discussions are given in Section 4.

2. Definitions and notations

2.1. Definition of multidimensional finite-difference derivatives

Let Z be the space of sequences $(x,u,u_1,u_2,\ldots)$, where $x=\{x^i;i=1,2,\ldots,n\}$, $u=\{u^k;k=1,2,\ldots,m\}$. $u_1=\{u^k_i\}$ is the set of $m\cdot n$ first partial derivatives. $u_2=\{u^k_{ij}\}$ is the set of second partial derivatives.

We restrict to the case $n=3$, i.e., $x=\{x^1,x^2,x^3\}$. Three types of differentiations are considered,

$$\begin{aligned}
D_1&=\frac{\partial}{\partial x^1}+u_1\frac{\partial}{\partial u}+u_{11}\frac{\partial}{\partial u_1}+u_{21}\frac{\partial}{\partial u_2}+u_{31}\frac{\partial}{\partial u_3}+\cdots,\\
D_2&=\frac{\partial}{\partial x^2}+u_2\frac{\partial}{\partial u}+u_{12}\frac{\partial}{\partial u_1}+u_{22}\frac{\partial}{\partial u_2}+u_{32}\frac{\partial}{\partial u_3}+\cdots,\\
D_3&=\frac{\partial}{\partial x^3}+u_3\frac{\partial}{\partial u}+u_{13}\frac{\partial}{\partial u_1}+u_{23}\frac{\partial}{\partial u_2}+u_{33}\frac{\partial}{\partial u_3}+\cdots,
\end{aligned}\tag{1}$$

where $u_1=\partial u/\partial x^1$, $u_{11}=\partial^2u/\partial(x^1)^2$, $u_{21}=\partial^2u/\partial x^2x^1,\cdots$, and summation over the omitted superscript k in Eq. (1) is assumed.

The operators D_1, D_2, and D_3 generate three commuting Taylor groups, whose finite transformations are determined by the action of the operators $T^1_a=\mathrm{e}^{aD_1},T^2_a=\mathrm{e}^{aD_2}$, and $T^3_a=\mathrm{e}^{aD_3}$. One fixes three arbitrary parameter values h_1, h_2, $h_3>0$ and forms three kinds of discrete shift operators

$$\underset{\pm h}{S_i}=\mathrm{e}^{\pm h_iDi}\equiv\sum_{s\geq0}\frac{(\pm h_i)^s}{s!}D^s_i,\quad i=1,2,3.\tag{2}$$

Accordingly, the discrete differentiation operators can be defined,

$$\underset{h}{D_i}=\pm\frac{1}{h_i}(\underset{\pm h}{S_i}-1),\quad i=1,2,3.\tag{3}$$

We introduce the difference derivatives $\underset{h}{u_i}=\underset{+h}{D_i}(u)$, $\underset{h}{u_{ij}}=\underset{+h}{D_j}\,\underset{+h}{D_i}(u)$, $\underset{h}{u_{i\bar{j}}}=\underset{-h}{D_j}\,\underset{+h}{D_i}(u)$. The analogs of tables of discrete shifts and differentiations can be formed accordingly.

In Tables 1 and 2, $\underset{h}{u_{k\cdot i}}=\underset{h}{u_{i\cdots i}}$, i.e., taking difference derivative with respect to the i-th variable k times. Before study the relations between one-parameter groups and difference meshes preserving their geometric structure under group transformations, we introduce the definition of potential system in the following.

Table 1. Action of the discrete shift operator on the point $z=(x^1,x^2,x^3,u,\underset{h}{u_i},\cdots)$.

$\underset{-h}{S_i}$ is the finite-difference left shift operator	$\underset{+h}{S_i}$ is the finite-difference right shift operator
$\underset{-h}{S_i}\,(x^i)=x^i-h_i$	$\underset{+h}{S_i}\,(x^i)=x^i+h_i$
$\underset{-h}{S_i}\,(u)=u-h_i\,\underset{h}{u_i}+h_i^2\,\underset{h}{u_{i\bar{i}}}$	$\underset{+h}{S_i}\,(u)=u+h_i\,\underset{h}{u_i}$
$\underset{-h}{S_i}\,(\underset{h}{u_j})=\underset{h}{u_j}-h_i\,\underset{h}{u_{j\bar{i}}}$	$\underset{+h}{S_i}\,(\underset{h}{u_j})=\underset{h}{u_j}+h_i\,\underset{h}{u_{ji}}+h_i^2\,\underset{h}{u_{j\bar{i}i}}$
$\underset{-h}{S_i}\,(\underset{h}{u_{i\bar{i}}})=\underset{h}{u_{i\bar{i}}}-h_i\,\underset{h}{u_{i\bar{i}i}}+h_i^2\,\underset{h}{u_{i\bar{i}i\bar{i}}}$	$\underset{+h}{S_i}\,(\underset{h}{u_{i\bar{i}}})=\underset{h}{u_{i\bar{i}}}+h_i\,\underset{h}{u_{i\bar{i}i}}$
$\cdots$	$\cdots$
$\underset{-h}{S_i}\,(\underset{h}{u_{k\cdot i}})=\underset{h}{u_{k\cdot i}}-h_i\,\underset{h}{u_{(k+1)\cdot i}}+h_i^2\,\underset{h}{u_{(k+1)\cdot i\bar{i}}}$	$\underset{+h}{S_i}\,(\underset{h}{u_{k\cdot i}})=\underset{h}{u_{k\cdot i}}+h_i\,\underset{h}{u_{k\cdot i\bar{i}}}+h_i^2\,\underset{h}{u_{k\cdot i\bar{i}i}}$
$\underset{-h}{S_j}\,(\underset{h}{u_{k\cdot i}})=\underset{h}{u_{k\cdot i}}-h_j\,\underset{h}{u_{k\cdot ij}}+h_j^2\,\underset{h}{u_{k\cdot ij\bar{j}}}$	$\underset{+h}{S_j}\,(\underset{h}{u_{k\cdot i}})=\underset{h}{u_{k\cdot i}}+h_j\,\underset{h}{u_{k\cdot i\bar{j}}}+h_j^2\,\underset{h}{u_{k\cdot i\bar{j}j}}$

Table 2. Action of the discrete differentiation operator on the coordinates of the point $\underset{h}{z} = (x^1, x^2, x^3, u, \underset{h}{u_i}, \cdots)$.

$\underset{-h}{D_i}$ is the finite-difference left differentiation operator	$\underset{+h}{D_i}$ is the finite-difference right differentiation operator
$\underset{-h}{D_i}(x^i) = 1$	$\underset{+h}{D_i}(x^i) = 1$
$\underset{-h}{D_i}(u) = \underset{h}{u_i} - h_i \underset{h}{u_{\bar{i}\bar{i}}}$	$\underset{+h}{D_i}(u) = \underset{h}{u_i}$
$\underset{-h}{D_i}(u_j) = \underset{h}{u_{j\bar{i}}}$	$\underset{+h}{D_i}(u_j) = \underset{h}{u_{j\bar{i}}} + h_i \underset{h}{u_{j\bar{i}\bar{i}}}$
$\underset{-h}{D_i}(\underset{h}{u_{i\bar{i}}}) = \underset{h}{u_{i\bar{i}\bar{i}}} - h_i \underset{h}{u_{i\bar{i}\bar{i}\bar{i}}}$	$\underset{+h}{D_i}(\underset{h}{u_{i\bar{i}}}) = \underset{h}{u_{i\bar{i}\bar{i}}}$
$\cdots$	$\cdots$
$\underset{-h}{D_i}(\underset{h}{u_{k\cdot i}}) = \underset{h}{u_{(k+1)\cdot i}} - h_i \underset{h}{u_{(k+1)\cdot i\bar{i}}}$	$\underset{+h}{D_i}(\underset{h}{u_{k\cdot i}}) = \underset{h}{u_{k\cdot i\bar{i}}} + h_i \underset{h}{u_{k\cdot i\bar{i}\bar{i}}}$
$\underset{-h}{D_j}(\underset{h}{u_{k\cdot i}}) = \underset{h}{u_{k\cdot ij}} - h_i \underset{h}{u_{k\cdot ij\bar{j}}}$	$\underset{+h}{D_j}(\underset{h}{u_{k\cdot i}}) = \underset{h}{u_{k\cdot i\bar{j}}} + h_i \underset{h}{u_{k\cdot i\bar{j}\bar{j}}}$

2.2. Potential system of multidimensional case

For an evolution equation,

$$f(t,x,y,u,u_t,u_x,u_y,\ldots) = 0, \tag{4}$$

we consider nonlocally related systems for a given system of differential equations using Bluman's method.[29,30] For PDE systems with $n > 2$ independent variables, the situation for obtaining and using nonlocally related PDE systems is considerably more complex than that in the two-dimensional case. In particular, every divergence-type conservation law gives rise to several potential variables, which are only defined to within arbitrary functions of the independent variables.

For a given PDE system, divergence-type conservation laws are given by expressions of the form

$$\mathrm{div}\,\Phi[u] = \Phi^1_t + \Phi^2_x + \Phi^3_y, \tag{5}$$

with a flux vector $\Phi = (\Phi^1(t,x,y), \Phi^2(t,x,y), \Phi^3(t,x,y))$ and independent variables $(t,\ x\ ,y)$. From Eq. (5) it immediately follows that $\Phi = \mathrm{curl}\,\Psi$ where $\Psi = (\Psi^1(t,x,y),\ \Psi^2(t,x,y), \Psi^3(t,x,y))$ is a vector potential, involving three scalar potential variables. Explicitly, the potential equations are given by

$$\begin{aligned} \Psi^3_x - \Psi^2_y &= \Phi^1, \\ \Psi^1_y - \Psi^3_t &= \Phi^2, \\ \Psi^2_t - \Psi^1_x &= \Phi^3. \end{aligned} \tag{6}$$

However, the system of potential equations (6) is underdetermined. In particular, the system of potential equations (6) is invariant under the transformations

$$\Psi \to \Psi + \mathrm{grad}\,\phi(t,x,y),$$

where $\phi(t,x,y)$ is an arbitrary smooth function of its arguments. A theorem is presented below.

Theorem 1[29] Each local symmetry of an underdetermined potential system (6) projects onto a local symmetry of the given system (4).

Let

$$\begin{aligned} X \ = \ & \xi^t \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \xi^y \frac{\partial}{\partial y} + \mu \frac{\partial}{\partial u} + \varphi^i \frac{\partial}{\partial \Psi^i} + \cdots, \\ & i = 1,2,3 \end{aligned} \tag{7}$$

be the generator of a one-parameter transformation group of system (6). Dots denote the prolongation of the operator to other variables used in the given differential equation.

The group generated by Eq. (7) transforms a point $(t,x,y,u,u_t,u_x,u_{xx},\ldots,\Psi^1,\Psi^2,\Psi^3)$ to a new one $(t^*,x^*,y^*, u^*,u^*_t,u^*_x,u^*_{xx},\ldots,\Psi^{1*},\Psi^{2*},\Psi^{2*})$ together with Eq. (5). This situation changes when applying Lie points transformations to difference equations.

Using the Theorem 1, one can obtain that

$$X' = \xi^t \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \xi^y \frac{\partial}{\partial y} + \mu \frac{\partial}{\partial u} \tag{8}$$

must be the symmetry of system (4).

Next, we study the relations between one-parameter groups and difference meshes preserving their geometric structure under group transformations.

2.3. Invariant difference meshes

The discretization of a PDE involving one dependent variable u and three independent ones, x, y, and t, consists of sampling the point (t,x,y) in the real space. Let $\underset{h}{Z}$ be the space of sequences of mesh variables $(t,x,y,u,\underset{h}{u_1},\ldots,\tau,h,\eta)$, and let $\underset{h}{A}$ be the space of analytic functions of finitely many coordinates z^i of a vector $z \in \underset{h}{Z}$. Then each finite-difference equation on the mesh $\underset{h}{w}$ can be written as

$$F_i(z) = 0, \quad i = 1,2,3, \tag{9}$$

where $F_i \in \underset{h}{A}$. In contrast to the point $(t,\ x,\ y,\ u,\ u_t,\ u_x,\ u_{xx},\ \ldots,\ \Psi^1,\ \Psi^2,\ \Psi^3)$, the difference stencil has its own geometrical structure. In the continous limit, all three equations reduce to the potential system. This equation is written on finitely many points of the difference mesh $\underset{h}{w}$, which may be uniform or nonuniform.

We assume that the mesh is determined by the equation

$$\Omega_j(z) = 0 \quad (j = 1,2,3), \tag{10}$$

where $\Omega_j \in \underset{h}{A}$. The function Ω_j is uniquely determined by the "discretization" of the space of independent variables, which is obtained by the action of the operator $\underset{\pm h}{S^\alpha}$ on the "starting point" x_0. Thus, we initially assume that Eq. (4) is invariant under the discrete shift operator $\underset{\pm h}{S}$ in $\underset{h}{Z}$. The function $\Omega(z)$

depends on finitely many variables in $\underset{h}{Z}$ and explicitly depends on the mesh spacing h^+. In the continuum limit, the mesh equation $\Omega(z)=0$ degenerates into an identity (for example, $0=0$).

If a transformation group acts on $\underset{h}{Z}$, then, in contrast to the continuous case, one should also include Eq. (4) in the invariance condition, because an arbitrary mesh $\underset{h}{w}$ does not necessarily admit this group.

The invariance of the difference equation (9) depends on the invariance of Eq. (8), since the latter must be included in the general condition of invariance:

$$\begin{cases} XF_i(z)\big|_{(9),(10)}=0, \\ X\Omega_j(z)\big|_{(9),(10)}=0. \end{cases} \tag{11}$$

Thus, what makes this approach special is the inclusion of the second equation of Eq. (10) into the invariance conditions.

Proposition 1 For the mesh $\underset{h}{w}$ to remain uniform ($\tau_+^*=\tau_-^*, h_+^*=h_-^*, \eta_+^*=\eta_-^*$) under the action of the transformation group, if it satisfies the following condition at each point $z\in\underset{h}{\tilde{Z}}$:

$$\underset{+h}{D}\,\underset{-h}{D}(\xi^t)=0,\quad \underset{+h}{D}\,\underset{-h}{D}(\xi^x)=0,\quad \underset{+h}{D}\,\underset{-h}{D}(\xi^y)=0, \tag{12}$$

where $\underset{+h}{D}$ and $\underset{-h}{D}$ are right and left difference operators.

Proposition 2 For an orthogonal mesh $\underset{h}{w}$ to preserve its orthogonality in the plane (t,x,y) under any transformation of the group G with the operator (7), it is necessary and sufficient that the following condition be satisfied at each mesh point

$$\begin{aligned} &\underset{\pm h}{D}(\xi^t)=-\underset{\pm h}{D}(\xi^x), \\ &\underset{\pm h}{D}(\xi^t)=-\underset{\pm h}{D}(\xi^y), \\ &\underset{\pm h}{D}(\xi^y)=-\underset{\pm h}{D}(\xi^x). \end{aligned} \tag{13}$$

The conditions (11)–(13) connect the geometry of grids with the Lie group symmetry. These conditions will be used in what follows.

2.4. Construct the symmetry-preserving models

In order to get the symmetry-preserving model of the original equation, one should construct the symmetry-preserving models of potential system firstly. Using the results and the following theorem, the symmetry-preserving model can be obtained directly. The final step is to verify this model meeting the symmetry of the original equation using the condition (11).

Theorem 2 If the invariant difference models of potential system (6), i.e., $F_i(z)=0,\ \ i=1,2,3$ satisfy the symmetries (7), then the difference model $\underset{+h}{D_1}F_1(z)+\underset{+h}{D_2}F_2(z)+\underset{+h}{D_3}F_3(z)=0$ satisfies the symmetries (8) of original equation.

Proof Let the invariant difference models $F_i(z)=0$, $i=1,2,3$ of potential system have the following forms

$$\begin{aligned} &\underset{h}{\Psi_2^3}-\underset{h}{\Psi_3^2}=\underset{h}{\Phi^1}, \\ &\underset{h}{\Psi_3^1}-\underset{h}{\Psi_1^3}=\underset{h}{\Phi^2}, \\ &\underset{h}{\Psi_1^2}-\underset{h}{\Psi_2^1}=\underset{h}{\Phi^3}, \end{aligned}$$

using the discrete differentiation operators $\underset{+h}{D_1},\underset{+h}{D_2},\underset{+h}{D_2}$ acting on the above formulas respectively, one can obtain,

$$\begin{aligned} &\underset{h}{\Psi_{21}^3}-\underset{h}{\Psi_{31}^2}=\underset{+h}{D_1}\,\underset{h}{\Phi^1}, \\ &\underset{h}{\Psi_{32}^1}-\underset{h}{\Psi_{12}^3}=\underset{+h}{D_2}\,\underset{h}{\Phi^2}, \\ &\underset{h}{\Psi_{13}^2}-\underset{h}{\Psi_{23}^1}=\underset{+h}{D_3}\,\underset{h}{\Phi^3}. \end{aligned}$$

One can easily prove the following equalities by the definition of the discrete shift operator,

$$\underset{h}{\Psi_{21}^3}=\underset{h}{\Psi_{12}^3},\quad \underset{h}{\Psi_{31}^2}=\underset{h}{\Psi_{13}^2},\quad \underset{h}{\Psi_{32}^1}=\underset{h}{\Psi_{23}^1}.$$

In the continous limit, the difference equation $\underset{+h}{D_1}\,\underset{h}{\Phi^1}+\underset{+h}{D_2}\,\underset{h}{\Phi^2}+\underset{+h}{D_3}\,\underset{h}{\Phi^3}=0$ reduces to the original system and preserves all of its Lie point symmetries.

In the following section, using the above method, an example of constructing symmetry-preserving model will be given.

3. Invariant model for the (2+1)-dimensional Burgers equation

We consider the (2+1)-dimensional Burgers equation,

$$(u_t+uu_x-u_{xx})_x+u_{yy}=0, \tag{14}$$

which describes weakly nonlinear two-dimensional shocks in dissipative media. The shocks described by Eq. (14) are weakly two-dimensional ones in the sense that the scale length of variation in the y direction is much larger than that in the x direction. A general derivation of Eq. (14) has been given by Bartucelli *et al.*[31] Equation (14) is sometimes referred to as the Zabolotskaya–Khoklov equation in nonlinear acoustics[32,33] with the u_{yy} term representing wave diffraction.

One can rewrite Eq. (14) as divergence-type conservation law by expressions of the form,

$$(u_x)_t+(uu_x-u_{xx})_x+(u_y)_y=0. \tag{15}$$

Following Eq. (4), the potential systems[29] of Eq. (15) are given by

$$\begin{cases} \varphi_x^3-\varphi_y^2=u_x, \\ \varphi_y^1-\varphi_t^3=uu_x-u_{xx}, \\ \varphi_t^2-\varphi_x^1=u_y, \end{cases} \tag{16}$$

where $\varphi^1(t,x,y)$, $\varphi^2(t,x,y)$, and $\varphi^3(t,x,y)$ are potential variables.

For system (16), using the Lie group method, it admits a 4-parameter symmetry group. This group can be represented by the following infinitesimal operators

$$X_1 = \frac{4}{3}t\frac{\partial}{\partial t}+\frac{2}{3}x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}-\frac{2}{3}u\frac{\partial}{\partial u} - \varphi^1\frac{\partial}{\partial \varphi^1}-\frac{1}{3}\varphi^2\frac{\partial}{\partial \varphi^2}-\frac{2}{3}\varphi^3\frac{\partial}{\partial \varphi^3}, \quad (17)$$

$$X_2 = \frac{\partial}{\partial y}, X_3 = \frac{\partial}{\partial x}, X_4 = \frac{\partial}{\partial t}. \quad (18)$$

The set (17) satisfies all conditions (10)–(11). In this case, there are virtually no constraints on the mesh and the difference equation. In particular, we can use an orthogonal mesh regular in both directions in the planes (t,x), (x,y), and (y,t) as long as the invariant orthogonality and uniformness conditions hold for operators (17).

Let us consider the set of operators (17) in the space $(t,x,y,u,u^{(x)+},\ u^{(x)-},u^{(y)+},\ \varphi^1,\ \varphi^{1(x)+},\ \varphi^{1(y)+},\ \varphi^2,\ \varphi^{2(y)+},\ \varphi^{2(t)+},\ \varphi^3,\ \varphi^{3(x)+},\ \varphi^{3(t)+},\ \tau,h,\eta)$, that corresponds to the stencil in the tx plane as shown in Fig. 1, where $u^{(x)-},u,u^{(x)+}$ represent the values of u at the points A,B,C, i.e., $u^{(x)-} = u(t,x-h,y)$, $u = u(t,x,y)$, $u^{(x)+} = u(t,x+h,y)$. $\varphi^{1(y)+}$ represents the value of φ^1 at the point $(t,x,y+\eta)$, and so on. The cases are same in the ty, xy planes, here we omit.

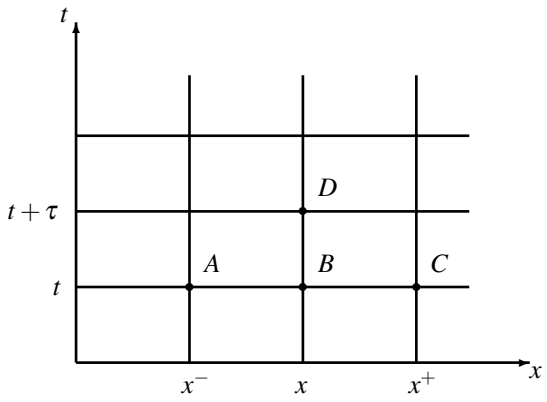


Fig. 1. The difference stencil in the tx plane.

The prolonged operator (17) in this subspace has the form,

$$\begin{aligned}\mathrm{pr}X = {} & \xi^t\frac{\partial}{\partial t}+\xi^x\frac{\partial}{\partial x}+\xi^y\frac{\partial}{\partial y}+(\hat{\xi}^t-\xi^t)\frac{\partial}{\partial \tau}+(\hat{\xi}^x-\xi^x)\frac{\partial}{\partial h}\\ & +(\hat{\xi}^y-\xi^y)\frac{\partial}{\partial \eta}+\psi^u\frac{\partial}{\partial u}+\psi^u_+\frac{\partial}{\partial u_+}+\psi^{\varphi^1}\frac{\partial}{\partial \varphi^1}\\ & +\psi^{\varphi^2}\frac{\partial}{\partial \varphi^2}+\psi^{\varphi^3}\frac{\partial}{\partial \varphi^3}+\cdots+\psi^{\varphi^3}_+\frac{\partial}{\partial \varphi^3_+}.\end{aligned}$$

The number of functionally independent invariants is given by

$$l = \dim M - \mathrm{rank}\, Z, \quad l \geq 0,$$

with $\dim M = 19$ and the matrix Z composed by the coefficients of the prolonged on the space M operators

$$Z = \begin{pmatrix} \xi^t_1 & \xi^x_1 & \xi^y_1 & ((\xi^x_1)_+ - \xi^x_1) & \eta^u_1 & \cdots & (\hat{\psi}^{\varphi^3}_1)_- \\ \vdots & & & & & & \vdots \\ \xi^t_n & \xi^x_n & \xi^y_n & ((\xi^x_n)_+ - \xi^x_n) & \eta^u_n & \cdots & (\hat{\psi}^{\varphi^3}_n)_- \end{pmatrix}.$$

Having found the finite-difference invariants as solutions of linear equations

$$\mathrm{pr}X_i I = 0, \quad i = 1,2,\ldots,n. \quad (19)$$

There are fifteen difference invariants by solving the above equation,

$$\begin{aligned} & I_1 = \frac{u^{(x)+}}{u}, \quad I_2 = \frac{u^{(x)-}}{u}, \quad I_3 = \frac{u^{(y)+}}{u}, \quad I_4 = \frac{\varphi^1}{u^{3/2}},\\ & I_5 = \frac{\varphi^{1(x)+}}{u^{3/2}}, \quad I_6 = \frac{\varphi^{1(y)+}}{u^{3/2}}, \quad I_7 = \frac{\varphi^2}{u^{1/2}}, \quad I_8 = \frac{\varphi^{2(y)+}}{u^{1/2}},\\ & I_9 = \frac{\varphi^{2(t)+}}{u^{1/2}}, \quad I_{10} = \frac{\varphi^3}{u}, \quad I_{11} = \frac{\varphi^{3(x)+}}{u}, \quad I_{12} = \frac{\varphi^{3(t)+}}{u},\\ & I_{13} = hu, \quad I_{14} = \eta u^{3/2}, \quad I_{15} = \tau u^2. \end{aligned} \quad (20)$$

By means of the invariants (20), we can write the following explicit scheme for system (16)

$$\begin{cases} \dfrac{I_{11}-I_{10}}{I_{13}} - \dfrac{I_8-I_7}{I_{14}} = \dfrac{I_1-1}{I_{13}},\\ \dfrac{I_6-I_4}{I_{14}} - \dfrac{I_{12}-I_{10}}{I_{15}} = \dfrac{I_1-1}{I_{13}} - \dfrac{1}{I_{13}^2}(I_1+I_2-2),\\ \dfrac{I_9-I_7}{I_{15}} - \dfrac{I_5-I_4}{I_{13}} = \dfrac{I_3-1}{I_{15}}, \end{cases}$$

i.e.,

$$\begin{cases} \underset{h}{\varphi^3_2} - \underset{h}{\varphi^2_3} = \underset{h}{u_2},\\ \underset{h}{\varphi^1_3} - \underset{h}{\varphi^3_1} = u\underset{h}{u_2} - \underset{h}{u_{2\bar{2}}},\\ \underset{h}{\varphi^2_1} - \underset{h}{\varphi^1_2} = \underset{h}{u_3}. \end{cases}$$

By using the Theorem 2, one can obtain the following results

$$\begin{aligned} & \underset{h}{u_{2\bar{1}}} + h_1\underset{h}{u_{2\bar{1}1}} + \underset{h}{u^2_2} + u\underset{h}{u_{2\bar{2}}} + h_2 u\underset{h}{u_{2\bar{2}2}}\\ & + h_2\underset{h}{u_2}\underset{h}{u_{2\bar{2}}} + h_2^2\underset{h}{u_2}\underset{h}{u_{2\bar{2}2}} - \underset{h}{u_{2\bar{2}2}} + \underset{h}{u_{3\bar{3}}} + h_3\underset{h}{u_{3\bar{3}3}} = 0, \end{aligned} \quad (21)$$

or we can rewrite the result as the following form

$$\underset{h}{u_{21}} + \underset{h}{u^2_2} + u\underset{h}{u_{22}} + h_2\underset{h}{u_2}\underset{h}{u_{22}} - \underset{h}{u_{2\bar{2}2}} + \underset{h}{u_{33}} = 0, \quad (22)$$

where $\underset{h}{u_{21}} = \underset{+h}{D_1}\underset{+h}{D_2}(u), \underset{h}{u_{2\bar{1}}} = \underset{-h}{D_1}\underset{+h}{D_2}(u), \ldots$.

Let us represent the operators (17) in $\underset{h}{Z}$ by extending X_i, $(i=1,2,3,4)$ to τ, h, and η as follows:

$$\tilde{X}_i = \xi^t\frac{\partial}{\partial t}+\xi^x\frac{\partial}{\partial x}+\xi^y\frac{\partial}{\partial y}+\mu\frac{\partial}{\partial u}+\sum_{l\geq 1}\underset{h}{\varsigma}^{i_1\cdots i_l}\frac{\partial}{\partial\, \underset{h}{u}_{i_1\cdots i_l}} + \tau\underset{+\tau}{D}(\xi^t)\frac{\partial}{\partial\tau}+h\underset{+h}{D}(\xi^x)\frac{\partial}{\partial h}+\eta\underset{+\eta}{D}(\xi^y)\frac{\partial}{\partial\eta}, \quad (23)$$

where

$$\begin{aligned}\underset{h}{\varsigma}^1 &= \underset{+\tau}{D}(\mu)-\underset{h}{u}_1\underset{+\tau}{D}(\xi^t)-\underset{+\tau}{S}(u_2)\underset{+\tau}{D}(\xi^x)-\underset{+\tau}{S}(u_3)\underset{+\tau}{D}(\xi^y),\\ \underset{h}{\varsigma}^{21} &= \underset{+h+\tau}{D}\underset{+h+\tau}{D}(\mu)-\underset{h}{u}_{21}(\underset{+\tau}{D}(\xi^t)+\underset{+h}{D}(\xi^x)+\underset{+\eta}{D}(\xi^y))\\ &\quad -\frac{1}{h}\underset{+\tau}{S}(u_2)\underset{+\tau}{D}(\xi^x)-\frac{1}{\tau}\underset{+h}{S}(u_1)\underset{+h}{D}(\xi^t),\\ &\quad \cdots.\end{aligned}$$

We check the invariance criterion (8) for Eq. (21) and Eq. (22),

$$\tilde{X}_i(20)\big|_{(20)}=0,\quad i=1,2,3,4, \quad (24)$$

or

$$\tilde{X}_i(21)\big|_{(21)}=0,\quad i=1,2,3,4. \quad (25)$$

From the above identical equation, one can obtain that the difference equation (21) or (22) admits the group G with operator (23). i.e., the original continuous symmetries are preserved in discrete models.

Remark We point out that the invariant approximation is still not unique. For example, by extending the stencil (i.e., by increasing the number of mesh points involved in the approximation) we can find invariant approximations of any higher order.

4. Conclusions

One system of differential equations can be approximated by infinitely many difference schemes. Hence, finite-difference modeling always involves the problem of selecting the schemes that are in some respect advantageous. The selection criteria are often given by fundamental physical principles present in the original model, such as conservation laws, variational principles, the existence of physically meaningful exact solutions, etc.

The invariance of differential equations under continuous transformation groups is certainly a fundamental property of these models and reflects the homogeneity and isotropy of space-time, the Galilean principle, and other symmetry properties that are intuitively taken into account by the creators of physical models. Therefore, it is apparently important in the theory of difference schemes to preserve the symmetry properties when passing to the finite-difference model, thus adequately representing the symmetry of the original differential model.

Our attention is focused on the problem of constructing difference equations and meshes of high dimensional PDEs such that the difference model preserves the symmetry of the original continuous model. We introduced an application of conservation laws to construct difference models. This method can be used to obtain the difference models preserving the symmetry of original PDEs, and construct difference models of other higher dimensional PDEs. We give an example in the paper, i.e., the (2+1)-dimensional Burgers equation. Through the verification, the difference models that we obtained inherited the symmetry of the original PDEs.

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