Inflationary dynamics in the braneworld scenarios*

Zhang Kai-Yuan(张开源), Wu Pu-Xun(吴普训)†, and Yu Hong-Wei(余洪伟)

Center of Nonlinear Science and Department of Physics, Ningbo University, Ningbo 315211, *China*

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We analyze the attractor behaviour of the inflation field in braneworld scenarios using the Hamilton–Jacobi formalism, where the Friedmann equation has the form of $H^2 = \rho + \varepsilon\sqrt{2\rho_0\rho}$ or $H^2 = \rho + \varepsilon\rho^2/2\sigma$, with $\varepsilon = \pm 1$. We find that in all models the linear homogeneous perturbation can decay exponentially as the scalar field rolls down its potential. However, in the case of a $-\rho^2$ correction to the standard cosmology with $\rho < \sigma$, the existence of an attractor solution requires $(\sigma - \rho)/\dot{\phi}^2 > 1$. Our results show that the perturbation decays more quickly in models with positive-energy correction than in the standard cosmology, which is opposite to the case of negative-energy correction. Thus, the positive-energy modification rather than the negative one can assist the inflation and widen the range of initial conditions.

Keywords: inflation, braneworld, attractor

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1. Introduction

Inflation is a transient and rapidly-accelerating expansion in the very early universe, and it is successful in understanding some of the mysteries of the standard big-bang cosmology, such as the horizon and flatness problems. At the same time, quantum fluctuations during the inflationary era provide a reasonable evidence for the present large-scale structure and the observed anisotropy in the cosmic microwave background. A simple inflation scenario is realized by a scalar field ϕ that rolls slowly down its self-interacting potential $V(\phi)$. This scalar field satisfies a second-order differential equation, so $\dot{\phi}$ can take, in principle, any initial values. As a result, a unique solution at each point of the potential is lacking. However, a successful inflation should be independent of the initial conditions. Otherwise, prediction from inflation would inevitably relate to these unknown initial conditions. Fortunately, an attractor behavior of the inflation equation in the standard cosmology model can rapidly cancel the difference of initial conditions, and ensure that inflation is predictable.[1–3]

Since in the early epoch, the universe is presumably under extreme physical conditions, new effects, such as those stemming from quantization of gravity, or extra dimension, may become important and modify the dynamics of the cosmological evolution. For example, the standard Friedmann equation is modified by the appearance of a ρ^2 term in the Randall–Sundrum (RS) braneworld scenario,[4,5] where the three branes with positive tension are embedded in a five-dimensional anti de Sitter space. This modification assists slowly-rolling inflation, increases the amount of inflation obtained in any given model, and drives the perturbations towards an exactly scale-invariant Harrison–Zeldovich spectrum.[6] It also leads to a more rapid decay of the initial kinetic term, and thus relaxes initial conditions for a successful inflation.[7] For a negative brane tension, Shtanov and Sahni (SS)[8] obtained a new brane scenario with a $-\rho^2$ correction to the standard Friedmann equation. Note that this $-\rho^2$ correction term can also arise from the loop quantum cosmology (LQC) with the inverse volume correction to gravity.[9,10] This negative-energy correction can induce a big bounce, which provides a nice resolution of the big-bang singularity.[8–10] After the bounce, a super-inflation phase without phantom matter can emerge in a natural way.[11] Here, we should note that by extending the spacetime before the big bang and writing the Einstein equation in a non-singular form, Stoica found that the big-bang singularity can also be avoided in the Friedmann–Robertson–Walker (FRW) context.[12–14]

Assuming that the bulk is a flat Minkowski spacetime and a reduced gravity term appears on the brane without tension, Dvali, Gabadadze, and Porrati (DGP)[15] proposed another interesting braneworld scenario. The DGP model induces a $\pm\rho^{1/2}$ rather than a ρ^2 correction term to the standard Friedmann equation, where $\pm$ depends on the way of embedding the four-dimensional brane universe into the five-dimensional spacetime.[16] This model can account for the present accelerated cosmic expansion since the $+$ branch is self-accelerating by slowly leaking the gravity from our four-dimensional world into an extra dimension.[17,18] Furthermore, the DGP model also displays some new characteristics in the very early universe. It has been found that the DGP inflation driven by a single-scalar field with an exponential potential and the DGP

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†Corresponding author. E-mail: wpx0227@gmail.com

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chaotic inflation yields much better consistency with the current observation data.[19,20] Recently, we found that, in the DGP scenario, the universe can undergo a series of infinite nonsingular oscillations, and then evolve to inflation.[21,22] Therefore, the big-bang singularity can be avoided successfully.

Apparently, these modified theories play important roles in dealing with the early universe, especially the inflation era. Recently, Guo *et al.*[7] studied the inflationary attractor in the RS braneworld scenario with high-energy approximation, and showed that the differences between solutions for different initial conditions vanish rapidly, just like that in the standard inflation model. It remains an open question as to whether other brane inflation models have this important property. In the present paper, using the Hamilton–Jacobi formalism,[23–26] we study the brane inflation dynamics in the framework of the Friedmann equation with a $\pm\rho^{1/2}$ or $\pm\rho^2$ correction. For comparison, the RS brane inflation is also considered, but we discard the restriction of the high-energy limit.

2. Attractor behavior for the DGP brane scenario

The DGP theory starts with the idea that our observed four-dimensional universe resides in a five-dimensional, infinite-volume Minkowski bulk, and the whole energy–momentum is confined in a three-dimensional spacial brane.[15] It induces a $\sqrt{\rho}$ modification to the standard Friedmann equation as follows:[16]

$$H^2 = \frac{1}{3\mu^2}(\rho + \varepsilon\sqrt{2\rho\rho_0}), \tag{1}$$

where $\varepsilon = \pm 1$, μ denotes the strength of the induced gravity on the brane, and $\rho_0 = m_\lambda^4 + 6m_5^6/\mu^2$. In the following, for simplicity, we set $3\mu^2 \equiv 1$. In the DGP braneworld, the $\varepsilon = 1$ branch is self-accelerating, and has been proposed to explain the present SNIa observation.[17,27,28] On the other hand, the $\varepsilon = -1$ branch is very different since it does not self-accelerate. The standard Friedmann equation is recovered for $\varepsilon = 0$.

Inflation on the DGP brane has been discussed in Refs. [19], [20], [29], and [30], and it has been found that the modified term loosens the condition for inflation when $\varepsilon = 1$ and tightens the condition when $\varepsilon = -1$.[19] While for a scalar field with exponential potential, the inflationary phase can exit naturally only in the branch of $\varepsilon = -1$ without the need of any other mechanisms, and the $\varepsilon = -1$ DGP inflation yields much better consistency with the current observation data.[19,20] Now, we analyze the attractor behavior of DGP inflation. Both cases of $\varepsilon = 1$ and $\varepsilon = -1$ will be considered. The inflation field satisfies the following evolution equation:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{2}$$

with energy density being $\rho = \dot{\phi}^2/2 + V$ and pressure $p = \dot{\phi}^2/2 - V$, where $V(\phi)$ is the self-interaction potential and $V' = \mathrm{d}V/\mathrm{d}\phi$.

Differentiating Eq. (1) with respect to time and using the scalar field equation, one can obtain

$$H'(\phi) = -\frac{3}{2}\left(1 + \frac{\varepsilon}{2}\sqrt{\frac{2\rho_0}{\rho}}\right)\dot{\phi}\,. \tag{3}$$

For convenience, we use the scalar field as a time variable, which requires that ϕ does not change sign during inflation. Thus, $\dot{\phi} > 0$ is chosen in this paper. For the case of $\dot{\phi} < 0$, this condition can be satisfied by redefining $\phi \to -\phi$. Using the above equation, we have

$$\frac{2}{9}\left(1 + \frac{\varepsilon}{2}\sqrt{\frac{2\rho_0}{\rho}}\right)^{-2} H'^2 + V = \rho\,, \tag{4}$$

where $\sqrt{\rho} = (\sqrt{4H^2 + 2\rho_0} - \varepsilon\sqrt{2\rho_0})/2$.

Equations (3) and (4) are the Hamilton–Jacobi equations, which are more convenient than Eqs. (1) and (2) to discuss the inflation attractor. Assuming that $H(\phi)$ is a solution of Eq. (3) and adding a linear homogeneous perturbation $\delta H(\phi)$ to $H(\phi)$, one can find a tractor behavior if $\delta H(\phi)$ becomes small as ϕ increases. To obtain the solution of $\delta H(\phi)$, we try to obtain the evolutionary equation of $\delta H(\phi)$ by substituting $H(\phi) \to H(\phi) + \delta H(\phi)$ into Eq. (3) and then linearizing it as follows:

$$H'\delta H' = 2(f_1 + f_2 H'^2)H\delta H, \tag{5}$$

where

$$f_1 = \frac{9}{2}\frac{\sqrt{2H^2 + \rho_0}}{\sqrt{2H^2 + \rho_0} - \varepsilon\sqrt{\rho_0}},$$
$$f_2 = -\frac{2\varepsilon\sqrt{\rho_0}}{(2H^2 + \rho_0)(\sqrt{2H^2 + \rho_0} - \varepsilon\sqrt{\rho_0})}. \tag{6}$$

Solving Eq. (5), we obtain

$$\delta H(\phi) = \delta H(\phi_i)\exp\left[\int_{\phi_i}^{\phi} 2\left(f_1\frac{H}{H'} + f_2 HH'\right)\mathrm{d}\phi\right], \tag{7}$$

where $\delta H(\phi_i)$ is the value at some initial point ϕ_i. It is easy to see that, when $\varepsilon = -1$, both f_1 and f_2 are positive. Since Eq. (3) shows that H' and $\mathrm{d}\phi$ have opposite signs, the solution decays exponentially as ϕ increases. This demonstrates that all perturbations die away and the attractor solution exists. For the $\varepsilon = 1$ case, $f_1 > 0$ while $f_2 < 0$. We have $f_1 + f_2 H'^2 > 0$, since $-f_1/(f_2 H'^2) = (2\rho/\dot{\phi}^2)\sqrt{1 + 2H^2/\rho_0} > 1$. Thus, the attractor behavior also occurs in this case.

Now, we use the phase space diagram to demonstrate the attractor behavior discussed above. In our numerical calculation, we consider two independent variables ϕ and $\dot{\phi}$. From Eqs. (1) and (2), we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \dot{\phi}, \tag{8}$$

$$\frac{\mathrm{d}\dot{\phi}}{\mathrm{d}t} = -3\dot{\phi}\sqrt{\frac{1}{2}\dot{\phi}^2+V(\phi)+\varepsilon\sqrt{2\rho_0\left(\frac{1}{2}\dot{\phi}^2+V(\phi)\right)}} - V'(\phi). \tag{9}$$

In the following, two different examples will be considered. The first example is the inflation with a potential of Coleman–Weinberg type which has the following form:

$$V(\phi) = \lambda\phi^4\left(\ln\frac{\phi^2}{\varphi^2}-\frac{1}{2}\right)+\frac{1}{2}\lambda\varphi^4. \tag{10}$$

This potential has a true vacuum at $\pm\varphi$. Setting $\lambda\sim 1$, $\varphi\sim 1$, and $\rho_0\sim 0.05$, we obtain the trajectories in the $(\phi,\dot{\phi})$ plane for this model with different initial conditions (see Fig. 1). The red and blue solid lines represent the results with $\varepsilon = 1,-1$, respectively. The black dashed lines represent the case of the standard cosmology. Apparently, the trajectories with different initial conditions contract rapidly into a curve, which corresponds to the slowly-rolling solution. The red solid, blue solid, and black dashed lines show that the perturbation decays more quickly in the DGP braneworld with $\varepsilon = 1$ than in the standard cosmology, while it is just the opposite for the case of $\varepsilon = -1$.

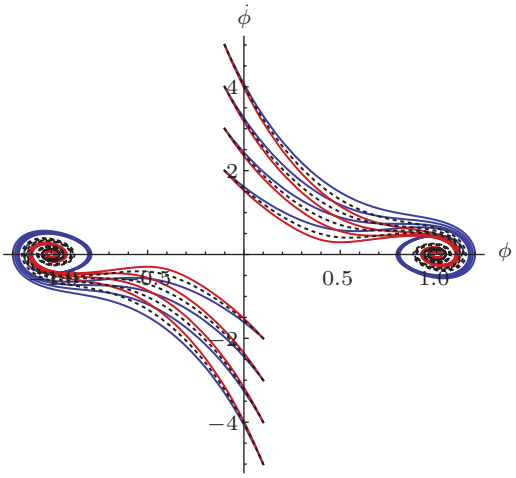


Fig. 1. (color online) Phase portrait for the scalar field with $V(\phi) = \lambda\phi^4(\ln(\phi^2/\varphi^2)-1/2)+\lambda\varphi^4/2$ in coordinates $(\phi,\dot{\phi})$ in the DGP brane. The red and blue solid lines correspond to $\varepsilon = 1$ and $\varepsilon = -1$, respectively, and the black dashed lines show the result of the standard cosmology (same representation is used in the following figures).

The second example is the simple chaotic inflation model, which has the potential

$$V(\phi) = \frac{1}{2}m^2\phi^2. \tag{11}$$

Figure 2 displays the trajectories in $(\phi,\dot{\phi})$ plane for this potential with $m^2\sim 1$ and $\rho_0\sim 0.02$. Similar to the first example, the perturbation decays more quickly in the DGP model with $\varepsilon = 1$ than in the standard cosmology, while for the $\varepsilon = -1$ case, it decays more slowly. From Figs. 1 and 2, one can see that a positive energy correction term assists the slowly-rolling inflation, while a negative one does not.

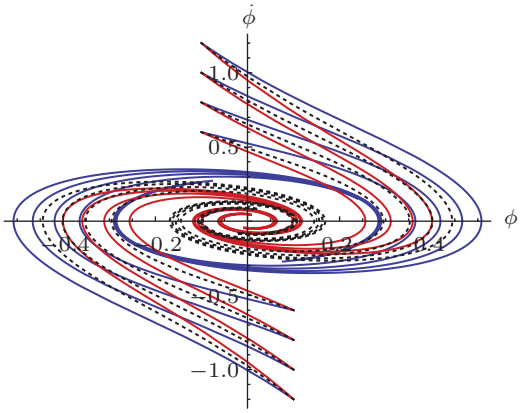


Fig. 2. (color online) Phase portrait for the scalar field with $V(\phi) = \frac{1}{2}m^2\phi^2$ in $(\phi,\dot{\phi})$ coordinates in the DGP brane.

3. Attractor behavior for the RS AND SS brane scenarios

The RS brane scenario induces a ρ^2 correction to the standard Friedmann equation,[4,5] while the SS model leads to a $-\rho^2$ modification term.[8] Thus, in the RS and SS brane scenarios, the modified Friedmann equation can be expressed as

$$H^2 = \rho+\varepsilon\frac{\rho^2}{2\sigma}, \tag{12}$$

where σ is the brane tension, and $\varepsilon = 1,-1$ correspond to the RS and SS models, respectively. When $\varepsilon = 1$, the high-energy limit gives $H^2 = \rho^2/2\sigma$, whose attractor solution has been studied in Ref. [7]. Here, we consider a general model. The $\varepsilon = -1$ model can also be obtained from the LQC, in which $2\sigma = \rho_c$ with ρ_c being a critical energy density.[9,10] When $\rho = 2\sigma$ and $\varepsilon = -1$, $H = 0$ and a big bounce scenario is obtained. Thus, in the inflation era, $\rho < 2\sigma$ is required.

Using the same method as the DGP model, we can obtain the following Hamilton–Jacobi equations in the RS and SS brane scenarios:

$$H'(\phi) = -\frac{3}{2}\left(1+\varepsilon\frac{\rho}{\sigma}\right)\dot{\phi}, \tag{13}$$

$$\frac{2}{9}H'^2\left(1+\varepsilon\frac{\rho}{\sigma}\right)^{-2}+V = \rho, \tag{14}$$

with $\rho = \sigma(-1+\sqrt{1+2H^2/\sigma})$ for $\varepsilon = 1$ and $\rho = \rho_\pm = \sigma(1\pm\sqrt{1-2H^2/\sigma})$ for $\varepsilon = -1$. Apparently, $\sigma < \rho_+ < 2\sigma$ and $\rho_- < \sigma$. Equation (13) indicates that $H'(\phi)$ and $\mathrm{d}\phi$ have opposite signs for the $\rho = \rho_+$ case, while for $\rho = \rho_-$, the signs are the same. Considering a linear homogeneous perturbation $\delta H(\phi)$ for any solution $H(\phi)$, and linearizing Eq. (14), one can obtain

$$\delta H' = \left(\frac{2\varepsilon H'^2/\sigma}{1+2\varepsilon H^2/\sigma}-e\frac{9}{2}\sqrt{1+2\varepsilon\frac{H^2}{\sigma}}\right)\frac{H}{H'}\delta H, \tag{15}$$

where $e=-1$ when $\varepsilon=1$, and $e=\pm1$ for $\varepsilon=-1$ with $\pm$ corresponding to $\rho=\rho_{\pm}$, respectively. Solving this equation leads to

$$\delta H(\phi) = \delta H(\phi_i)\exp\left[\int_{\phi_i}^{\phi}\left(\frac{2\varepsilon H'^2/\sigma}{1+2\varepsilon H^2/\sigma} - e\frac{9}{2}\sqrt{1+2\varepsilon\frac{H^2}{\sigma}}\right)\frac{H}{H'}\mathrm{d}\phi\right]. \quad (16)$$

For the $\varepsilon=1$ case, because both terms in parentheses are positive, and $\mathrm{d}\phi$ and $H'(\phi)$ have opposite signs, $\delta H(\phi)$ decays exponentially as ϕ increases. When $\varepsilon=-1$, $\rho=\rho_+$, $e=1$, and $\sigma<\rho<2\sigma$, both terms in parentheses are negative, and $\mathrm{d}\phi$ and $H'(\phi)$ have same signs. Thus, there is an attractor solution. If $\rho=\rho_-$, then $e=-1$ and $\rho<\sigma$. An attractor behavior requires that the sum of two terms in parentheses is positive because $\mathrm{d}\phi$ and $H'(\phi)$ have opposite signs. Since the first term in parentheses is negative, while the second one is positive, we find that the condition for an attractor is

$$\frac{\sigma-\rho}{\dot\phi^2}>1\,, \quad (17)$$

which is obtained from the ratio between the second term and the first one.

Considering a potential given by Eq. (10), we display the trajectories in the $(\phi,\dot\phi)$ plane in Fig. 3. Parameters are set as $\sigma\sim2.5$, $\lambda\sim1$, and $\varphi\sim1$. We can see that all trajectories with different initial conditions contract into the slowly-rolling curve. The perturbation decays more rapidly in the $\varepsilon=1$ case than in the standard cosmology. When $\varepsilon=-1$, it is just the opposite.

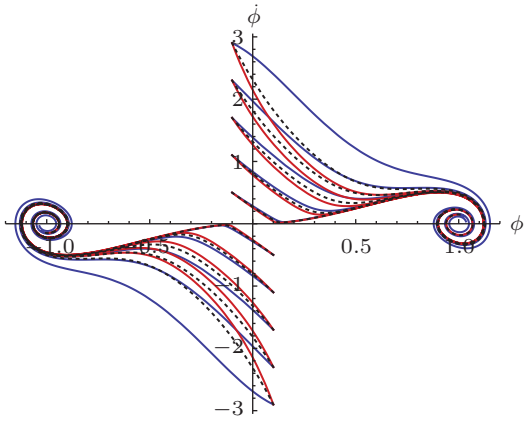


Fig. 3. (color online) Phase portrait for the scalar field with $V(\phi)=\lambda\phi^4(\ln(\phi^2/\varphi^2)-1/2)+\lambda\varphi^4/2$ in $H^2=\rho+\varepsilon\rho^2/2\sigma$ model.

For another potential mentioned in Eq. (11), the results are shown in Fig. 4, which are similar to the first-type potential. The attractor behavior appears, and the perturbation decays most rapidly in the RS brane scenario.

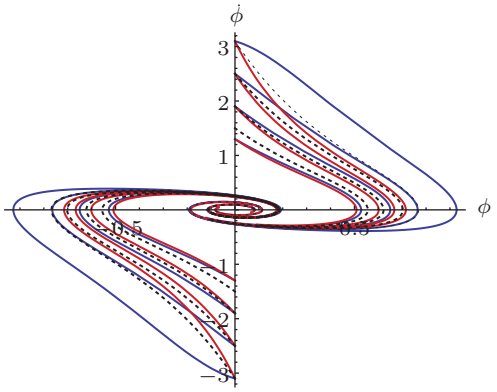


Fig. 4. (color online) Phase portrait for the scalar field with $V(\phi)=m^2\phi^2/2$ in $H^2=\rho+\varepsilon\rho^2/2\sigma$ model.

4. Conclusion

The braneworld scenarios induce some modifications to the standard Friedmann equation. For example, there is a $\pm\rho^{1/2}$ correction term in the DGP brane model. The RS braneworld scenario leads to a ρ^2 correction, while the SS brane induces a $-\rho^2$ one. This $-\rho^2$ modification can also be obtained from the LQC. These modifications may lead to some new characters in the slowly-rolling inflation, and thus spur an increasing deal of interest. In this paper, using the Hamilton–Jacobi formalism, we discuss the attractor behavior of the inflation field in these brane models. We find that in all models the linear homogeneous perturbations can decay exponentially as the scalar field rolls along its potential. However, in the SS brane inflation with $\rho<\sigma$, the existence of an attractor solution requires $(\sigma-\rho)/\dot\phi^2>1$, which shows that the energy density of inflaton field cannot be close to the value of the brane tension, i.e., the kinetic energy of the scalar field cannot be too large. Otherwise, there is not an attractor inflation solution, and predictions from inflation would depend on the unknown initial conditions. Thus, a successful inflation cannot be achieved. In addition, we find that the perturbation decays more quickly in models with positive-energy correction than in the standard cosmology, while it is just the opposite for the case of negative-energy correction. This is because the positive-energy correction term increases the Hubble parameter so that the friction term $3H\dot\phi$ in the evolution equation of the scalar field becomes larger, leading to a more rapid decay of the initial kinetic term. Thus, the positive-energy correction can assist the inflation and widen the range of initial conditions, while the negative-energy one does not although the $\varepsilon=-1$ DGP inflation yields much better consistency with the current observation data.

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