Dynamic aggregation evolution of competitive societies of cooperative and noncooperative agents*

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We propose an evolution model of cooperative agent and noncooperative agent aggregates to investigate the dynamic evolution behaviors of the system and the effects of the competing microscopic reactions on the dynamic evolution. In this model, each cooperative agent and noncooperative agent are endowed with integer values of cooperative spirits and noncooperative spirits, respectively. The cooperative spirits of a cooperative agent aggregate and the noncooperative spirits of a noncooperative agent aggregate change via four competing microscopic reaction schemes: the win–win reaction between two cooperative agents, the lose–lose reaction between two noncooperative agents, the win–lose reaction between a cooperative agent and a noncooperative agent (equivalent to the migration of spirits from cooperative agents to noncooperative agents), and the cooperative agent catalyzed decline of noncooperative spirits. Based on the generalized Smoluchowski's rate equation approach, we investigate the dynamic evolution behaviors such as the total cooperative spirits of all cooperative agents and the total noncooperative spirits of all noncooperative agents. The effects of the three main groups of competition on the dynamic evolution are revealed. These include: (i) the competition between the lose–lose reaction and the win–lose reaction, which gives rise to respectively the decrease and increase in the noncooperative agent spirits; (ii) the competition between the win–win reaction and the win–lose reaction, which gives rise to respectively the increase and decrease in the cooperative agent spirits; (iii) the competition between the win–lose reaction and the catalyzed-decline reaction, which gives rise to respectively the increase and decrease in the noncooperative agent spirits.

Keywords: dynamic evolution in competitive societies, cooperative and noncooperative agents, win–win and lose–lose reaction, win–lose reaction

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1. Introduction

Aggregation growth phenomena are popular in nature and important for abundant kinetic evolutional behaviors arising from a variety of complex mechanisms.[1–4] The kinetic evolutional behaviors of aggregates are of theoretical and practical significance for understanding aggregation growth phenomena. Originally studied mechanisms in pure aggregation processes have been steadily extended to aggregation phenomena with more complex mechanisms, such as fragmentation, annihilation, their various combinations, and so on.[5,6]

Recently, the concepts and techniques used in aggregation growth phenomena in nature have been generalized to quantitatively model some generalized aggregation growth phenomena in social and economic systems to investigate the kinetic behaviors of their aggregation growth. Ispolatov *et al.* introduced several asset exchange models for the evolution of the wealth distribution in the economical interaction population.[7] Leyvraz and Redner proposed a migration-driven irreversible aggregate growth model for the evolution of city population.[8] In these models, irreversible growth of aggregates takes place through biased migration or unbiased exchange mechanisms. In the biased migration model, there exists preferential migration of a monomer (equivalently, one unit of assets in asset exchange models or one person in population migration models) from a smaller aggregate to a larger aggregate. This microscopic evolution mechanism can be described by an irreversible reaction scheme, $A_k + A_l \xrightarrow{K(k,l)} A_{k-1} + A_{l+1}$ $(k \leq l)$, where A_k denotes an aggregate characterized only by its size k (an aggregate of k units of assets in asset exchange models or k persons in population migration models) and the migration rate kernel $K(k,l)$ represents the rate of a monomer migrating from an aggregate of size k to another aggregate of size l, which generally depends on the size of the two reactant aggregates. While in an unbiased exchange model, an aggregate is equally likely to gain or to loss a monomer. These processes exhibit much more abundant kinetic behaviors than those in the single aggregation, annihilation, fragmentation processes or their various combinations. Furthermore, in the migration-driven city population growth model, Leyvraz and Redner found a qualitative correspondence between model predictions and recent data on U.S. cities.

Subsequently, we investigated the kinetics of the general unbiased exchange-driven growth system.[9] We further generalized the research to the exchange-driven aggregation

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with birth and death to mimic the evolution of city population and individual wealth.[10,11] Finally, Ben-Naim and Krapivsky made general research on exchange-driven growth with product rate kernel $K(k,l) = K(l,k) = (kl)^{\lambda}$ and the generalized homogeneous rate kernel $K(k,l) = K(l,k) = (k^{\nu}l^{\mu} + k^{\mu}l^{\nu})$.[12] The kinetic behavior was found to fall into three categories: growth, gelation, and instant gelation.

Later, we further generalized those studies to the aggregation growth systems of more than one species with interactions between two aggregates of different species to provide better aggregation growth models for the real social aggregation phenomena. As an example, we studied the mutually catalyzed birth of population and assets in exchange-driven growth to investigate the interaction between the population and the asset aggregates.[13] The competition between the two growth mechanisms, exchange and catalyzed birth, gives rise to much more abundant kinetic behaviors. Furthermore, the complex forms of the competition between the two growth mechanisms and the effects of the two competing mechanisms on the kinetic evolution behaviors of the system are revealed. Those studies were also generalized to a three-species exchange-driven aggregate growth model with competition between catalyzed birth and catalyzed death,[14,15] and a catalysis-select migration driven evolution model of two-species (A- and B-species) aggregates where one unit of species A migrates to species B under the catalysis by a species C aggregate, while under the catalysis by a species D aggregate the reaction will become one unit of species B migrating to species A.[16] Moreover, we also generalized the study to the kinetics of migration-driven aggregate growth on completely connected scale-free network. The analytic results were found to be in good agreement with the data of the population distributions of all U.S. counties.[17]

On the other hand, Ben-Naim and Redner introduced and solved analytically a simple advancement-decline model based on the interplay between advancement and decline to study the formation of the social hierarchies that are ubiquitously observed in animal populations and in human societies.[18] In this model, each agent is endowed with an integer fitness value and individual agents advance their fitness by competing against those in lower classes, and in parallel, individuals decline due to inactivity. The reaction scheme can be described as: (i) the fitness strength of stronger competitor increases by one unit, $A_k + A_l \xrightarrow{K(k,l)} A_{k+1} + A_l$ (for $k \geq l$) with rate kernel $K(k,l) = 1$; (ii) the fitness of each agent declines by one unit with rate r, $A_k \xrightarrow{r(k)=r} A_{k-1}$. The latter is just the self-death as in former city population evolution models. The parameter r fixes the balance of advancement and decline. The results successfully demonstrate a phase transition from a homogeneous, single-class society to a hierarchical, multi-class society. Later, Ben-Naim *et al.* generalized the model by introducing a stochastic advancement dynamics, in which the stronger competitor of a pair of interacting agents wins advancement with a probability of p and the agent of less fitness wins advancement with a probability of $1-p$.[19] This model yields a richer phase diagram.

We considering here that there are always many species of individual agents in animal populations and in human societies, and the interplay between agents of different species may play important roles in the dynamic evolution processes. Later, we proposed an aggregation evolution model of two-species (A- and B-species) aggregates to study the complex aggregation evolution phenomena in social and economic systems, where agents of each species perform self-exchange and self-birth processes, while the interaction between two aggregates of different species causes a monomer lose–lose scheme, $A_k + B_l \xrightarrow{H(k,l)} A_{k-1} + B_{l-1}$ with the rate kernel $H(k,l) = Hkl$.[20] The results showed that the evolution behaviors are crucially dominated by the competition between the two self-birth processes, while the lose–lose scheme plays important roles in some special cases.

Looking further into the competition-driven dynamic evolutions in animal populations and in human societies, where there are many species of individual agents, their cooperation and competition as a whole drive the dynamic evolutions of the systems and bring into force richer evolutionary behaviors. Inspired by the constant situations in game theory, namely, win–win, lose–lose, and win–lose, we introduce a dynamic evolutionary model of a system consisting of cooperative agents and noncooperative agents, and investigate their evolution behaviors driven by the microscopic interaction schemes such as win–win, lose–lose, and win–lose.

In this model, each cooperative agent A_k of species A is endowed with an integer value k ($k \geq 0$) of cooperative spirits, and each noncooperative agent B_k of species B is endowed with an integer value k ($k \geq 0$) of noncooperative spirits. On our consideration, the cooperative spirit value of a cooperative agent and the noncooperative spirit value of a noncooperative agent change due to the following four processes. (i) The win–win interaction between two cooperative agents: when two cooperative agents meet they will both profit from their cooperation and encourage their cooperative spirits to increase by one unit. (ii) The lose–lose interaction between two noncooperative agents: when two noncooperative agents meet they will both lose from their noncooperation and make their noncooperative spirits decrease by one unit. When a cooperative agent meets a noncooperative agent, two different reaction schemes will occur with some probabilities: (iii) the win–lose interaction with probability p ($0 \leq p \leq 1$), where a cooperative agent loses from the reaction and his cooperative spirits decrease by one unit, while the noncooperative agent profits from the reaction and he is encouraged to increase his noncooperative spirits by one; (iv) the catalyzed decline of noncooperative spir-

its with probability $q = 1 - p$, where the cooperative agent leads or urges the noncooperative agent to increase his cooperative spirits, and thus his noncooperative spirits decrease by one unit. We believe that our present model may provide a better description of a wide variety of the dynamic evolutionary phenomena of social and economic agent systems and exhibit effects of these reaction schemes on the aggregation evolution behaviors.

Our investigation is based on the mean-field theory, where the fluctuations in densities of the reactant agent aggregates are assumed to be ignorable and the aggregates are considered to be homogeneously distributed in the space throughout the whole aggregation evolution process.

The rest of this paper is organized as follows. In Section 2, we introduce the aggregation evolution model of the cooperative and the noncooperative agents in the competitive societies. The dynamic evolution behaviors of the agent aggregates of the two species are studied by the generalized Smoluchowski's rate equation approach. In Section 3, we investigate the dynamic evolution behaviors such as the total cooperative spirits of all cooperative agents and the total noncooperative spirits of all noncooperative agents. The effects of the three main groups of the competitions on the dynamic evolution are revealed. Finally, a brief summary is given in Section 4.

2. Model of the aggregation evolution of cooperative and noncooperative agents in competitive societies

The interaction schemes in our model can be described by the following formulas: (i) win–win interaction between two cooperative agents, $A_k + A_l \xrightarrow{K(k,l)} A_{k+1} + A_{l+1}$ with the reactive rate kernel $K(k,l)$; (ii) lose–lose interaction between two noncooperative agents, $B_k + B_l \xrightarrow{L(k,l)} B_{k-1} + B_{l-1}$ with the reactive rate kernel $L(k,l)$; (iii) win–lose interaction with probability p, $A_k + B_l \xrightarrow{I(k,l),p} A_{k-1} + B_{l+1}$ with reactive rate kernel $I(k,l)$, which is equivalent to the migration of one unit from an A-species aggregate to a B-species aggregate; (iv) catalyzed decline of noncooperative spirits with probability $q = 1 - p$, $B_k + A_l \xrightarrow{J(k,l),q} B_{k-1} + A_l$ with rate kernel $J(k,l)$, which is equivalent to the catalyzed-death of species B by species A. As a reasonable assumption, here we investigate the case of $p = q = 1/2$.

In general, the rate kernels of those reactions are dependent on the reactant aggregates' sizes. For the convenience of solving the rate equations, we focus on the most reasonable case: the rate kernels of the above four reactions are all proportional to the reactant aggregates' sizes. Then those rate kernels are $K(k,l) = Kkl$, $L(k,l) = Lkl$, $I(k,l) = Ikl$, and $J(k,l) = Jkl$, where K, L, I, and J are positive constants.

The investigation on the kinetic behaviors of the system is focused mainly on total mass (the total number of the cooperative spirits and the total number of the noncooperative spirits) of the two species agent aggregates and the evolution of the concentrations of the reactant aggregates. Here, they are the cooperative agent aggregates' size distribution $a_k(t)$ and that of the noncooperative agent aggregates $b_k(t)$, which denote the concentrations or the numbers of the cooperative agent aggregates A_k and the noncooperative agent aggregates B_k at time t, respectively.

In this model, we assume that the system has spatial homogeneity, so that the fluctuations in the densities of the reactant aggregates (cooperative agents or noncooperative agents) are ignored and the aggregates of each species are considered to be homogeneously distributed in the space throughout the whole evolution process. The theoretical approach to investigate the kinetics of the aggregation process can then be based on the rate equations in the mean-field frame, which assumes a reaction proceeds at a rate proportional to the reactant concentrations. We generalize the rate equation of the exchange or migration-driven aggregation process[7,13] and write the corresponding rate equations for our system as follows:

$$\begin{aligned}\frac{\mathrm{d}a_k}{\mathrm{d}t} &= \sum_{l=1}^{\infty} K(k-1,l)a_{k-1}a_l - \sum_{l=1}^{\infty} K(k,l)a_k a_l \\ &\quad + \frac{1}{2}\sum_{l=1}^{\infty} I(k+1,l)a_{k+1}b_l - \frac{1}{2}\sum_{l=1}^{\infty} I(k,l)a_k b_l, \quad (1)\\ \frac{\mathrm{d}b_k}{\mathrm{d}t} &= \sum_{l=1}^{\infty} L(k+1,l)b_{k+1}b_l - \sum_{l=1}^{\infty} L(k,l)b_k b_l \\ &\quad + \frac{1}{2}\sum_{l=1}^{\infty} I(l,k-1)b_{k-1}a_l - \frac{1}{2}\sum_{l=1}^{\infty} I(l,k)b_k a_l \\ &\quad + \frac{1}{2}\sum_{l=1}^{\infty} J(k+1,l)b_{k+1}a_l - \frac{1}{2}\sum_{l=1}^{\infty} J(k,l)b_k a_l. \quad (2)\end{aligned}$$

In Eq. (1), the first two terms account for the gain in $a_k(t)$ due to the win–win processes $A_{k-1} + A_l \xrightarrow{K(k-1,l)} A_k + A_{l+1}$ and the lose in $a_k(t)$ due to $A_k + A_l \xrightarrow{K(k,l)} A_{k+1} + A_{l+1}$. The third and fourth terms account for the gain and lose in $a_k(t)$ due to the migration processes $A_{k+1} + B_l \xrightarrow{I(k+1,l),p=\frac{1}{2}} A_k + B_{l+1}$ and $A_k + B_l \xrightarrow{I(k,l),p=\frac{1}{2}} A_{k-1} + B_{l+1}$, respectively.

We assume that there exist only monomer aggregates of cooperative agents or noncooperative agents at initial time $t = 0$ (more precisely, at $t = 0^+$). Consequently, the monodisperse initial conditions for the two species agent aggregates can be written as

$$a_k(0) = A_0\delta_{k1}, \quad b_k(0) = B_0\delta_{k1}, \quad (3)$$

where A_0 and B_0 are the concentrations (or the total numbers) of the initial monomer cooperative agent aggregates and initial monomer cooperative agent aggregates, respectively.

The υ-th moments of the aggregate size distribution $a_k(t)$ of species A and $b_k(t)$ of species B can be expressed as

$$M_\upsilon^{\rm A}(t)=\sum_{k=1}^{\infty}k^\upsilon a_k(t),\quad M_\upsilon^{\rm B}(t)=\sum_{k=1}^{\infty}k^\upsilon b_k(t). \tag{4}$$

In particular, the first two moments $M_0^{\rm A}(t)=\sum\limits_{k=1}^{\infty}a_k(t)$ and $M_1^{\rm A}(t)=\sum\limits_{k=1}^{\infty}ka_k(t)$ are the total number and the total mass of A-species aggregates, respectively. They are similar for B-species aggregates.

With the aid of the expressions of the moments of the aggregate size distributions, the above rate given by Eqs. (1) and (2) can be rewritten in the following form:

$$\begin{aligned}\frac{{\rm d}a_k}{{\rm d}t}&=K[(k-1)a_{k-1}-ka_k]M_1^{\rm A}\\&\quad+\frac{1}{2}I[(k+1)a_{k+1}-ka_k]M_1^{\rm B},\end{aligned} \tag{5}$$

$$\begin{aligned}\frac{{\rm d}b_k}{{\rm d}t}&=L[(k+1)b_{k+1}-kb_k]M_1^{\rm B}\\&\quad+\frac{1}{2}I[(k-1)b_{k-1}-kb_k]M_1^{\rm A}\\&\quad+\frac{1}{2}J[(k+1)b_{k+1}-kb_k]M_1^{\rm A}.\end{aligned} \tag{6}$$

Equations (5) and (6) include a large number of correlated nonlinear equations. Several methods have been developed to solve the rate equations in different aggregation processes. One is to make a special ansatz $a_k(t)=A(t)[a(t)]^{k-1}$ to find some special solutions, whose applicability was thoroughly discussed in our former work.[11] Here we find current rate equations of this model can be solved with the help of the ansatz

$$a_k(t)=A(t)[a(t)]^{k-1},\quad b_k(t)=B(t)[b(t)]^{k-1}. \tag{7}$$

By applying ansatz (7), we write the first two moments of the aggregate size distribution $a_k(t)$ of species A as

$$\begin{aligned}M_0^{\rm A}(t)&=\sum_{k=1}^{\infty}a_k(t)=\frac{A}{(1-a)},\\M_1^{\rm A}(t)&=\sum_{k=1}^{\infty}ka_k(t)=\frac{A}{(1-a)^2},\end{aligned} \tag{8}$$

with their relation $M_0^{\rm A}=M_1^{\rm A}(1-a)$. Those for species B are

$$\begin{aligned}M_0^{\rm B}(t)&=\sum_{k=1}^{\infty}b_k(t)=\frac{B}{(1-b)},\\M_1^{\rm B}(t)&=\sum_{k=1}^{\infty}kb_k(t)=\frac{B}{(1-b)^2},\end{aligned} \tag{9}$$

and their relation is $M_0^{\rm B}=M_1^{\rm B}(1-b)$.

Substituting ansatz (7) into Eqs. (5) and (6), they can be transformed into the following differential equations:

$$\frac{1}{1-a}\frac{{\rm d}a}{{\rm d}t}=KM_1^{\rm A}-\frac{1}{2}IM_1^{\rm B}a, \tag{10}$$

$$\frac{1}{A}\frac{{\rm d}A}{{\rm d}t}=-KM_1^{\rm A}-\frac{1}{2}IM_1^{\rm B}(1-2a), \tag{11}$$

$$\frac{1}{1-b}\frac{{\rm d}b}{{\rm d}t}=\frac{1}{2}IM_1^{\rm A}-\left[\frac{1}{2}JM_1^{\rm A}+LM_1^{\rm B}\right]b, \tag{12}$$

$$\frac{1}{B}\frac{{\rm d}B}{{\rm d}t}=-\frac{1}{2}IM_1^{\rm A}-\left[\frac{1}{2}JM_1^{\rm A}+LM_1^{\rm B}\right](1-2b). \tag{13}$$

Correspondingly, the monodisperse initial conditions (3) at $t=0$ are transformed as follows:

$$a(0)=0,\quad A(0)=A_0,\quad b(0)=0,\quad B(0)=B_0. \tag{14}$$

From these equations, we can derive the equations for the first two moments of the aggregate size distributions of species A and B as follows:

$$\begin{aligned}\frac{{\rm d}\ln M_0^{\rm A}}{{\rm d}t}&=\frac{1}{1-a}\frac{{\rm d}a}{{\rm d}t}+\frac{1}{A}\frac{{\rm d}A}{{\rm d}t}\\&=-\frac{1}{2}IM_1^{\rm B}(1-a)=-\frac{1}{2}IM_1^{\rm B}\frac{M_0^{\rm A}}{M_1^{\rm A}},\end{aligned} \tag{15}$$

$$\frac{{\rm d}\ln M_1^{\rm A}}{{\rm d}t}=\frac{2}{1-a}\frac{{\rm d}a}{{\rm d}t}+\frac{1}{A}\frac{{\rm d}A}{{\rm d}t}=KM_1^{\rm A}-\frac{1}{2}IM_1^{\rm B}, \tag{16}$$

$$\begin{aligned}\frac{{\rm d}\ln M_0^{\rm B}}{{\rm d}t}&=\frac{1}{1-b}\frac{{\rm d}b}{{\rm d}t}+\frac{1}{B}\frac{{\rm d}B}{{\rm d}t}=-\left[\frac{1}{2}JM_1^{\rm A}+LM_1^{\rm B}\right](1-b)\\&=-\left[\frac{1}{2}JM_1^{\rm A}+LM_1^{\rm B}\right]\frac{M_0^{\rm B}}{M_1^{\rm B}},\end{aligned} \tag{17}$$

$$\frac{{\rm d}\ln M_1^{\rm B}}{{\rm d}t}=\frac{2}{1-b}\frac{{\rm d}b}{{\rm d}t}+\frac{1}{B}\frac{{\rm d}B}{{\rm d}t}=\frac{1}{2}(I-J)M_1^{\rm A}-LM_1^{\rm B}. \tag{18}$$

The corresponding initial conditions at $t=0$ are given by

$$M_0^{\rm A}(0)=M_1^{\rm A}(0)=A_0,\quad M_0^{\rm B}(0)=M_1^{\rm B}(0)=B_0. \tag{19}$$

Those equations can be rewritten as

$$\frac{{\rm d}(1/M_0^{\rm A})}{{\rm d}t}=\frac{1}{2}I\frac{M_1^{\rm B}}{M_1^{\rm A}}, \tag{20}$$

$$\frac{{\rm d}(1/M_1^{\rm A})}{{\rm d}t}=-K+\frac{1}{2}I\frac{M_1^{\rm B}}{M_1^{\rm A}}, \tag{21}$$

$$\frac{{\rm d}(1/M_0^{\rm B})}{{\rm d}t}=L+\frac{1}{2}J\frac{M_1^{\rm A}}{M_1^{\rm B}}, \tag{22}$$

$$\frac{{\rm d}(1/M_1^{\rm B})}{{\rm d}t}=L-\frac{1}{2}(I-J)\frac{M_1^{\rm A}}{M_1^{\rm B}}. \tag{23}$$

By introducing four new functions

$$n_{\rm A}=\frac{1}{M_0^{\rm A}},\quad m_{\rm A}=\frac{1}{M_1^{\rm A}},\quad n_{\rm B}=\frac{1}{M_0^{\rm B}},\quad m_{\rm B}=\frac{1}{M_1^{\rm B}}, \tag{24}$$

the above equations can be transformed into

$$\frac{{\rm d}n_{\rm A}}{{\rm d}t}=\frac{1}{2}I\frac{m_{\rm A}}{m_{\rm B}}, \tag{25}$$

$$\frac{{\rm d}m_{\rm A}}{{\rm d}t}=-K+\frac{1}{2}I\frac{m_{\rm A}}{m_{\rm B}}, \tag{26}$$

$$\frac{{\rm d}n_{\rm B}}{{\rm d}t}=L+\frac{1}{2}J\frac{m_{\rm B}}{m_{\rm A}}, \tag{27}$$

$$\frac{\mathrm{d}m_{\mathrm{B}}}{\mathrm{d}t} = L - \frac{1}{2}(I-J)\frac{m_{\mathrm{B}}}{m_{\mathrm{A}}}, \tag{28}$$

with initial conditions

$$n_{\mathrm{A}}(0) = m_{\mathrm{A}}(0) = \frac{1}{A_0}, \quad n_{\mathrm{B}}(0) = m_{\mathrm{B}}(0) = \frac{1}{B_0}. \tag{29}$$

In the next section, we study the kinetics of the system in various cases to reveal the competition among the above reaction mechanisms.

3. Dynamic aggregation evolution of cooperative and noncooperative agents in competitive societies

At first, we can derive the relation between n_{A} and m_{A} from Eqs. (25) and (26) as

$$n_{\mathrm{A}} - m_{\mathrm{A}} = Kt, \tag{30}$$

and that between n_{B} and m_{B} from Eqs. (27) and (28) as

$$(I-J)n_{\mathrm{B}} + Jm_{\mathrm{B}} = \frac{I}{B_0}(1+LB_0t). \tag{31}$$

From Eqs. (26) and (28), we derive an equation of m_{A} and m_{B} as

$$\frac{\mathrm{d}m_{\mathrm{A}}}{\mathrm{d}m_{\mathrm{B}}} = \frac{-K + Im_{\mathrm{A}}/2m_{\mathrm{B}}}{L - m_{\mathrm{B}}(I-J)/2m_{\mathrm{A}}}. \tag{32}$$

Introducing a new function $u(t) = m_{\mathrm{A}}(t)/m_{\mathrm{B}}(t)$, equation (32) can be transformed into

$$m_{\mathrm{B}}\frac{\mathrm{d}u}{\mathrm{d}m_{\mathrm{B}}} = \frac{1}{L}\frac{au^2 - bu}{u - c}, \tag{33}$$

where the constants $a = I/2 - L$, $b = K - (I-J)/2$, and $c = (I-J)/2L$.

In the next subsections, we study the kinetic evolution behaviors of the system in various cases with different probability p and constants a, b, and c, and further reveal the competition among different mechanisms.

3.1. Case of $I = J$

First we study the simplest case. In this case, we have $c = (I-J)/2L = 0$ and can solve $m_{\mathrm{B}}(t)$ directly from Eq. (28)

$$\frac{\mathrm{d}m_{\mathrm{B}}}{\mathrm{d}t} = L. \tag{34}$$

Under the initial conditions (29), the solution is

$$m_{\mathrm{B}} = \frac{1}{B_0}(1+LB_0t). \tag{35}$$

Inserting it into Eq. (26), we obtain the equation of $m_{\mathrm{A}}(t)$

$$\frac{\mathrm{d}m_{\mathrm{A}}}{\mathrm{d}t} - \frac{IB_0}{2}(1+LB_0t)^{-1}m_{\mathrm{A}} = -K. \tag{36}$$

This equation can be solved in two cases.

(i) Case of $a = I/2 - L > 0$. In this case, equation (36) yields

$$m_{\mathrm{A}}(t) = \frac{1}{(I/2-L)A_0B_0}(1+LB_0t)\Big\{KA_0 + \left[\left(\frac{I}{2}-L\right)B_0 - KA_0\right](1+LB_0t)^{I/2L-1}\Big\}. \tag{37}$$

From this solution, we can derive $n_{\mathrm{A}}(t)$ directly from Eq. (30) as

$$n_{\mathrm{A}}(t) = \frac{K}{(I/2-L)B_0}(1+LB_0t) + \frac{(I/2-L)B_0 - KA_0}{(I/2-L)A_0B_0}(1+LB_0t)^{I/2L} + Kt. \tag{38}$$

Equation (27) becomes

$$\frac{\mathrm{d}n_{\mathrm{B}}}{\mathrm{d}t} = L + \frac{(J/2)(I/2-L)A_0}{KA_0 + [(I/2-L)B_0 - KA_0](1+LB_0t)^{I/2L-1}}. \tag{39}$$

According to the initial conditions (29), $n_{\mathrm{B}}(t)$ can be obtained as

$$n_{\mathrm{B}}(t) = \frac{1}{B_0}(1+LB_0t) + \int_0^t \frac{(J/2)(I/2-L)A_0}{KA_0 + [(I/2-L)B_0 - KA_0](1+LB_0t)^{I/2L-1}}\,\mathrm{d}t. \tag{40}$$

Now we can discuss the kinetic evolution behaviors of the two species in the following several cases.

Firstly, in the case of $(I/2-L)B_0 - KA_0 > 0$, we derive the function $a(t)$ and $A(t)$ from the expressions (8) of the first two moments of A-species aggregates as follows:

$$a(t) = 1 - \frac{M_0^{\mathrm{A}}}{M_1^{\mathrm{A}}} = 1 - \frac{m_{\mathrm{A}}}{n_{\mathrm{A}}} = 1 - \frac{m_{\mathrm{A}}}{m_{\mathrm{A}} + Kt}, \tag{41}$$

and

$$A(t) = \frac{(M_0^{\mathrm{A}})^2}{M_1^{\mathrm{A}}} = \frac{m_{\mathrm{A}}}{n_{\mathrm{A}}^2}. \tag{42}$$

In the long-time limit ($t \gg 1$), $a(t)$ is approximated as

$$a(t) \simeq \frac{(I/2-L)KA_0B_0}{(I/2-L)B_0 - KA_0}(LB_0)^{-I/2L}t^{-(I/2L-1)}, \tag{43}$$

and $A(t)$ is approximated as

$$A(t) \simeq \frac{(I/2-L)A_0B_0}{(I/2-L)B_0 - KA_0}(LB_0)^{-I/2L}t^{-I/2L}. \tag{44}$$

In the end, the aggregate size distribution of A-species aggregates in the long-time limit is obtained using the ansatz $a_k(t) = A(t)[a(t)]^{k-1}$ as

$$a_k(t) \simeq \frac{(I/2-L)A_0B_0}{(I/2-L)B_0 - KA_0}(LB_0)^{-I/2L} \times \left[\frac{(I/2-L)KA_0B_0}{(I/2-L)B_0 - KA_0}(LB_0)^{-I/2L}\right]^{k-1} \times t^{-I/2L}t^{-(I/2L-1)(k-1)}. \tag{45}$$

It does not exhibit any scaling form.

In the long-time limit ($t \gg 1$), equation (40) can be integrated

$$n_{\rm B}(t) \simeq \begin{cases} Lt[1+C_1 t^{-(I/2L-1)}], & \left(0 < \dfrac{I}{2L} - 1 < 1\right), \\ Lt[1+C_2 \ln t], & \left(\dfrac{I}{2L} - 1 = 1\right), \\ Lt + (1/B_0 + C_3), & \left(\dfrac{I}{2L} - 1 > 1\right), \end{cases} \tag{46}$$

where

$$C_1 = \frac{(J/2)(I/2-L)A_0}{(2L-I/2)[(I/2-L)B_0 - KA_0](LB_0)^{I/2L-1}} > 0,$$
$$C_2 = \frac{(J/2)(I/2-L)A_0}{[(I/2-L)B_0 - KA_0]L^2 B_0} > 0,$$
$$C_3 = \frac{(J/2)(I/2-L)A_0}{(I/2L-2)[(I/2-L)B_0 - KA_0]LB_0} > 0.$$

In the same way, we derive the expressions of $b(t)$ and $B(t)$ in the long-time limit from Eq. (9) as

$$b(t) = 1 - \frac{m_{\rm B}}{n_{\rm B}} \simeq \begin{cases} C_1 t^{-(I/2L-1)}, & \left(0 < \dfrac{I}{2L} - 1 < 1\right), \\ C_2 \dfrac{\ln t}{t}, & \left(\dfrac{I}{2L} - 1 = 1\right), \\ \dfrac{C_3}{L} t^{-1}, & \left(\dfrac{I}{2L} - 1 > 1\right), \end{cases} \tag{47}$$

$$B(t) = \frac{m_{\rm B}}{n_{\rm B}^2} \simeq (Lt)^{-1}. \tag{48}$$

In the end, the aggregate size distribution of noncooperative agent aggregates of species B in the long-time limit is obtained using the ansatz $b_k(t) = B(t)[b(t)]^{k-1}$ as

$$b_k(t) \simeq \begin{cases} (Lt)^{-1} C_1^{k-1} t^{-(I/2L-1)(k-1)}, & \left(0 < \dfrac{I}{2L} - 1 < 1\right), \\ (Lt)^{-1} C_2^{k-1} \left(\dfrac{\ln t}{t}\right)^{k-1}, & \left(\dfrac{I}{2L} - 1 = 1\right), \\ (Lt)^{-1} \left(\dfrac{C_3}{L}\right)^{k-1} t^{-(k-1)}, & \left(\dfrac{I}{2L} - 1 > 1\right). \end{cases} \tag{49}$$

It also does not exhibit any scaling form.

Secondly, in the case of $(I/2-L)B_0 - KA_0 = 0$, $m_{\rm A}(t)$ in Eq. (37) can be simplified into

$$m_{\rm A}(t) = \frac{1}{A_0}(1 + LB_0 t), \tag{50}$$

and $n_{\rm A}(t)$ is obtained as

$$n_{\rm A}(t) = m_{\rm A}(t) + Kt = \frac{1}{A_0}[1 + (KA_0 + LB_0)t]. \tag{51}$$

We then derive the functions $a(t)$ and $A(t)$ in the same way as that in the above case as follows:

$$a(t) = \frac{KA_0 t}{1 + (KA_0 + LB_0)t}, \tag{52}$$

$$A(t) = \frac{m_{\rm A}}{n_{\rm A}^2} = \frac{A_0(1 + LB_0 t)}{[1 + (KA_0 + LB_0)t]^2}. \tag{53}$$

In the long-time limit, $a(t)$ becomes

$$a(t) \simeq \frac{KA_0}{KA_0 + LB_0}\left[1 - \frac{1}{(KA_0 + LB_0)t}\right] \simeq \frac{KA_0}{KA_0 + LB_0} \exp\left[-\frac{1}{(KA_0 + LB_0)t}\right]. \tag{54}$$

In the end, the aggregate size distribution of A-species aggregates in the long-time limit and for larger aggregates is obtained using ansatz (7)

$$a_k(t) \simeq \frac{LA_0 B_0}{(KA_0 + LB_0)^2} \left(\frac{KA_0}{KA_0 + LB_0}\right)^k t^{-1} \times \exp\left[-\frac{k}{(KA_0 + LB_0)t}\right]. \tag{55}$$

It approaches the modified scaling form in the scaling region of $k \gg 1$ and $t \gg 1$[15]

$$a_k(t) \simeq \lambda^k t^{-w} \Phi[k/S(t)], \quad S(t) \propto t^z, \tag{56}$$

with the scaling function $\Phi(x) = \exp(-x)$. The characteristic size of the cooperative agent aggregates is

$$S_{\rm A}(t) = (KA_0 + LB_0)t. \tag{57}$$

It is well known that usual scaling forms, such as the conventional scaling form

$$a_k(t) \simeq t^{-w} \Phi[k/S(t)], \quad S(t) \propto t^z, \tag{58}$$

have only one scale $S(t)$. Meanwhile, the modified scaling form comprises two scales, the growing scale $S(t)$ and the time-independent scale

$$S = \lim_{t\to\infty} \sum_k k^2 a_k(t) \Big/ \sum_k k a_k(t) = 1/(1-\lambda).$$

For the noncooperative agent aggregates of species B, equation (40) can be integrated into

$$n_{\rm B}(t) = \frac{1}{B_0}\left[1 + \left(\frac{J}{2}A_0 + LB_0\right)t\right]. \tag{59}$$

In the same way, we derive the expressions of $b(t)$ and $B(t)$ from Eq. (9) as

$$b(t) = 1 - \frac{m_{\rm B}}{n_{\rm B}} = 1 - \frac{1 + LB_0 t}{1 + (JA_0/2 + LB_0)t}, \tag{60}$$

$$B(t) = \frac{m_{\rm B}}{n_{\rm B}^2} = \frac{B_0(1 + LB_0 t)}{[1 + (JA_0/2 + LB_0)t]^2}. \tag{61}$$

In the long-time limit, $b(t)$ is approximated as

$$b(t) \simeq \frac{(J/2)A_0}{JA_0/2 + LB_0}[1 - (JA_0/2 + LB_0)^{-1} t^{-1}]. \tag{62}$$

Therefore, in the kinetic evolution, the noncooperative agent aggregates approach the same modified scaling form as species A in the scaling region of $k \gg 1$ and $t \gg 1$[15]

$$b_k(t) \simeq \frac{LB_0^2}{(JA_0/2+LB_0)^2}\left(\frac{(J/2)A_0}{JA_0/2+LB_0}\right)^k t^{-1} \times \exp\left[-\frac{k}{(JA_0/2+LB_0)t}\right], \quad (63)$$

with the scaling function $\Phi(x) = \exp(-x)$. The characteristic size of the noncooperative agent aggregates is

$$S_B(t) = (JA_0/2+LB_0)t. \quad (64)$$

Thirdly, in the case of $(I/2-L)B_0 - KA_0 < 0$, $m_A(t)$ of Eq. (37) can be rewritten as

$$m_A(t) = \frac{1}{(I/2-L)A_0B_0}(1+LB_0t) \times \left\{KA_0 - \left[KA_0 - \left(\frac{I}{2}-L\right)B_0\right] \times (1+LB_0t)^{I/2L-1}\right\}. \quad (65)$$

It reveals that m_A decreases with time, and then the total mass $M_1^A = m_A^{-1}$ of aggregates of species A increases with time. Here, it must be noticed that m_A will decrease monotonously to zero at a finite time t_c

$$t_c = \frac{1}{LB_0}\left\{\left[\frac{KA_0}{KA_0-(I/2-L)B_0}\right]^{L/(I/2-L)} - 1\right\}. \quad (66)$$

This shows that equation (65) is valid only in the time region $t < t_c$. When t increases to t_c, M_1^A increases to an infinite value, and a gelation transition of cooperative agent aggregates of species A takes place.

(ii) Case of $a = I/2 - L = 0$. In this case, equation (36) yields

$$m_A(t) = \frac{KA_0+LB_0}{LA_0B_0}(1+LB_0t) \times \left[1 - \frac{KA_0}{KA_0+LB_0}\ln(1+LB_0t)\right]. \quad (67)$$

Here, the total mass $M_1^A = m_A^{-1}$ of aggregates of species A also increases with time and reaches an infinite value at time

$$t_c = \frac{1}{LB_0}\left[\exp\left(\frac{KA_0+LB_0}{KA_0}\right) - 1\right]. \quad (68)$$

Thus, equation (67) is valid only in the time region $t < t_c$. At $t = t_c - \Delta t$ ($\Delta t \to 0^+$), M_1^A can be written as

$$M_1^A(t_c-\Delta t) = m_A^{-1}(t_c-\Delta t) \simeq \left(1+\frac{LB_0}{KA_0}\right)\frac{1}{\Delta t}. \quad (69)$$

Thus, at a finite time $t = t_c$, a gelation transition of cooperative agent aggregates of species A takes place.

(iii) Case of $a = I/2 - L < 0$. In this case, equation (36) yields

$$m_A(t) = \frac{1}{(L-I/2)A_0B_0}(1+LB_0t)^{I/2L} \times \left\{\left[\left(L-\frac{I}{2}\right)B_0 + KA_0\right] - KA_0(1+LB_0t)^{1-I/2L}\right\}. \quad (70)$$

Here, the total mass $M_1^A = m_A^{-1}$ of aggregates of species A also increases with time and reaches an infinite value at time

$$t_c = \frac{1}{LB_0}\left\{\left[\frac{(L-I/2)B_0+KA_0}{KA_0}\right]^{L/(L-I/2)} - 1\right\}. \quad (71)$$

A gelation transition of cooperative agent aggregates of species A also takes place at this finite time t_c.

It further reveals that in this case the win–win processes of cooperative agent aggregate of species A dominate the kinetic evolution of the system.

3.2. Case of $a = 0$ and $b \neq 0$

In this case, $a = I/2 - L = 0$, and then we have $I = 2L$ and $b = K - (2L-J)/2$. Because the case of $c = (I-J)/2L = 0$ has been discussed in the above subsection, we only consider $c = (2L-J)/2L \neq 0$ case. In this case, equation (33) becomes

$$m_B\frac{du}{dm_B} = -\frac{b}{L}\frac{u}{u-c}. \quad (72)$$

It is apparent that the constant solution $u = u(0)$ is not the solution of this equation. Thus, equation (72) can be transformed into

$$\frac{dm_B}{m_B} = -\frac{L}{b}\left(1-\frac{c}{u}\right)du. \quad (73)$$

Under the initial conditions (29), this equation gives the solution

$$m_B(t) = \frac{C_4}{B_0}u^{Lc/b}e^{-Lu/b}, \quad (74)$$

with constant

$$C_4 = [u(0)]^{-Lc/b}e^{Lu(0)/b} = (B_0/A_0)^{-Lc/b}e^{LB_0/bA_0} > 0.$$

Substituting this solution into Eq. (28), we derive the equation for $u(t)$ as follows:

$$\frac{du}{dt} = -\frac{B_0b}{C_4}u^{-Lc/b}e^{Lu/b}. \quad (75)$$

It reveals that the growth or decrease of $u(t)$ is determined by $b < 0$ or $b > 0$. Then we discuss the evolution behaviors in subcases of $b > 0$ and $b < 0$, respectively.

(i) Case of $b = K - (I-J)/2 = K - Lc > 0$. In this case, equation (75) determines that $du/dt < 0$, so $u(t)$ decreases monotonously with time and we have $u \ll 1$ in the long-time limit. Equation (75) is transformed into

$$u^{Lc/b}e^{-Lu/b}du = -\frac{B_0b}{C_4}dt. \quad (76)$$

Since $Lc/b+1=K/(K-Lc)>0$, this equation can be integrated in the case of $u\ll 1$

$$\theta_1(u)-\theta_1[u(0)]=-\frac{KB_0}{C_4}t, \tag{77}$$

where

$$\theta_1(u)=u^{K/b}\,\mathrm{e}^{-Lu/b}\left[1+\frac{L}{K+b}u+\phi_1(u)\right], \tag{78}$$

and $\phi_1(u)$ denotes the integration

$$\begin{aligned}\phi_1(u) &= \left(\frac{L}{b}\right)^2\frac{b}{K+b}u^{-K/b}\,\mathrm{e}^{Lu/b}\int u^{K/b+1}\,\mathrm{e}^{-Lu/b}\,\mathrm{d}u\\ &= O(u^2),\end{aligned} \tag{79}$$

$$\begin{aligned}\theta_1[u(0)] &= [u(0)]^{K/b}\,\mathrm{e}^{-Lu(0)/b}\left\{1+\frac{L}{K+b}u(0)+\phi_1[u(0)]\right\}\\ &= \left(\frac{B_0}{A_0}\right)^{K/b}\mathrm{e}^{-LB_0/bA_0}\\ &\quad\times\left\{1+\frac{LB_0}{(K+b)A_0}+\phi_1[u(0)]\right\}.\end{aligned} \tag{80}$$

Apparently, we have $\theta_1[u(0)]>0$.

In the long-time limit, considering $u\ll 1$, we can obtain a zeroth-order approximate solution for $u(t)$ from Eq. (77) as follows:

$$u\simeq\left\{\theta_1[u(0)]-\frac{KB_0}{C_4}t\right\}^{b/K}. \tag{81}$$

Then, we further obtain $m_{\mathrm{B}}(t)$ from Eq. (74) as

$$\begin{aligned}m_{\mathrm{B}}(t) &= \frac{C_4}{B_0}u^{Lc/b}\,\mathrm{e}^{-Lu/b}\simeq\frac{C_4}{B_0}\frac{\theta_1(u)}{u}\\ &= \frac{C_4}{B_0}\left\{\theta_1[u(0)]-\frac{KB_0}{C_4}t\right\}^{Lc/K},\end{aligned} \tag{82}$$

$$\begin{aligned}n_{\mathrm{B}}(t) &= \frac{1}{I-J}\left[\frac{I}{B_0}(1+LB_0t)-Jm_{\mathrm{B}}(t)\right]\\ &= \frac{1}{2L-J}\left[\frac{2L}{B_0}(1+LB_0t)-Jm_{\mathrm{B}}(t)\right].\end{aligned} \tag{83}$$

For the cooperative agent species, we obtain

$$\begin{aligned}m_{\mathrm{A}}(t) &= u(t)m_{\mathrm{B}}(t)\simeq\frac{C_4}{B_0}\theta_1(u)\\ &= \frac{C_4}{B_0}\left\{\theta_1[u(0)]-\frac{KB_0}{C_4}t\right\},\end{aligned} \tag{84}$$

$$n_{\mathrm{A}}(t) = m_{\mathrm{A}}(t)+Kt\simeq\frac{C_4}{B_0}\theta_1[u(0)], \tag{85}$$

which reveals that in the zeroth-order approximation n_{A} approaches a constant.

If we take second-order approximation, we have

$$\begin{aligned}m_{\mathrm{B}}(t) &= \frac{C_4}{B_0}u^{Lc/b}\,\mathrm{e}^{-Lu/b}\\ &\simeq\frac{C_4}{B_0}\frac{\theta_1(u)}{u}\left(1+\frac{L}{K+b}u\right)^{-1},\end{aligned} \tag{86}$$

$$\begin{aligned}m_{\mathrm{A}}(t) &\simeq \frac{C_4}{B_0}\left\{\theta_1[u(0)]-\frac{KB_0}{C_4}\right\}\left(1+\frac{L}{K+b}u\right)^{-1}\\ &\simeq \frac{C_4}{B_0}\left\{\theta_1[u(0)]-\frac{KB_0}{C_4}t\right\}\left(1-\frac{L}{K+b}u\right)\\ &\simeq \frac{C_4}{B_0}\left\{\theta_1[u(0)]-\frac{KB_0}{C_4}t\right\}\\ &\quad-\frac{C_4L}{(K+b)B_0}\left\{\theta_1[u(0)]-\frac{KB_0}{C_4}t\right\}^{1+b/K}.\end{aligned} \tag{87}$$

We finally obtain the second-order approximate solution for n_{A} as follows:

$$\begin{aligned}n_{\mathrm{A}}(t) &= m_{\mathrm{A}}(t)+Kt\simeq\frac{C_4}{B_0}\theta_1[u(0)]\\ &\quad-\frac{C_4L}{(K+b)B_0}\left\{\theta_1[u(0)]-\frac{KB_0}{C_4}\right\}^{1+b/K},\end{aligned} \tag{88}$$

which is no longer a constant.

From Eq. (84), we find that the total mass of cooperative agent aggregates $M_1^{\mathrm{A}}=m_{\mathrm{A}}^{-1}$ increases with time and reaches an infinite value at a finite time t_{c}

$$t_{\mathrm{c}}=\frac{C_4}{KB_0}\theta_1[u(0)]. \tag{89}$$

Thus, equation (84) is valid only in the time region $t<t_{\mathrm{c}}$, and at time $t=t_{\mathrm{c}}$, a gelation transition of cooperative agent aggregates of species A takes place.

For the noncooperative agent aggregates of species B, the kinetic evolution behavior of the total mass of aggregates $M_1^{\mathrm{B}}=m_{\mathrm{B}}^{-1}$ is dominated by the value of $c=(I-J)/2L$, which is consistent with Eq. (28). This reflects the competition between the two different reaction schemes between a cooperative agent aggregate and a noncooperative agent aggregate. One is the one unit of spirit migrating from a cooperative agent aggregate to a noncooperative agent aggregate $A_k+B_l\xrightarrow{I(k,l)}A_{k-1}+B_{l+1}$, which causes the increase in the total mass M_1^{B} of B-species aggregates, and the other is the cooperative agent catalyzed decrease in the spirit of a noncooperative agent $B_k+A_l\xrightarrow{J(k,l)}B_{k-1}+A_l$, which causes the decrease in the total mass M_1^{B} of B-species aggregates. When $c=(I-J)/2L>0$, the effect of a spirit migrating from a cooperative agent aggregate to a noncooperative agent aggregate $A_k+B_l\xrightarrow{I(k,l)}A_{k-1}+B_{l+1}$ is stronger than that of the cooperative agent catalyzed-decrease of the spirit of a noncooperative agent $B_k+A_l\xrightarrow{J(k,l)}B_{k-1}+A_l$, and the net effect is the increase in the total mass M_1^{B} of B-species aggregates. In this situation, as for the species A, a gelation transition of B-species aggregates will occur at the same time $t_{\mathrm{c}}=(C_4/KB_0)\theta_1[u(0)]$. On the contrary, if $c=(I-J)/2L<0$, the effect of the cooperative agent catalyzed-decrease of the spirit of a noncooperative agent $B_k+A_l\xrightarrow{J(k,l)}B_{k-1}+A_l$ is stronger, and the net effect is the decrease in the total mass of B-species aggregates.

(ii) Case of $b=K-(I-J)/2=K-Lc<0$. In this case, it is apparent that we have $c=(I-J)/2L>0$. Here,

equation (75) determines that $\mathrm{d}u/\mathrm{d}t > 0$, so $u(t)$ increases monotonously with time and we have $u \gg 1$ in the long-time limit $t \gg 1$. Equation (76) can be integrated in case of

$$\theta_2(u) - \theta_2[u(0)] = \frac{LB_0}{C_4}t, \tag{90}$$

where

$$\theta_2(u) = u^{Lc/b}\,\mathrm{e}^{-Lu/b}[1 + cu^{-1} + \phi_2(u)], \tag{91}$$

and $\phi_2(u)$ denotes the integration

$$\begin{aligned}\phi_2(u) &= c\left(1 - \frac{Lc}{b}\right)u^{-Lc/b}\,\mathrm{e}^{Lu/b}\int u^{\frac{Lc}{b}-2}\,\mathrm{e}^{-Lu/b}\,\mathrm{d}u \\ &= O\left[\left(\frac{1}{u}\right)^2\right], \qquad (92)\\ \theta_2[u(0)] &= [u(0)]^{Lc/b}\,\mathrm{e}^{-Lu(0)/b}\{1 + c[u(0)]^{-1} + \phi_2[u(0)]\} \\ &= \left(\frac{B_0}{A_0}\right)^{Lc/b} \\ &\quad \times \mathrm{e}^{-LB_0/bA_0}\left\{1 + \frac{cA_0}{B_0} + \phi_2[u(0)]\right\}. \qquad (93)\end{aligned}$$

In the long-time limit, considering $u \gg 1$, we can simplify Eq. (90) into

$$u \simeq -\frac{b}{L}\ln t + c\ln u. \tag{94}$$

Considering $u \gg \ln u$ in the situation of $u \gg 1$, we obtain a zeroth-order approximate solution for $u(t)$ from Eq. (94) as follows:

$$u \simeq -\frac{b}{L}\ln t. \tag{95}$$

Substituting this zeroth-order approximate solution of $u(t)$ into the right hand of Eq. (94), we then further obtain the first-order approximate solution for $u(t)$ as follows:

$$u \simeq -\frac{b}{L}\ln t + c\ln\left(-\frac{b}{L}\ln t\right). \tag{96}$$

From Eq. (74), we further obtain $m_{\mathrm{B}}(t)$ as

$$\begin{aligned}m_{\mathrm{B}}(t) &= \frac{C_4}{B_0}u^{Lc/b}\,\mathrm{e}^{-Lu/b} = \frac{C_4}{B_0}\frac{\theta_2(u)}{1 + cu^{-1} + \phi_2(u)} \\ &= \frac{C_4}{B_0}\frac{\theta_2[u(0)] + LB_0t/C_4}{1 + cu^{-1} + \phi_2(u)}. \qquad (97)\end{aligned}$$

In the long-time limit, considering $u \gg 1$, we obtain a zeroth-order approximate solution for $m_{\mathrm{B}}(t)$,

$$m_{\mathrm{B}}(t) \simeq Lt. \tag{98}$$

From Eq. (31), we can further obtain $n_{\mathrm{B}}(t)$ as

$$n_{\mathrm{B}}(t) = \frac{1}{I-J}\left[\frac{I}{B_0}(1 + LB_0t) - Jm_{\mathrm{B}}(t)\right] \simeq Lt, \tag{99}$$

which is approximately equal to $m_{\mathrm{B}}(t)$.

For the cooperative agent species A, we obtain

$$m_{\mathrm{A}}(t) = u(t)m_{\mathrm{B}}(t) \simeq -bt\ln t, \tag{100}$$

$$n_{\mathrm{A}}(t) = m_{\mathrm{A}}(t) + Kt \simeq -bt\ln t + Kt. \tag{101}$$

We can then derive the functions $a(t)$ and $A(t)$ to investigate the kinetic evolution behavior of the aggregate size distribution $a_k(t)$ in the long-time limit

$$\begin{aligned}a(t) &= 1 - \frac{M_0^{\mathrm{A}}}{M_1^{\mathrm{A}}} = 1 - \frac{m_{\mathrm{A}}}{n_{\mathrm{A}}} \\ &\simeq 1 - \frac{-bt\ln t}{-bt\ln t + Kt} \simeq -\frac{K}{b}(\ln t)^{-1}, \qquad (102)\\ A(t) &= \frac{(M_0^{\mathrm{A}})^2}{M_1^{\mathrm{A}}} = \frac{m_{\mathrm{A}}}{n_{\mathrm{A}}^2} \\ &\simeq \frac{-bt\ln t}{(-bt\ln t + Kt)^2} \simeq -\frac{1}{b}t^{-1}(\ln t)^{-1}. \qquad (103)\end{aligned}$$

In the long-time limit, $a_k(t)$ is approximated to be

$$a_k(t) \simeq -\frac{1}{b}\left(-\frac{K}{b}\right)^{k-1}t^{-1}(\ln t)^{-k}, \tag{104}$$

which does not exhibit any scaling form.

To investigate the kinetic evolution behavior of the aggregate size distribution $b_k(t)$ in the long-time limit, we can further take the first-order approximation for $m_{\mathrm{B}}(t)$,

$$\begin{aligned}m_{\mathrm{B}}(t) &= \frac{C_4}{B_0}\frac{\theta_2[u(0)] + LB_0t/C_4}{1 + cu^{-1} + \phi_2(u)} \\ &\simeq Lt\frac{1 + C_4\theta_2[u(0)]t^{-1}/LB_0}{1 - Lc/b(\ln t)^{-1}} \\ &\simeq Lt\left[1 + \frac{Lc}{b}(\ln t)^{-1}\right], \qquad (105)\end{aligned}$$

and the first-order approximation for $m_{\mathrm{B}}(t)$ can then be obtained as

$$\begin{aligned}n_{\mathrm{B}}(t) &= \frac{1}{I-J}\left[\frac{I}{B_0}(1 + LB_0t) - Jm_{\mathrm{B}}(t)\right] \\ &\simeq Lt\left[1 - \frac{J}{2b}(\ln t)^{-1}\right]. \qquad (106)\end{aligned}$$

Now we can derive the functions $b(t)$ and $B(t)$ to investigate the aggregate size distribution $b_k(t)$ in the long-time limit

$$\begin{aligned}b(t) &= 1 - \frac{M_0^{\mathrm{B}}}{M_1^{\mathrm{B}}} = 1 - \frac{m_{\mathrm{B}}}{n_{\mathrm{B}}} \\ &\simeq 1 - \frac{Lt[1 + (Lc/b)(\ln t)^{-1}]}{Lt[1 - (J/2b)(\ln t)^{-1}]} \\ &\simeq -\frac{Lc + J/2}{b}(\ln t)^{-1}, \qquad (107)\\ B(t) &= \frac{(M_0^{\mathrm{B}})^2}{M_1^{\mathrm{B}}} = \frac{m_{\mathrm{B}}}{n_{\mathrm{B}}^2} \simeq \frac{Lt}{(Lt)^2} \simeq L^{-1}t^{-1}. \qquad (108)\end{aligned}$$

In the long-time limit ($t \gg 1$), the aggregate size distribution $b_k(t)$ is approximated to be

$$b_k(t) \simeq L^{-1}\left(-\frac{Lc + J/2}{b}\right)^{k-1}t^{-1}(\ln t)^{-(k-1)}, \tag{109}$$

which does not exhibit any scaling form.

3.3. Case of $a \neq 0$ and $b = 0$

In this case, $b = K - (I-J)/2 = 0$, and then we have $I - J = 2K$ and $c = (I-J)/2L = K/L > 0$. Equation (33) becomes

$$m_{\rm B}\frac{{\rm d}u}{{\rm d}m_{\rm B}} = \frac{1}{L}\frac{au^2}{u-c}. \tag{110}$$

It can be directly solved to obtain $m_{\rm B}(u)$ under the initial conditions (29)

$$m_{\rm B}(t) = \frac{C_5}{B_0}u^{L/a}\,{\rm e}^{K/au}, \tag{111}$$

with constant

$$C_5 = [u(0)]^{-L/a}\,{\rm e}^{-K/au(0)} = \left(\frac{A_0}{B_0}\right)^{L/a}{\rm e}^{-KA_0/aB_0} > 0.$$

Substituting this solution into Eq. (28), we derive the equation of $u(t)$ as follows:

$$\begin{aligned}\frac{{\rm d}m_{\rm B}}{{\rm d}t} &= \frac{LC_5}{aB_0}u^{L/a-1}\,{\rm e}^{K/au}\left(1-\frac{K}{L}\frac{1}{u}\right)\frac{{\rm d}u}{{\rm d}t}\\ &= L\left(1-\frac{K}{L}\frac{1}{u}\right).\end{aligned} \tag{112}$$

It indicates that there is a constant solution for $u = u(0)$ under the condition $u = u(0) = B_0/A_0 = K/L$. Otherwise, equation (112) can be reduced to

$$\frac{{\rm d}u}{{\rm d}t} = \frac{B_0 a}{C_5}u^{-(L/a-1)}\,{\rm e}^{-K/au}, \tag{113}$$

which can be solved in cases of $a > 0$ and $a < 0$.

(i) Constant solution $u = u(0)$ under a special condition $KA_0 = LB_0$. Under the condition $KA_0 = LB_0$, equation (112) has a constant solution $u = u(0) = B_0/A_0$. We then obtain $m_{\rm B}$ and $m_{\rm A}$ as

$$m_{\rm B} = m_{\rm B}(0) = \frac{1}{B_0}, \tag{114}$$

$$m_{\rm A} = m_{\rm A}(0) = \frac{1}{A_0}, \tag{115}$$

which reveal that both the total masses $M_1^{\rm A} = m_{\rm A}^{-1}$ and $M_1^{\rm B} = m_{\rm B}^{-1}$ are conserved.

From $m_{\rm B}$ and $m_{\rm A}$, we can further obtain $n_{\rm A}$ and $n_{\rm B}$ as

$$n_{\rm A} = m_{\rm A} + Kt = \frac{1}{A_0}(1+KA_0t), \tag{116}$$

$$\begin{aligned}n_{\rm B} &= \frac{1}{I-J}\left[\frac{I}{B_0}(1+LB_0t) - Jm_{\rm B}\right]\\ &= \frac{1}{B_0} + \frac{IL}{2K}t = \frac{1}{B_0}\left(1+\frac{I}{2}A_0t\right).\end{aligned} \tag{117}$$

It can be found that in this case the total numbers $M_0^{\rm A} = n_{\rm A}^{-1}$ and $M_0^{\rm B} = n_{\rm B}^{-1}$ of the aggregates of both species decrease with time, which signifies that the aggregates of both species grow with time.

Using those solutions, we can obtain the functions $a(t)$, $A(t)$, $b(t)$, and $B(t)$ to further investigate the kinetic evolution behaviors of the aggregate size distributions $a_k(t)$ and $b_k(t)$.

$$\begin{aligned}a(t) &= 1 - \frac{m_{\rm A}}{n_{\rm A}} = \frac{KA_0t}{1+KA_0t},\\ A(t) &= \frac{m_{\rm A}}{n_{\rm A}^2} = \frac{A_0}{(1+KA_0t)^2},\end{aligned} \tag{118}$$

$$\begin{aligned}b(t) &= 1 - \frac{m_{\rm B}}{n_{\rm B}} = \frac{IA_0t/2}{1+IA_0t/2},\\ B(t) &= \frac{m_{\rm B}}{n_{\rm B}^2} = \frac{B_0}{(1+IA_0t/2)^2}.\end{aligned} \tag{119}$$

In the long-time limit, they become

$$a(t) \simeq 1 - (KA_0t)^{-1}, \quad A(t) \simeq K^{-2}A_0^{-1}t^{-2}, \tag{120}$$

$$b(t) \simeq 1 - \left(\frac{I}{2}A_0t\right)^{-1}, \quad B(t) \simeq \left(\frac{I}{2}\right)^{-2}A_0^{-2}B_0t^{-2}. \tag{121}$$

In the scaling region of $k \gg 1$ and $t \gg 1$, the aggregate size distributions $a_k(t)$ and $b_k(t)$ become

$$\begin{aligned}a_k(t) &\simeq K^{-2}A_0^{-1}t^{-2}\exp[-k/S_{\rm A}(t)],\\ S_{\rm A}(t) &= KA_0t,\end{aligned} \tag{122}$$

$$\begin{aligned}b_k(t) &\simeq \left(\frac{I}{2}\right)^{-2}A_0^{-2}B_0t^{-2}\exp[-k/S_{\rm B}(t)],\\ S_{\rm B}(t) &= \frac{I}{2}A_0t.\end{aligned} \tag{123}$$

They both satisfy the following conventional scaling form:

$$a_k(t) \simeq t^{-w}\Phi[k/S(t)], \quad S(t) \propto t^z, \tag{124}$$

with the same scaling function $\Phi(x) = \exp(-x)$, and the same scaling exponents $w = 2$ and $z = 1$. Here, $S_{\rm A}(t)$ and $S_{\rm B}(t)$ are the characteristic aggregate sizes of the aggregates of species A and B, respectively. In this cooperative agent and noncooperative agent system, they are the characteristic numbers of the cooperative spirit units of the cooperative agents and the noncooperative spirit units of the noncooperative agents, respectively. From the scaling forms of $a_k(t)$ and $b_k(t)$, we find that the cooperative agent aggregates of species A play a more important role in the evolution of the whole system. The evolution behavior of species A is dominated by the win–win interaction scheme $A_k + A_l \xrightarrow{K(k,l)} A_{k+1} + A_{l+1}$. The evolution behavior of species B is dominated by the scheme of a spirit migrating from a cooperative agent aggregate to a noncooperative agent aggregate $A_k + B_l \xrightarrow{I(k,l)} A_{k-1} + B_{l+1}$, whose effect not only depends on the rate kernel $I(k,l)$, but also relies on the initial number of the cooperative agent monomers A_0.

(ii) Case of $a = I/2 - L > 0$. Substituting function $u(t)$ into a new function $v(t)$, we can transform Eq. (113) into a differential equation of $v(t)$

$$v^{-L/a-1}\,{\rm e}^{Kv/a}\,{\rm d}v = -\frac{B_0 a}{C_5}\,{\rm d}t. \tag{125}$$

From Eq. (113), we find that in the case of $a = I/2 - L > 0$, we have $\mathrm{d}u/\mathrm{d}t > 0$. Hence, in the whole evolution process, u increases monotonously with time t. In the long-time limit, we have $u \gg 1$, and thus $v \ll 1$.

When $a \neq L$, equation (125) can be integrated in the case of $v \ll 1$

$$\theta_3(v) - \theta_3[v(0)] = \frac{LB_0}{C_5}t, \tag{126}$$

where

$$\theta_3(v) = v^{-L/a}\,\mathrm{e}^{Kv/a}\left[1 - \frac{K}{a-L}v + \phi_3(v)\right], \tag{127}$$

and $\phi_3(v)$ denotes the integration

$$\phi_3(v) = \frac{K^2}{a(a-L)}v^{L/a}\,\mathrm{e}^{-Kv/a}\int v^{-L/a+1}\,\mathrm{e}^{Kv/a}\mathrm{d}v. \tag{128}$$

In the long-time limit, we have $\phi_3(v) = O(v^2)$.

Transforming the function $v(t)$ back to the function $u(t)$, the solution of $u(t)$ in this case can be obtained and satisfy

$$\theta_3'(u) - \theta_3'[u(0)] = \frac{LB_0}{C_5}t, \tag{129}$$

where

$$\theta_3'(u) = u^{L/a}\,\mathrm{e}^{K/au}\left[1 - \frac{K}{a-L}u^{-1} + \phi_3'(u)\right], \tag{130}$$

and $\phi_3'(u)$ denotes the integration

$$\phi_3'(u) = -\frac{K^2}{a(a-L)}u^{-L/a}\,\mathrm{e}^{-K/au}\int u^{L/a-3}\,\mathrm{e}^{K/au}\mathrm{d}u. \tag{131}$$

In the long-time limit, we have $\phi_3'(u) = O(1/u^2)$.

When $a = L$, equation (125) can also be integrated in the case of $v \ll 1$

$$\theta_4(v) - \theta_4[v(0)] = \frac{B_0 a}{C_5}t = \frac{LB_0}{C_5}t, \tag{132}$$

with

$$\theta_4(v) = v^{-1}\,\mathrm{e}^{Kv/L}\left[1 - \frac{K}{L}v\ln v + \phi_4(v)\right], \tag{133}$$

and $\phi_4(v)$ denotes the integration

$$\phi_4(v) = \left(\frac{K}{L}\right)^2 v\,\mathrm{e}^{-Kv/L}\int \ln v\,\mathrm{e}^{Kv/L}\mathrm{d}v. \tag{134}$$

In the long-time limit, we have $\phi_4(v) \ll 1$.

According to the solutions (129) and (132), considering $u \gg 1$ or $v \ll 1$ in the long-time limit, we can obtain the solution for $u(t)$ in either $a \neq L$ or $a = L$ case as follows:

$$u \simeq t^{a/L}. \tag{135}$$

We can further obtain $m_{\mathrm{B}}(t)$ and $m_{\mathrm{A}}(t)$ in the long-time limit

$$m_{\mathrm{B}}(t) = \frac{C_5}{B_0}u^{L/a}\,\mathrm{e}^{K/au} \simeq Lt, \tag{136}$$

$$m_{\mathrm{A}} = u(t)m_{\mathrm{B}}(t) \simeq Lt^{1+a/L}. \tag{137}$$

We find that both the total masses $M_1^{\mathrm{A}} = m_{\mathrm{A}}^{-1}$ and $M_1^{\mathrm{B}} = m_{\mathrm{B}}^{-1}$ decrease with time.

From the solutions of $m_{\mathrm{A}}(t)$ and $m_{\mathrm{B}}(t)$, we can further obtain $n_{\mathrm{A}}(t)$ and $n_{\mathrm{B}}(t)$ as

$$n_{\mathrm{A}} = m_{\mathrm{A}} + Kt \simeq Lt^{1+a/L} + Kt, \tag{138}$$

$$n_{\mathrm{B}} = \frac{1}{I-J}\left[\frac{I}{B_0}(1+LB_0t) - Jm_{\mathrm{B}}\right] \simeq \frac{I}{2KB_0} + Lt. \tag{139}$$

It can be found that in this case the total numbers $M_0^{\mathrm{A}} = n_{\mathrm{A}}^{-1}$ and $M_0^{\mathrm{B}} = n_{\mathrm{B}}^{-1}$ of the aggregates of both species decrease with time.

Now we derive the functions $a(t)$ and $A(t)$ to investigate the kinetic evolution behavior of the aggregate size distribution $a_k(t)$ in the long-time limit

$$\begin{aligned} a(t) &= 1 - \frac{m_{\mathrm{A}}}{n_{\mathrm{A}}} \simeq 1 - \frac{Lt^{1+a/L}}{Lt^{1+a/L} + Kt} \\ &\simeq \frac{K}{L}t^{-a/L}\left(1 - \frac{K}{L}t^{-a/L}\right), \end{aligned} \tag{140}$$

$$A(t) = \frac{m_{\mathrm{A}}}{n_{\mathrm{A}}^2} \simeq \frac{Lt^{1+a/L}}{(Lt^{1+a/L} + Kt)^2} \simeq L^{-1}t^{-1-a/L}. \tag{141}$$

In the long-time limit ($t \gg 1$), large aggregates $a_k(t)$ is approximated to be

$$a_k(t) \simeq K^{-1}\left(\frac{K}{L}\right)^k t^{-1-ak/L}\exp\left[-k/\left(\frac{L}{K}t^{a/L}\right)\right], \tag{142}$$

which does not exhibit any scaling form.

In the same way, we investigate the kinetic evolution behavior of the aggregate size distribution $b_k(t)$ in the long-time limit

$$\begin{aligned} b(t) &= 1 - \frac{m_{\mathrm{B}}}{n_{\mathrm{B}}} \simeq 1 - \frac{Lt}{I/(2KB_0) + Lt} \\ &\simeq \frac{I}{2KLB_0t}\left(1 - \frac{I}{2KLB_0t}\right), \end{aligned} \tag{143}$$

$$B(t) = \frac{m_{\mathrm{B}}}{n_{\mathrm{B}}^2} \simeq \frac{Lt}{(I/(2KB_0) + Lt)^2} \simeq L^{-1}t^{-1}. \tag{144}$$

In the long-time limit, large aggregates $b_k(t)$ is approximated to be

$$b_k(t) \simeq \frac{2K}{I}B_0\left(\frac{I}{2KLB_0}\right)^k t^{-k}\exp\left(-k/\frac{2KL}{I}B_0t\right), \tag{145}$$

which also does not exhibit any scaling form.

(iii) Case of $a = I/2 - L < 0$. From Eq. (113), we find $\mathrm{d}u/\mathrm{d}t < 0$ in this case. Thus, in the whole evolution process, u decreases monotonously with time t. In the long-time limit, we have $u \ll 1$, and thus $v \gg 1$.

Considering $v \gg 1$ in the long-time limit, equation (125) can be integrated to be

$$\theta_5(v) - \theta_5[v(0)] = -\frac{KB_0}{C_5}t, \tag{146}$$

where

$$\theta_5(v) = v^{-L/a-1}\,\mathrm{e}^{Kv/a}\left[1+\frac{L+a}{K}v^{-1}+\phi_5(v)\right], \quad (147)$$

and $\phi_5(v)$ denotes the integration

$$\phi_5(v) = \frac{L+a}{K}\left(\frac{L}{a}+2\right)v^{L/a+1}\,\mathrm{e}^{-Kv/a}\int v^{-L/a-3}\,\mathrm{e}^{Kv/a}\,\mathrm{d}v. \quad (148)$$

In the long-time limit, we have $\phi_5(v) = O(v^{-2})$.

Considering $v \gg 1$ and a large time t, we have $\theta_5(v) > 0$. It is apparent that $\theta_5[v(0)] > 0$. For large v, we have $v \gg \ln v$, and thus we obtain the zero-order approximate solution for v at large time from Eq. (146) as

$$v \simeq \frac{a}{K}\ln(t_\mathrm{c}-t), \quad (149)$$

with finite $t_\mathrm{c} = (C_5/KB_0)\theta_5[v(0)]$. The asymptotic solution for u can then be obtained

$$u = \frac{1}{v} \simeq \frac{K}{a}\frac{1}{\ln(t_\mathrm{c}-t)}. \quad (150)$$

It is apparent that the solutions (149) and (150) are valid only in the time region $t_\mathrm{c}-1 < t < t_\mathrm{c}$. When the evolution progresses and the time t increases to approach the finite time t_c, u decreases.

We can then obtain $m_\mathrm{B}(t)$ and $m_\mathrm{A}(t)$ in the time region $t_\mathrm{c}-1 < t < t_\mathrm{c}$ as

$$m_\mathrm{B}(t) = \frac{C_5}{B_0}u^{L/a}\,\mathrm{e}^{K/au} \simeq (I/2-L)(t_\mathrm{c}-t)\ln(t_\mathrm{c}-t), \quad (151)$$

$$m_\mathrm{A} = u(t)m_\mathrm{B}(t) \simeq K(t_\mathrm{c}-t). \quad (152)$$

We find that both the total masses of the cooperative agent aggregates $M_1^\mathrm{A} = m_\mathrm{A}^{-1}$ and the noncooperative agent aggregates $M_1^\mathrm{B} = m_\mathrm{B}^{-1}$ increase to infinite values when time approaches the finite time t_c, and gelation transitions of both species take place at time $t = t_\mathrm{c}$.

From the solutions of $m_\mathrm{A}(t)$ and $m_\mathrm{B}(t)$, we can further obtain $n_\mathrm{A}(t)$ and $n_\mathrm{B}(t)$ as

$$n_\mathrm{A} = m_\mathrm{A} + Kt \simeq Kt_\mathrm{c}, \quad (153)$$

$$n_\mathrm{B} = \frac{1}{I-J}\left[\frac{I}{B_0}(1+LB_0t)-Jm_\mathrm{B}\right] \simeq \frac{1}{2K}\left[ILt+J\left(L-\frac{I}{2}\right)(t_\mathrm{c}-t)\ln(t_\mathrm{c}-t)\right]. \quad (154)$$

3.4. General case of $a \neq 0$ and $b \neq 0$

We now consider the general case, $a = I/2-L \neq 0$, $b = K-(I-J)/2 \neq 0$, and $c = (I-J)/2L \neq 0$. Equation (33) becomes

$$m_\mathrm{B}\frac{\mathrm{d}u}{\mathrm{d}m_\mathrm{B}} = \frac{a}{L}\frac{u(u-b/a)}{u-c}. \quad (155)$$

(i) Constant solution $u = u(0) = b/a > 0$. It is apparent that there is a constant solution $u = b/a$ under a special condition. The constant value solution is

$$u = u(0) = B_0/A_0 = b/a = \frac{K-(I-J)/2}{I/2-L} > 0,$$

which is based on the special condition $[K-(I-J)/2]A_0 = (I/2-L)B_0$.

In this special case, the aggregates of species A and B have the same evolution kinetic behaviors as those in the case of $a \neq 0$ and $b = 0$ under the special condition $KA_0 = LB_0$.

(ii) Case of $b/a = \frac{K-(I-J)/2}{I/2-L} < 0$. In a general case, $u \neq b/a$, and equation (155) becomes

$$\frac{\mathrm{d}m_\mathrm{B}}{m_\mathrm{B}} = \frac{L}{a}\frac{u-c}{u(u-b/a)}\mathrm{d}u = \frac{L}{a}\left[\frac{ac}{b}\frac{1}{u}+\left(1-\frac{ac}{b}\right)\frac{1}{u-b/a}\right]\mathrm{d}u. \quad (156)$$

Integrating this equation, we obtain $m_\mathrm{B} = m_\mathrm{B}(u)$ as

$$m_\mathrm{B} = C_6u^{K/b-1}|u-b/a|^{(bL-aK)/ab+1}, \quad (157)$$

with constant

$$C_6 = B_0{}^{-1}[u(0)]^{-K/b+1}\left|u(0)-\frac{b}{a}\right|^{-(bL-aK)/ab-1} > 0.$$

To derive the solution $u(t)$ by substituting Eq. (157) into Eq. (28), we consider simpler cases with $b/a = \frac{K-(I-J)/2}{I/2-L} < 0$. In this case, we derive the equation of $u(t)$ by substituting Eq. (157) into Eq. (28) in the form of

$$u^{K/b-1}\left(u-\frac{b}{a}\right)^{(bL-aK)/ab}(1-cu^{-1})\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{a}{C_6}(1-cu^{-1}). \quad (158)$$

When there is a special relation $B_0/A_0 = (I-J)/2L$ among the initial monomer aggregate numbers A_0 and B_0 of species A and B, there is a constant solution $u = u(0) = B_0/A_0 = c = (I-J)/2L$ for Eq. (158). In this special case, the aggregates of species A and B also have the same evolution kinetic behaviors as those in the case of $a \neq 0$ and $b = 0$ under the special condition $KA_0 = LB_0$.

When this special relation $B_0/A_0 = (I-J)/2L$ is not satisfied, equation (158) can be reduced to

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{a}{C_6}u^{-K/b+1}\left(u-\frac{b}{a}\right)^{(aK-bL)/ab}. \quad (159)$$

It is apparent that the increase or decrease in $u(t)$ depends on whether $a > 0$ or $a < 0$. Equation (159) can be transformed into

$$u^{K/b-1}\left(u-\frac{b}{a}\right)^{-(aK-bL)/ab}\mathrm{d}u = \frac{a}{C_6}\mathrm{d}t. \quad (160)$$

Then, we discuss the evolution behaviors for $a > 0$ or $a < 0$, respectively.

Firstly, we investigate the cases of $a = I/2 - L > 0$ and $b = K - (I-J)/2 < 0$. In this case, $\mathrm{d}u/\mathrm{d}t > 0$. Thus, the function $u(t)$ will increase monotonously, and in the long-time limit ($t \gg 1$), we have $u \gg 1$. In this limiting case, the left side of Eq. (160) can be further rewritten as

$$\begin{aligned}
&u^{K/b-1}\left(u-\frac{b}{a}\right)^{-(aK-bL)/ab}\mathrm{d}u \\
&= u^{K/b-1-(aK-bL)/ab}\left(1-\frac{b}{a}\frac{1}{u}\right)^{-(aK-bL)/ab}\mathrm{d}u \\
&= u^{L/a-1}\left[1+\frac{b}{a}\frac{aK-bL}{ab}\frac{1}{u}\right. \\
&\quad \left.+\sum_{n=2}^{\infty}\frac{(-(aK-bL)/ab)!}{n!(-(aK-bL)/ab-n)!}\left(-\frac{b}{a}\frac{1}{u}\right)^n\right]\mathrm{d}u \\
&= \left[u^{L/a-1}+\frac{aK-bL}{a^2}u^{L/a-2}+u^{L/a-1}\right. \\
&\quad \left.\times\sum_{n=2}^{\infty}\frac{(-(aK-bL)/ab)!}{n!(-(aK-bL)/ab-n)!}\left(-\frac{b}{a}\frac{1}{u}\right)^n\right]\mathrm{d}u. \qquad (161)
\end{aligned}$$

Equation (160) can then be integrated in the region of $u \gg 1$ to be

$$\theta_6(u) - \theta_6[u(0)] = \frac{L}{C_6}t, \qquad (162)$$

with function

$$\theta_6(u) = \begin{cases} u\left[1+\dfrac{aK-bL}{a^2}\dfrac{\ln u}{u}+\dfrac{1}{u}\phi_6(u)\right], & (a=L), \\ u^{L/a}\left[1+\dfrac{aK-bL}{a^2}\dfrac{L}{L-a}\dfrac{1}{u}\right. & \\ \quad \left.+\dfrac{L}{a}u^{-L/a}\phi_6(u)\right], & (a\neq L), \end{cases} \qquad (163)$$

where

$$\begin{aligned}
\phi_6(u) &= \int u^{L/a-1}\sum_{n=2}^{\infty}\frac{(-(aK-bL)/ab)!}{n!(-(aK-bL)/ab-n)!} \\
&\quad \times\left(-\frac{b}{a}\frac{1}{u}\right)^n\mathrm{d}u. \qquad (164)
\end{aligned}$$

In the region of $u \gg 1$, we obtain a zeroth-order approximate solution for $u(t)$ from Eq. (162) as

$$u \simeq \left(\frac{L}{C_6}t\right)^{a/L}. \qquad (165)$$

We then further obtain m_{B} and m_{A} as

$$\begin{aligned}
m_{\mathrm{B}}(t) &= C_6 u^{K/b-1}\left(u-\frac{b}{a}\right)^{(bL-aK)/ab+1} \\
&\simeq C_6 u^{K/b-1}u^{(bL-aK)/ab+1} = C_6 u^{L/a} \simeq Lt, \qquad (166) \\
m_{\mathrm{A}}(t) &= u(t)m_{\mathrm{B}}(t) \simeq L\left(\frac{L}{C_6}\right)^{a/L}t^{a/L+1}. \qquad (167)
\end{aligned}$$

We find that both the total mass $M_1^{\mathrm{A}} = m_{\mathrm{A}}^{-1}$ and $M_1^{\mathrm{B}} = m_{\mathrm{B}}^{-1}$ decrease with time.

From the solution of $m_{\mathrm{A}}(t)$, we can further obtain $n_{\mathrm{A}}(t)$ as

$$n_{\mathrm{A}}(t) = m_{\mathrm{A}}(t) + Kt \simeq L\left(\frac{L}{C_6}\right)^{a/L}t^{a/L+1} + Kt. \qquad (168)$$

We can then derive the functions $a(t)$ and $A(t)$ to investigate the kinetic evolution behavior of the aggregate size distribution $a_k(t)$ in the long-time limit

$$\begin{aligned}
a(t) &= 1 - \frac{m_{\mathrm{A}}}{n_{\mathrm{A}}} \simeq \frac{K}{L}\left(\frac{L}{C_6}\right)^{-a/L}t^{-a/L}, \qquad (169) \\
A(t) &= \frac{m_{\mathrm{A}}}{n_{\mathrm{A}}^2} \simeq \frac{1}{L}\left(\frac{L}{C_6}\right)^{-a/L}t^{-a/L-1}. \qquad (170)
\end{aligned}$$

In the long-time limit, $a_k(t)$ is approximated to be

$$a_k(t) \simeq K^{-1}\left[\frac{K}{L}\left(\frac{L}{C_6}\right)^{-a/L}\right]^k t^{-1}t^{-ak/L}, \qquad (171)$$

which does not exhibit any scaling form.

For the noncooperative agent species B, we obtain

$$\begin{aligned}
n_{\mathrm{B}}(t) &= \frac{1}{I-J}\left[\frac{I}{B_0}(1+LB_0t) - Jm_{\mathrm{B}}\right] \\
&\simeq Lt + \frac{I}{(I-J)B_0}. \qquad (172)
\end{aligned}$$

We can obtain the functions $b(t)$ and $B(t)$ to investigate the kinetic evolution behavior of the aggregate size distribution $b_k(t)$ in the long-time limit

$$\begin{aligned}
b(t) &= 1 - \frac{m_{\mathrm{B}}}{n_{\mathrm{B}}} \simeq \frac{I}{(I-J)LB_0t}, \qquad (173) \\
B(t) &= \frac{m_{\mathrm{B}}}{n_{\mathrm{B}}^2} \simeq \frac{1}{Lt}. \qquad (174)
\end{aligned}$$

In the long-time limit, $b_k(t)$ is approximated to be

$$b_k(t) \simeq L^{-1}\left[\frac{I}{(I-J)LB_0}\right]^{k-1}t^{-k}, \qquad (175)$$

which also does not exhibit any scaling form.

Secondly, we investigate the case of $a = I/2 - L < 0$ and $b = K - (I-J)/2 > 0$. In this case, $\mathrm{d}u/\mathrm{d}t < 0$. The function $u(t)$ will decrease monotonously, and in the long-time limit we have $u \ll 1$. In this limiting case, the left side of Eq. (160) can be further rewritten as

$$\begin{aligned}
&u^{K/b-1}\left(u-\frac{b}{a}\right)^{(bL-aK)/ab}\mathrm{d}u \\
&= \left(-\frac{b}{a}\right)^{(bL-aK)/ab}u^{K/b-1}\left(1-\frac{a}{b}u\right)^{(bL-aK)/ab}\mathrm{d}u \\
&= \left(-\frac{b}{a}\right)^{(bL-aK)/ab}\left[u^{K/b-1}-\frac{bL-aK}{b^2}u^{K/b}\right. \\
&\quad \left.+u^{K/b-1}\sum_{n=2}^{\infty}\frac{((bL-aK)/ab)!}{n!((bL-aK)/ab-n)!}\left(-\frac{a}{b}u\right)^n\right]\mathrm{d}u. \qquad (176)
\end{aligned}$$

Equation (160) can then be integrated in the $u \ll 1$ region

$$\theta_7(u) - \theta_7[u(0)] = -C_7 t, \tag{177}$$

with function

$$\theta_7(u) = u^{K/b}\left[1 - \frac{K(bL-aK)}{b^2(K+b)}u + \phi_7(u)\right], \tag{178}$$

and the constant $C_7 = -(aK/bC_6)(-b/a)^{-(bL-aK)/ab} > 0$, where

$$\begin{aligned}\phi_7(u) &= \frac{K}{b}u^{-K/b}\int u^{K/b-1} \\ &\times \sum_{n=2}^{\infty}\frac{((bL-aK)/ab)!}{n!((bL-aK)/ab-n)!}\left(-\frac{a}{b}u\right)^n \mathrm{d}u.\end{aligned} \tag{179}$$

In the region of $u \ll 1$, we obtain a zeroth-order approximate solution for $u(t)$ from Eq. (177) as

$$u \simeq C_7^{b/K}(t_c - t)^{b/K}, \tag{180}$$

where the time $t_c = \theta_7[u(0)]/C_7$.

It is apparent that Eq. (177) is valid only in the time region $t < t_c$. From Eq. (180), we found that u decreases with time and $u \ll 1$ when t approaches t_c.

From Eq. (180), we then obtain $m_B(t)$ in the form of

$$\begin{aligned}m_B &= C_6 u^{K/b-1}\left(u - \frac{b}{a}\right)^{(bL-aK)/ab+1} \\ &= C_6\left(-\frac{b}{a}\right)^{(bL-aK)/ab+1} u^{K/b-1}\left(1 - \frac{a}{b}u\right)^{(bL-aK)/ab+1} \\ &\simeq C_6\left(-\frac{b}{a}\right)^{(bL-aK)/ab+1} u^{K/b-1} \\ &\simeq C_7 K(t_c - t)^{(I-J)/2K}.\end{aligned} \tag{181}$$

For the cooperative agent species A, we obtain

$$m_A(t) = u(t)m_B(t) \simeq C_7^{b/K+1}K(t_c - t), \tag{182}$$

$$n_A(t) = m_A(t) + Kt \simeq C_7^{b/K+1}K(t_c - t) + Kt. \tag{183}$$

From Eqs. (181) and (182) we derive the total masses of the aggregates of the cooperative agent species A and the noncooperative agent species B as

$$M_1^A(t) = m_A^{-1} \simeq C_7^{-b/K-1}K^{-1}(t_c - t)^{-1}, \tag{184}$$

$$M_1^B(t) = m_B^{-1} \simeq C_7^{-1}K^{-1}(t_c - t)^{-(I-J)/2K}. \tag{185}$$

From those solutions, we find that there is a gelation transition at time t_c for the cooperative agent species A. For the noncooperative agent species B, when $I > J$, a gelation transition takes place at time t_c. When $I < J$ and time approaches t_c, we have $M_1^B \to 0$, which means the noncooperative agent species B will finally disappear.

4. Summary

In summary, we have proposed an evolution model of cooperative agent and noncooperative agent aggregates to investigate the competitions among the microscopic mechanisms. In this model, between two cooperative agent aggregates, a win–win interaction $A_k + A_l \xrightarrow{K(k,l)} A_{k+1} + A_{l+1}$ occurs, which gives rise to the increase in the cooperative spirits of both agents or the sizes of both agent aggregates. A lose–lose reaction $B_k + B_l \xrightarrow{L(k,l)} B_{k-1} + B_{l-1}$ occurs between two noncooperative agents, which causes the decrease in the noncooperative spirits of either agent. Between two agent aggregates of different species, a win–lose reaction (similar to the migration reaction: migration of sprits from cooperative agents to noncooperative agents) $A_k + B_l \xrightarrow{I(k,l),p} A_{k-1} + B_{l+1}$ and a catalyzed-decline reaction $B_k + A_l \xrightarrow{J(k,l),q} B_{k-1} + A_l$ occur with the same probability $p = q = 1/2$. The former of the last two reactions gives rise to the decrease in the cooperative agent's size of spirits and the increase in the noncooperative agent's size of spirits. The last reaction gives rise to the decrease in the noncooperative agent's size of spirits.

Based on the mean-field rate equations, we analyzed the kinetic evolution behaviors of two species? aggregates. It is found that there are several groups of competitions which dominate the evolution behaviors of the system. The first important or dominating competition is between the lose–lose reaction $B_k + B_l \xrightarrow{L(k,l)} B_{k-1} + B_{l-1}$ and the win–lose reaction $A_k + B_l \xrightarrow{I(k,l),p} A_{k-1} + B_{l+1}$, where the former and last reactions give rise to the decrease and increase in the noncooperative agent spirits, respectively. This competition appears in the form $a = I/2 - L > 0$, $a = I/2 - L = 0$, and $a = I/2 - L < 0$. The second is the competition between the win–win reaction $A_k + A_l \xrightarrow{K(k,l)} A_{k+1} + A_{l+1}$ and the win–lose reaction $A_k + B_l \xrightarrow{I(k,l),p} A_{k-1} + B_{l+1}$, where the former and last reactions give rise to the increase and decrease in the cooperative agent spirits, respectively. It appears in the form of $b = K - (I-J)/2 > 0$, $b = K - (I-J)/2 = 0$, and $b = K - (I-J)/2 < 0$. The third is the competition between the win–lose reaction $A_k + B_l \xrightarrow{I(k,l),p} A_{k-1} + B_{l+1}$ and the catalyzed-decline reaction $B_k + A_l \xrightarrow{J(k,l),q} B_{k-1} + A_l$, where the former and last reactions give rise to the increase and decrease in the noncooperative agent spirits, respectively. It appears in the form of $c = (I-J)/2L > 0$, $c = (I-J)/2L = 0$, and $c = (I-J)/2L < 0$.

(i) Case of $a = I/2 - L > 0$. When $b = K - (I-J)/2 \leq 0$ and $c = (I-J)/2L \neq 0$, both the total mass M_1^A and M_1^B decrease with time. This result indicates that in this case because the win–win reaction which leads to the increase in the cooperative agent spirits is weak (the strength is represented by the proportional constant K of the win–win reaction rate ker-

nel), and the effect of the win–lose reaction which gives rise to the decrease in the cooperative agent spirits is strong (whose strength is described by the proportional constant I of the win–lose reaction rate kernel), the win–win reaction dominates the dynamic evolution and net effects of the win–win reaction and the win–lose reaction is the decrease in total spirits of the cooperative agents and further gives rise to the the decrease in total spirits of the noncooperative agents.

(ii) Case of $a = I/2 - L = 0$. When $b = K - (I-J)/2 > 0$ and $c = (I-J)/2L \neq 0$, the total mass M_1^{A} increases with time and approaches infinite at a finite time, which reflects that the effects of the win–win reaction which gives rise to the increase in the cooperative spirits dominates the evolution of the cooperative agent aggregates. Meanwhile, the evolution behavior of the total mass M_1^{B} is dominated by the the competition between the migration reaction and the catalyzed-decline reaction. When $c = (I-J)/2L > 0$, the effect of the migration reaction is stronger than that of the catalyzed-decline reaction, and the total mass M_1^{B} increases with time. When $c = (I-J)/2L < 0$, the effect of the catalyzed-decline reaction is stronger than that of the migration reaction, and the total mass M_1^{B} decreases with time. When $b = K - (I-J)/2 < 0$ and $c = (I-J)/2L \neq 0$, both the total mass M_1^{A} and M_1^{B} decrease with time.

(iii) Case of $a = I/2 - L < 0$. When $b = K - (I-J)/2 > 0$ and $c = (I-J)/2L \neq 0$, the evolution behaviors of the two species are the same as those in the above case of $a = I/2 - L = 0$, $b = K - (I-J)/2 > 0$, and $c = (I-J)/2L \neq 0$. When $b = K - (I-J)/2 = 0$ and $c = (I-J)/2L \neq 0$, both the total mass M_1^{A} and M_1^{B} increase with time, and approach to infinite at a finite time.

(iv) Case of $c = (I-J)/2L = 0$. In the case of $a = I/2 - L > 0$, the evolution behaviors of the two species are dominated by the competition between the win–win reaction, the migration reaction, and the catalyzed-decline reaction, and the initial numbers of agent monomers also influence the evolution. When $(I/2 - L)B_0 - KA_0 \geq 0$, the total mass M_1^{A} decreases with time, and the aggregate size distribution $a_k(t)$ approaches a modified scaling form in the case of $(I/2 - L)B_0 - KA_0 = 0$. When $(I/2 - L)B_0 - KA_0 < 0$, the total mass M_1^{A} increases with time and approaches infinite at a finite time.

(v) Special cases with constant total masses M_1^{A} and M_1^{B}. In the general case of $a = I/2 - L \neq 0$, $b = K - (I-J)/2 \neq 0$, and $c = (I-J)/2L \neq 0$ under the special condition $[K - (I-J)/2]A_0 = (I/2 - L)A_0$, and in the case of $a = I/2 - L \neq 0$, $b = K - (I-J)/2 = 0$, and $c = (I-J)/2L \neq 0$ under the special condition $KA_0 = LB_0$, the total masses M_1^{A} and M_1^{B} converge to A_0 and B_0, respectively. Both the aggregate size distributions $a_k(t)$ and $b_k(t)$ approach the conventional scaling form.

Indeed, the competitions give rise to surprisingly rich dynamic evolution behaviors.

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