Cross-diffusive effects on the onset of double-diffusive convection in a horizontal saturated porous fluid layer heated and salted from above

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Double-diffusive stationary and oscillatory instabilities at the marginal state in a saturated porous horizontal fluid layer heated and salted from above are investigated theoretically under the Darcy's framework for a porous medium. The contributions of Soret and Dufour coefficients are taken into account in the analysis. Linear stability analysis shows that the critical value of the Darcy-Rayleigh number depends on cross-diffusive parameters at marginally stationary convection, while the marginal state characterized by oscillatory convection does not depend on the cross-diffusion terms even if the condition and frequency of oscillatory convection depends on the cross-diffusive parameters. The critical value of the Darcy–Rayleigh number increases with increasing value of the solutal Darcy–Rayleigh number in the absence of cross-diffusive parameters. The critical Darcy–Rayleigh number decreases with increasing Soret number, resulting in destabilization of the system, while its value increases with increasing Dufour number, resulting in stabilization of the system at the marginal state characterized by stationary convection. The analysis reveals that the Dufour and Soret parameters as well as the porosity parameter play an important role in deciding the type of instability at the onset. This analysis also indicates that the stationary convection is followed by the oscillatory convection for certain fluid mixtures. It is interesting to note that the roles of cross-diffusive parameters on the double-diffusive system heated and salted from above are reciprocal to the double-diffusive system heated and salted from below.

Keywords: double-diffusive convection, porous media, Soret and Dufour effects, linear stability analysis

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1. Introduction

The well-known Rayleigh–Bénard problem is not a simple low-temperature phenomenon. The problem is of potential importance in many real life applications. The study of convective motions in the presence of more than one diffusing component with different diffusivities is an interesting phenomenon in the field of convection, known as multi-convection.[1] In particular, the double-diffusive convection in the classical Rayleigh–Bénard type problem has been the subject of extensive theoretical and experimental interest because of its applications in various fields, such as high-quality crystal production, liquid gas storage, production of pure medication, solidification of molten alloys, coating of electronic materials, oceanography and geothermally heated lakes and magmas.[2] The defects in the formation of crystal depend on heat and mass transfer rates during solidification. These fluxes are mainly governed by the convective phenomenon of the liquid phase during processing. The analysis of double-diffusive convection becomes complicated when the diffusivity of one is much greater than the other. Furthermore, when two transport processes take place simultaneously, they interfere with each other, producing cross-diffusion effects. The cross-diffusion effects are significant in the stability of the system. The flux of concentration caused by temperature gradient and the flux of heat caused by concentration gradient are known as Soret and Dufour effects, respectively. In binary fluid mixtures this cross-coupling between thermal and solutal diffusions should not be eliminated in the case of a steep temperature gradient.

The study of the onset of double-diffusive convection in a porous medium due to temperature and concentration gradients has become an interesting field of study because of its numerous fundamental and industrial applications, especially the study of the behaviour of fluids in Earth's crust, in geology, geophysics, metallurgy, material sciences, and petroleum engineering. The appearance of convective motion is characterized by the critical value of the Rayleigh number. A great deal of work in this field is well documented in the book by Neild and Bejan.[3]

The problem of the onset of convective instability was first analyzed by Horton, Rogers,[4] and Lapwood[5] in a Newtonian fluid model heated uniformly from below in a saturated porous medium under Darcy's law. For the stationary thermal convection under the principle of exchange of stabilities they obtained the critical value of the Darcy–Rayleigh number $(4\pi^2)$. This limit determines the particular point at which the convective instability may start to occur in a fluid layer. Thereafter, many theoretical results were elucidated by Katto and Masuoka,[6] Combarnous and Bories,[7] and Cheng.[8] Combarnous and Bories[7] noted that the thermo-convective motion in a stationary mode becomes oscillatory with increasing value

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of the Darcy–Rayleigh number beyond six or seven times of its critical value. The occurrence of oscillatory instabilities will be expected at the threshold of the stationary convection in a horizontal porous layer.

The double-diffusive convection was observed first by Stern,[9] and the mechanism of instability at the onset of thermohaline convection in a porous medium have been described by Nield.[10] Thermohaline convection and its pattern of instabilities are important in many situations, and have wider applications in chemical and industrial flows, geological and environmental flows, and many others. The linear stability analysis was used to investigate the onset of convection in a double-diffusive fluid flow.[11,12] However, Motsa[13] investigated the effects of both the Soret and Dufour parameters on the onset of double-diffusive convection heated from below.

Recently, Kuznetsov and Nield[14] studied the onset of double-diffusive convection in a horizontal Newtonian fluid layer of a saturated porous medium uniformly heated from below using Brinkman model with combined effects of horizontal and vertical heterogeneity. Kuznetsov and Nield[15] also studied the onset of double-diffusive nanofluid convection in layer of a saturated porous medium using the Darcy model. Nield and Kuznetsov[16] studied the effects of cross-diffusion terms, as well as the effects of Brownian motion and thermophoresis on the onset of double-diffusive convection in a horizontal layer of nanofluid. Kuznetsov and Nield[17] studied these effects on the double-diffusive natural convective boundary-layer flow of nanofluid past a vertical plate. Straughan[18] studied the onset of thermohaline convection in a saturated porous medium using the Cattaneo–Christov model, where instead of the classical Fourier law, the Cattaneo–Christov heat flux law with energy balance is considered.

As mentioned above, there are several studies on the onset of double-diffusive convection with and without cross-diffusion effects in a horizontal fluid layer for a saturated porous medium heated and salted uniformly from below. However, there are quite a few studies on the onset of double-diffusive convection with cross-diffusion effects in a horizontal fluid layer for a saturated porous medium heated and salted uniformly from above. In this work, we studied the double-diffusive convection based on linear stability analysis taking into account the Soret and Dufour effects in a horizontal saturated porous fluid layer, heated and salted uniformly from above under the framework of Darcy. Linear stability analysis is made to find the critical value of the Darcy–Rayleigh number for the onset of the stationary mode of convection. The onset of oscillatory instabilities at the threshold of convection in this system is also investigated. This mechanism can be of interest in the transport of contaminants in ground water. Since the ground water flow is relatively slow, the Darcy model considered in this analysis is quite appropriate.

2. Basic equations of the problem

We consider an infinite horizontal saturated porous fluid layer of vertical depth h, confined between two stress-free boundaries at constant temperature and salinity. The hot salty water is above the cold fresh water so that the upper boundary is kept at a temperature $(T_0+\Delta T)$ and salinity $(C_0+\Delta C)$ with T_0 being the lower temperature boundary and C_0 the lower concentration boundary. It is assumed that the thermal diffusivity $\kappa_{\rm T}$ is much greater (100 times) than solutal diffusivity $\kappa_{\rm S}$. Physically, the instabilities develop as follows. The rising water blob absorbs heat much more quickly than salt. Hence, it becomes lighter than the surrounding fluids, and continues to rise. If a fluid blob drops by Δz, the faster heat loss can make it heavier than its surrounding fluids, and sinking continues. Thus, the fluid system can be unstable. This yields an instability known as fingering instability at marginally stationary state.

The porous medium with specific permeability K under the assumptions of homogeneous and isotropic medium, and the validity of Darcy's law for the fluid flow is assumed. The momentum equation may be written under the framework of Darcy model for simple flow as

$$u_i = -\frac{K}{\mu}\left(\frac{\partial p}{\partial x_i} - \rho g \delta_{i3}\right), \tag{1}$$

where u_i is the Darcy velocity, K is the permeability of the porous media, μ is the viscosity of fluid, p is the pressure, g is the acceleration due to gravity, $\delta_{ij}=1$ for $i=j$ and $\delta_{ij}=0$ for $i\neq j$, and ρ is the fluid density. The fluid density decreases with the increase in temperature, but slightly increases with the increase in concentration of the salt so that the thermal expansion coefficient $\alpha>0$ and the solutal expansion coefficient $\beta>0$. The equation of state is then given by

$$\rho = \rho_0[1-\alpha(T-T_0)+\beta(C-C_0)], \tag{2}$$

where the subscript 0 represent variable at the bottom layer. The appropriate governing equations, using the Boussinesq approximation, are expressed as follows:

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{3}$$

$$u_i = -\frac{K}{\mu}\left[\frac{\partial p}{\partial x_i} - \rho_0\{1-\alpha(T-T_0)+\beta(C-C_0)\}g\delta_{i3}\right], \tag{4}$$

$$\sigma\frac{\partial T}{\partial t} + u_i\frac{\partial T}{\partial x_i} = \kappa_{\rm T}\nabla^2 T + \beta_{\rm T}\nabla^2 C, \tag{5}$$

$$\phi\frac{\partial C}{\partial t} + u_i\frac{\partial C}{\partial x_i} = \kappa_{\rm S}\nabla^2 C + \beta_{\rm S}\nabla^2 T, \tag{6}$$

where $i=1,2,3$, C is the solute concentration, T the temperature, σ the ratio of specific heat capacities of solid and fluid, ϕ the porosity of the porous media, $\beta_{\rm T}$ the Dufour coefficient, and $\beta_{\rm S}$ Soret coefficient.

The boundary conditions are specified as

$$\begin{aligned} &T = T_0, \quad C = C_0(z=0),\\ &T = T_0+\Delta T, \quad C = C_0+\Delta C(z=h), \end{aligned} \tag{7}$$

where h is the dimensional depth of the fluid layer. A physical sketch of the problem is depicted in Fig. 1.

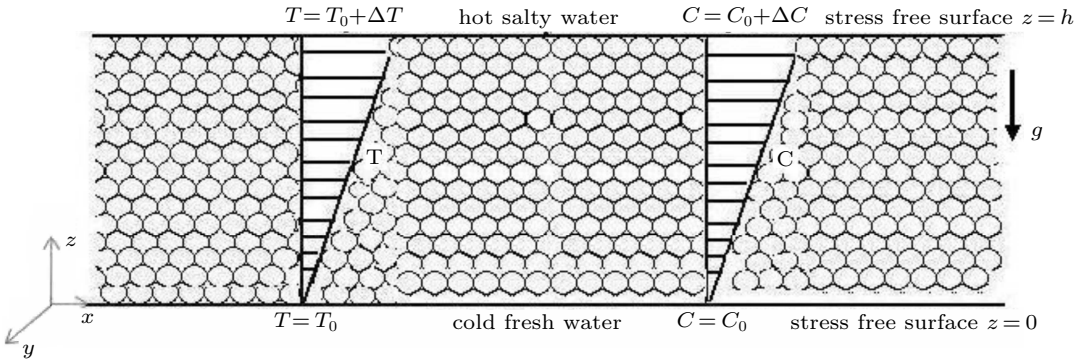


Fig. 1. Physical sketch of the problem representing an infinite horizontal saturated porous fluid layer with linear distribution of temperature and solute.

3. Linear stability analysis

The basic state of the system is perturbed and linear perturbation quantities are taken into consideration for the stability analysis of the system. The perturbation quantities are considered as follows:

$$\begin{aligned} u_i &= 0 + u_i'(x,t),\\ p &= p_{\rm b}(z) + p'(x,t),\\ T &= T_{\rm b}(z) + T'(x,t),\\ C &= C_{\rm b}(z) + C'(x,t), \end{aligned} \tag{8}$$

where the z axis is taken vertically upwards, the superscript $'$ represents perturbation variable, and the subscript b represents basic state variable.

The basic state satisfies the following equations:

$$\frac{\partial p_{\rm b}}{\partial z} = \rho_0\left[1-\alpha(T_{\rm b}-T_0)+\beta(C_{\rm b}-C_0)\right]g, \tag{9}$$

$$\kappa_{\rm T}\frac{{\rm d}^2T_{\rm b}}{{\rm d}z^2}+\beta_{\rm T}\frac{{\rm d}^2C_{\rm b}}{{\rm d}z^2}=0, \tag{10}$$

$$\kappa_{\rm S}\frac{{\rm d}^2C_{\rm b}}{{\rm d}z^2}+\beta_{\rm S}\frac{{\rm d}^2T_{\rm b}}{{\rm d}z^2}=0. \tag{11}$$

From Eqs. (10) and (11), we have

$$T_{\rm b}=T_0+\frac{\Delta T}{h}z=T_0+\Gamma_1 z,\ C_{\rm b}=C_0+\frac{\Delta C}{h}z=C_0+\Gamma_2 z, \tag{12}$$

where $\Gamma_1=\dfrac{{\rm d}T_{\rm b}}{{\rm d}z}>0$, and $\Gamma_2=\dfrac{{\rm d}C_{\rm b}}{{\rm d}z}>0$.

Using the perturbation quantities (8) in Eqs. (3)–(6), considering only the linear perturbation terms and using Eqs. (9)–(12) the perturbation equations are

$$\frac{\partial u_i'}{\partial x_i}=0, \tag{13}$$

$$u_i'=-\frac{K}{\mu}\left[\frac{\partial p'}{\partial x_i}+\rho_0 g(\alpha T'-\beta C')\delta_{i3}\right], \tag{14}$$

$$\sigma\frac{\partial T'}{\partial t}+\Gamma_1 w'=\kappa_{\rm T}\nabla^2T'+\beta_{\rm T}\nabla^2C', \tag{15}$$

$$\phi\frac{\partial C'}{\partial t}+\Gamma_2 w'=\kappa_{\rm S}\nabla^2C'+\beta_{\rm S}\nabla^2T'. \tag{16}$$

Expressing the perturbation equations in terms of w', T', and C' by eliminating the unknown pressure term, using certain mathematical manipulations, we obtain

$$\nabla^2w'=-\frac{K}{\mu}(\rho_0 g\alpha\nabla_H^2T'-\rho_0 g\beta\nabla_H^2C'), \tag{17}$$

$$\sigma\frac{\partial T'}{\partial t}+\Gamma_1 w'=\kappa_{\rm T}\nabla^2T'+\beta_{\rm T}\nabla^2C', \tag{18}$$

$$\phi\frac{\partial C'}{\partial t}+\Gamma_2 w'=\kappa_{\rm S}\nabla^2C'+\beta_{\rm S}\nabla^2T', \tag{19}$$

where

$$\nabla_{\rm H}^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

is the planar Laplacian.

The boundary conditions on the free surfaces $z=h$ and $z=0$ are expressed as

$$w'=T'=C'=\mu\left(\frac{\partial u'}{\partial z}+\frac{\partial w'}{\partial x}\right)=\mu\left(\frac{\partial v'}{\partial z}+\frac{\partial w'}{\partial y}\right)=0.$$

The last two conditions result from zero stress. As $w'=0$ for all (x, y) on the boundaries, then $(\partial u'/\partial z)=(\partial v'/\partial z)=0$ at the boundaries. Differentiating the following equation of continuity with respect to z:

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0,$$

and using the condition

$$\partial u'/\partial z=\partial v'/\partial z=0,$$

we have $(\partial^2w'/\partial z^2)=0$ at the free surfaces.

Finally, the boundary conditions are given by

$$w'=T'=C'=\frac{\partial^2w'}{\partial z^2}=0, \tag{20}$$

at $z=0$ and $z=h$.

Using the transformations $(x,y,z)\to(x^*h,y^*h,z^*h)$, $t\to(h^2\sigma/\kappa_{\rm T})\tau$, $w'\to(\kappa_{\rm T}/h)w^*$, $T'\to\Delta T\theta^*$, $C'\to\Delta CC^*$, and dropping superscript '*' (dimensionless variable) to the coordinates (x,y,z) for brevity, equations (17)–(19) become

$$\nabla^2w^*=-\frac{Kg\alpha\Delta Th}{\nu\kappa_{\rm T}}\nabla_{\rm H}^2\theta^*+\frac{Kg\beta\Delta Ch}{\nu\kappa_{\rm T}}\nabla_{\rm H}^2C^*, \tag{21}$$

$$\frac{\partial\theta^*}{\partial\tau}+w^*=\nabla^2\theta^*+\frac{\beta_{\rm T}\Delta C}{\kappa_{\rm T}\Delta T}\nabla^2C^*, \tag{22}$$

$$\frac{\phi}{\sigma}\frac{\partial C^*}{\partial\tau}+w^*=\frac{\kappa_{\rm S}}{\kappa_{\rm T}}\nabla^2C^*+\frac{\beta_{\rm S}\Delta T}{\kappa_{\rm T}\Delta C}\nabla^2\theta^*. \tag{23}$$

The boundary conditions are

$$w^*=C^*=\theta^*=\frac{\partial^2w'}{\partial z^2}=0, \tag{24}$$

at $z=0$ and $z=h$.

4. Normal mode analysis

Under the normal mode analysis the time-dependent periodic disturbances in a horizontal plane may take the following form:

$$(w^*,\theta^*,C^*)=(W(z),\theta(z),C(z))\,\mathrm{e}^{\mathrm{i}(a_x x+a_y y)+s\tau}, \tag{25}$$

where a_x and a_y are dimensionless wave numbers in the x–y plane, and τ is the dimensionless time. The temporal rate of change of disturbance s can be expressed as $s=s_\mathrm{r}+\mathrm{i}\omega$. If s_r is positive for any value of the wave number, the system is unstable. While the system with $s_\mathrm{r}<0$ is always stable. When $s=0$ and thus $s_\mathrm{r}=\omega=0$, then the system is marginally stable under the principle of exchange of stabilities. With $s_\mathrm{r}=0$ and $\omega\neq 0$, the overstability of periodic motion is possible at the marginal state and oscillatory motion appears with the dimensionless frequency ω.

The dimensionless numbers are defined as follows: the Darcy–Rayleigh number $\mathrm{Ra_D}=(Kg\alpha h\Delta T)/(\nu\kappa_\mathrm{T})$, the solutal Darcy–Rayleigh number $\mathrm{Ra_{D,S}}=(Kg\beta h\Delta C)/(\nu\kappa_\mathrm{S})$, the Lewis number $\mathrm{Le}=\kappa_\mathrm{T}/\kappa_\mathrm{S}$, the Soret number $S_\mathrm{r}=(\beta_\mathrm{S}\Delta T)/(\kappa_\mathrm{T}\Delta C)$, the Dufour number $D_\mathrm{f}=(\beta_\mathrm{T}\Delta C)/(\kappa_\mathrm{T}\Delta T)$. Then, based on Eq. (25), equations (21)–(23) are transformed to

$$(D^2-a^2)W=a^2\mathrm{Ra_D}\theta-\frac{1}{\mathrm{Le}}a^2\mathrm{Ra_{D,S}}C, \tag{26}$$

$$W=(D^2-a^2-s)\theta+D_\mathrm{f}(D^2-a^2)C, \tag{27}$$

$$W=\Big[\frac{1}{\mathrm{Le}}(D^2-a^2)-\frac{\phi}{\sigma}s\Big]C+S_\mathrm{r}(D^2-a^2)\theta. \tag{28}$$

The boundary conditions (24) become

$$W=\theta=C=0,\quad D^2W=0,\ \text{at } z=0 \text{ and } z=h, \tag{29}$$

where $D(\equiv \mathrm{d}/\mathrm{d}z)$ denotes the differential operator. Equations (26)–(28) are combined to get a single equation, characterizing the instability of the system as follows:

$$\begin{aligned}&\Big[a^2\mathrm{Ra_D}-(D^2-a^2)(D^2-a^2-s)\Big]\\&\times\Big[(1-\mathrm{Le}D_\mathrm{f})(D^2-a^2)-\mathrm{Le}\frac{\phi}{\sigma}s\Big]\theta\\=&\Big[a^2\frac{1}{\mathrm{Le}}\mathrm{Ra_{D,S}}+D_\mathrm{f}(D^2-a^2)^2\Big]\\&\times\Big[\mathrm{Le}(1-S_\mathrm{r})(D^2-a^2)-\mathrm{Le}s\Big]\theta.\end{aligned} \tag{30}$$

If $S_\mathrm{r}\to 0$ and $D_\mathrm{f}\to 0$, then the foregoing problem approaches the classical Darcy–Rayleigh problem with Newtonian fluid in a saturated porous fluid layer as studied by Horton, Rogers, and Lapwood. The boundary conditions become

$$\theta=D^2\theta=0,\quad \text{at } z=0 \text{ and } z=h. \tag{31}$$

Both boundaries are isothermal and viscous stress-free in Darcy's law. The solution of the governing equation (30) with boundary conditions (31) can be regarded as the usual eigenvalue problem involving eigen parameters $\mathrm{Ra_D}$, Le, N, S_r, D_f, a, and s as the governing parameters. In order to solve the present differential Eq. (30) with boundary conditions (31), the required solution is assumed as $\theta=\theta_0\sin(n\pi z)$, where θ_0 denotes the integration constant subject to the boundary conditions, and n is an integer.

Substituting this form of solution into Eq. (30) results in the following characteristic equation:

$$\begin{aligned}&\Big[a^2\mathrm{Ra_D}-(n^2\pi^2+a^2)(n^2\pi^2+a^2+s)\Big]\\&\times\Big[(1-\mathrm{Le}D_\mathrm{f})(n^2\pi^2+a^2)+\mathrm{Le}\frac{\phi}{\sigma}s\Big]\\=&\Big[a^2\frac{1}{\mathrm{Le}}\mathrm{Ra_{D,S}}+D_\mathrm{f}(n^2\pi^2+a^2)^2\Big]\\&\times\Big[\mathrm{Le}(1-S_\mathrm{r})(n^2\pi^2+a^2)+\mathrm{Le}s\Big].\end{aligned} \tag{32}$$

This algebraic equation can be rearranged into the form of second degree polynomials of s, with the consideration of $(n^2\pi^2+a^2)=J$, as

$$\begin{aligned}&s^2+\frac{1}{J\Phi}\big[(1+\Phi)J^2-a^2(\mathrm{LeRa_D}-\mathrm{Ra_{D,S}})\big]s\\&+\frac{1}{\Phi}\big\{(1-\mathrm{Le}D_\mathrm{f}S_\mathrm{r})J^2\\&-a^2\big[(1-\mathrm{Le}D_\mathrm{f})\mathrm{Ra_D}-(1-S_\mathrm{r})\mathrm{Ra_{D,S}}\big]\big\}=0,\end{aligned} \tag{33}$$

where a $(=\sqrt{a_x^2+a_y^2})$ is the horizontal wave number and $\Phi=\mathrm{Le}(\phi/\sigma)$. This is a quadratic equation of s, where the coefficients involving the governing parameters are all real. It has two roots, so if complex roots exist they must be in pair otherwise they are a pair of real roots. We are interested in the marginal stability analysis and in that case $s_\mathrm{r}=0$, where s_r is the real part of the complex roots $(s=s_\mathrm{r}\pm\mathrm{i}\omega)$. It is apparent that the marginal state $(s_\mathrm{r}=0)$ occurs for two cases of $\omega=0$ and $\omega\neq 0$, where ω is the imaginary part of the complex roots.

When the marginal state $(s_\mathrm{r}=0)$ is characterized by a stationary pattern of convection for which $\omega=0$, so we have $s=0$. Then, one of the roots of the quadratic equation (33) is zero and the condition for that is

$$\mathrm{Ra_D^{(steady)}}=\frac{(1-\mathrm{Le}D_\mathrm{f}S_\mathrm{r})}{(1-\mathrm{Le}D_\mathrm{f})}\frac{J^2}{a^2}+\frac{(1-S_\mathrm{r})}{(1-\mathrm{Le}D_\mathrm{f})}\mathrm{Ra_{D,S}}. \tag{34}$$

Thus, the marginal state with cellular convection ($\omega=0$ and so $s=0$), i.e., the stationary convection is governed by the relation (34). The critical value of Darcy–Rayleigh number for the stationary convection or monotonic instability is obtained using this equation.

It is apparent that a marginally oscillatory mode $(\omega\neq 0)$ occurs only when there exist a pair of purely imaginary roots of the quadratic equation (33) and then at the onset the instability sets in as oscillations of growing amplitude, known as "overstability". In the case of oscillatory instability $s=\mathrm{i}\omega$, where ω is a non-zero real number, the Rayleigh number is obtained from Eq. (33), according to Chandrasekhar,[19] as

$$\begin{aligned}&a^2\big[(1-\mathrm{Le}D_\mathrm{f})^2J^2+\Phi^2\omega^2\big]\mathrm{Ra_D}\\=&\,(1-\mathrm{Le}D_\mathrm{f})\big[(1-\mathrm{Le}D_\mathrm{f}S_\mathrm{r})J^2+a^2(1-S_\mathrm{r})\mathrm{Ra_{D,S}}\big]J^2\\&+\big[(\Phi+\mathrm{Le}D_\mathrm{f})+a^2\mathrm{Ra_{D,S}}\big]\Phi\omega^2\\&+\mathrm{i}\big\{\big[(1-\mathrm{Le}D_\mathrm{f})-\Phi(1-S_\mathrm{r})\mathrm{Le}D_\mathrm{f}\big]J^2\\&+a^2\big[(1-\mathrm{Le}D_\mathrm{f})-\Phi(1-S_\mathrm{r})\big]\mathrm{Ra_{D,S}}+\Phi^2\omega^2\big\}J\omega.\end{aligned} \tag{35}$$

From physical considerations, the Darcy–Rayleigh number $(\mathrm{Ra_D})$ is always a real number. Then, from the relation (35), the imaginary part must be zero, which gives the square of the

frequency of the periodic convection as

$$\Phi^2\frac{\omega^2}{a^2} = \left[\Phi(1-S_{\rm r}){\rm Le}D_{\rm f}-(1-{\rm Le}D_{\rm f})\right]\frac{J^2}{a^2} + \left[\Phi(1-S_{\rm r})-(1-{\rm Le}D_{\rm f})\right]{\rm Ra}_{\rm D,S}, \quad (36)$$

and the Darcy–Rayleigh number for oscillatory convection is given by

$${\rm Ra}_{\rm D}^{\rm (osc)} = \frac{(1-{\rm Le}D_{\rm f})^2J^2}{(1-{\rm Le}D_{\rm f})^2J^2+\Phi^2\omega^2} \times\left[\frac{(1-{\rm Le}D_{\rm f}S_{\rm r})}{(1-{\rm Le}D_{\rm f})}\frac{J^2}{a^2}+\frac{(1-S_{\rm r})}{(1-{\rm Le}D_{\rm f})}{\rm Ra}_{\rm D,S}\right] + \frac{1}{\Phi}\frac{\Phi^2\omega^2}{(1-{\rm Le}D_{\rm f})^2J^2+\Phi^2\omega^2} \times\left[(\Phi+{\rm Le}D_{\rm f})\frac{J^2}{a^2}+{\rm Ra}_{\rm D,S}\right]. \quad (37)$$

By means of the relation (37) and Eq. (36), the Darcy–Rayleigh number for the oscillatory convection can be expressed as

$$\Phi{\rm Ra}_{\rm D}^{\rm (osc)} = (\Phi+1)\frac{J^2}{a^2}+{\rm Ra}_{\rm D,S}. \quad (38)$$

Thus, the marginal state with oscillatory convection, i.e., overstability ($\omega\neq 0$) is governed by the relation (38) and the critical value of Darcy–Rayleigh number for oscillatory instability is obtained from this relation. Given that ω is always real, the condition for oscillatory convection is obtained from the relation (36) as

$${\rm Ra}_{\rm D,S} > \frac{(1-{\rm Le}D_{\rm f})-\Phi(1-S_{\rm r}){\rm Le}D_{\rm f}}{\Phi(1-S_{\rm r})-(1-{\rm Le}D_{\rm f})}\frac{J^2}{a^2}. \quad (39)$$

Equation (39) leads to an important result for a porous fluid layer heated and salted from above in double-diffusive convection that the oscillatory type of instability is possible for certain fluid mixtures. It reveals that in a double-diffusive system, steady convection strongly depends on the cross-diffusive terms, while the oscillatory convection remains independent of the cross-diffusive parameters, but depends on the porosity parameter Φ.

5. Results and discussion

We are mainly interested in the instability at marginally steady and oscillatory modes. The characteristic equation (33) may be represented as

$$s^2+A_1s+A_2=0, \quad (40)$$

where

$$A_1=\frac{1}{J\Phi}\left[(1+\Phi)J^2-a^2({\rm Le}{\rm Ra}_{\rm D}-{\rm Ra}_{\rm D,S})\right],$$

$$A_2=\frac{1}{\Phi}\left\{(1-{\rm Le}D_{\rm f}S_{\rm r})J^2-a^2\left[(1-{\rm Le}D_{\rm f}){\rm Ra}_{\rm D}-(1-S_{\rm r}){\rm Ra}_{\rm D,S}\right]\right\}.$$

For linear stability of the system, the necessary and sufficient conditions, according to Routh–Hurwitz criteria, are $A_1>0$ and $A_2>0$. If any one of the inequalities is not true then the instability sets in. Thus, the double-diffusive system with cross-diffusive effects, the stability or instability of the system depends upon the coefficients of the quadratic equation (40) that is Eq. (33), containing a large number of dimensionless parameters. The task of discussing stability of the system or the instabilities can be simplified by fixing some of the parameters. Accordingly, some of the parameters are fixed, and it is observed that for certain values of the governing parameters, the instability occurs in the above system. We are interested in finding the critical value of the Darcy–Rayleigh number at the onset of instability with cross-diffusive effects and the type of the instability at the onset.

When both the Soret and Dufour effects are present in the double-diffusive system, the marginal state characterized by the stationary pattern of convection is given by the relation (34). It is observed from the relation (34) that the stationary or monotonic instability strongly depends on the cross-diffusion terms, namely the Soret and Dufour coefficients. Minimizing firstly over n that is at the threshold of convection ($n=1$), and then over the square of wave number (a^2), we obtain the critical value of the Darcy–Rayleigh number at the onset of stationary convection. Thus, the boundary for monotonic or stationary instability is obtained from relation (34) as

$${\rm Ra}_{D_{\rm cr}}^{\rm (steady)} = \frac{(1-{\rm Le}D_{\rm f}S_{\rm r})}{(1-{\rm Le}D_{\rm f})}4\pi^2+\frac{(1-S_{\rm r})}{(1-{\rm Le}D_{\rm f})}{\rm Ra}_{\rm D,S}. \quad (41)$$

When $S_{\rm r}$ and $D_{\rm f}$ tend to be zero, the relation (41) becomes ${\rm Ra}_{D_{\rm cr}}^{\rm (steady)}=4\pi^2+{\rm Ra}_{\rm D,S}$. In the absence of solute (${\rm Ra}_{\rm D,S}=0$), the critical Darcy–Rayleigh number is given by ${\rm Ra}_{D_{\rm cr}}^{\rm (steady)}=4\pi^2$, which is the critical value of the Darcy–Rayleigh number for the onset of thermal convection in porous medium, previously reported by Nield.[20] The critical value of the Darcy–Rayleigh number for steady convection obtained in the relation (41) is identical but with an opposite sign to the result obtained by Motsa[13] for the double-diffusive convection in a porous fluid layer heated and salted from below. This result is also identical with the result obtained by Kuznetsov and Nield[15] considering the regular fluid instead of nanofluid. It is interesting to note that when $S_{\rm r}=1$, the critical value of the Darcy–Rayleigh number for steady double-diffusive convection becomes ${\rm Ra}_{D_{\rm cr}}^{\rm (steady)}=4\pi^2$, and is independent of the Dufour number and solutal Darcy–Rayleigh number, which is also the critical value of the Darcy–Rayleigh number for thermal convection in porous media.

Considering the minimum value of $J^2/a^2=4\pi^2$, the condition for the oscillatory instability in this double-diffusive system obtained from relation (39), satisfying the Solutal Darcy–Rayleigh number is

$${\rm Ra}_{\rm D,S} > \frac{(1-{\rm Le}D_{\rm f})-\Phi(1-S_{\rm r}){\rm Le}D_{\rm f}}{\Phi(1-S_{\rm r})-(1-{\rm Le}D_{\rm f})}4\pi^2. \quad (42)$$

If the condition (42) is satisfied, the marginal state is characterized by the oscillatory pattern of convection ($\omega\neq 0$), which is given by the relation (38), and then the critical value of the Darcy–Rayleigh number for the oscillatory instability at the threshold of convection is given by

$$\Phi{\rm Ra}_{D_{\rm cr}}^{\rm (osc)} = (\Phi+1)4\pi^2+{\rm Ra}_{\rm D,S}. \quad (43)$$

In the absence of cross-diffusive effects, namely the Soret and Dufour parameters, the critical value of the Darcy–Rayleigh number for steady convection becomes ${\rm Ra}_{D_{\rm cr}}^{\rm (steady)}=4\pi^2+{\rm Ra}_{\rm D,S}$. Thus, in the absence of cross-diffusion parameters,

the critical value of the Darcy–Rayleigh number increases with increasing solutal Darcy–Rayleigh number, so the solutal Darcy–Rayleigh number tends to stabilize the system. The critical value of Darcy–Rayleigh number for oscillatory convection, from the relation (43), is $\Phi \mathrm{Ra}_{D_{\mathrm{cr}}}^{(\mathrm{osc})} = (\Phi+1)4\pi^2 + \mathrm{Ra}_{\mathrm{D,S}}$. For all practical values of the parameters, the porosity parameter must be $\Phi = (\phi/\sigma)\mathrm{Le} > 1$,[18] and in that case the boundary lines for stationary and oscillatory convections can never be parallel. In this case, the points of intersection of the boundary lines for stationary and oscillatory convections are given by

$$\mathrm{Ra_D} = \frac{\Phi}{\Phi-1}4\pi^2, \quad \mathrm{Ra_{D,S}} = \frac{1}{\Phi-1}4\pi^2. \tag{44}$$

The oscillatory convection is not possible for $\Phi \leq 1$ due to the relation (36) with the consideration that ω is always real. In those cases, stationary convection prevails beyond the threshold of convection. When $(\mathrm{Ra_D} - \mathrm{Ra_{D,S}})$ exceeds $4\pi^2$, long thin salt fingers are observed. If the salinity gradient is large, then experiments as well as calculations show that a deep layer salt fingers become unstable and break down into a series of convecting layers with fingers confined to the interfaces. It is observed that as Φ increases to exceed 1, the oscillatory convection prevails earlier but is always preceded by the stationary convection for all values of Φ greater than 1, and for correspondingly small values of the solutal Darcy–Rayleigh number in the absence of cross-diffusion terms (e.g., $\Phi = 100$, $\mathrm{Ra_{D,S}} = 0.3984$).[21]

For a special type of porous medium when $\Phi = 1$, the value of the critical Darcy–Rayleigh number $(\mathrm{Ra_D})$ for oscillatory convection at the marginal state becomes $\mathrm{Ra_D^{osc}} = 8\pi^2 + \mathrm{Ra_{D,S}}$. In that case, the boundary lines for the monotonic instability and the oscillatory instability are parallel in the plane $(\mathrm{Ra_D}, \mathrm{Ra_{D,S}})$, so steady convection is always followed by oscillatory convection. When $\Phi \neq 1$, the boundary lines for monotonic and oscillatory instability are intersecting. The points of intersection are given by

$$\mathrm{Ra_D} = \frac{\Phi(1-S_{\mathrm{r}}) - (1-\mathrm{Le}D_{\mathrm{f}})S_{\mathrm{r}}}{\Phi(1-S_{\mathrm{r}}) - (1-\mathrm{Le}D_{\mathrm{f}})}4\pi^2,$$
$$\mathrm{Ra_{D,S}} = \frac{(1-\mathrm{Le}D_{\mathrm{f}}) - \Phi(1-S_{\mathrm{r}})\mathrm{Le}D_{\mathrm{f}}}{\Phi(1-S_{\mathrm{r}}) - (1-\mathrm{Le}D_{\mathrm{f}})}4\pi^2. \tag{45}$$

It is worth noting that the Dufour effects are prominent in gases. In liquids, its value is negligible, smaller than the Soret term by one order. Now we shall study the influences of the Soret effect on the double-diffusive convection in the absence of Dufour effects or with its constant value. In both cases, we observe the same result. The critical value of the Darcy–Rayleigh number for stationary convection and the condition for oscillatory convection with the frequency of the periodic convection depend on the Soret coefficient when the Dufour parameter is held constant, while the value of critical Darcy–Rayleigh number for oscillatory convection does not depend on the Soret or Dufour coefficient. Now in the presence of solute ($\mathrm{Ra_{D,S}} \neq 0$), the critical value of the Darcy–Rayleigh number for stationary convection decreases when the Soret number increases, but less than 1, which is observed in Fig. 2. Hence, the Soret effect serves to advance the onset of convection, i.e., it destabilizes the flow. For the Soret number greater than or equal to 1, the oscillatory convection is not observed, as follows from the relation (36) in which ω is always real. Also for the Soret number greater than or equal to 1, the stationary convection does not prevail. As the Soret number increases, the region of stability decreases in the $\mathrm{Ra_D}$–$\mathrm{Ra_{D,S}}$ plane, which is justified in Fig. 3. Thus, the Soret effect destabilizes the flow, so that the convection sets in earlier.

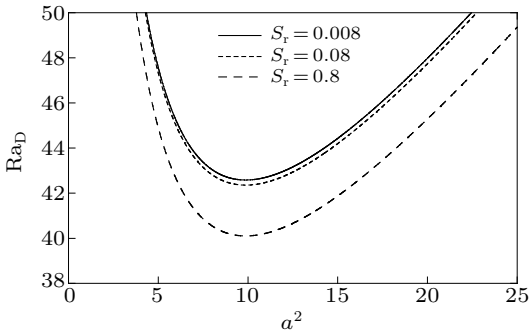


Fig. 2. Variations of Darcy–Rayleigh number with the square of the wave number for different values of the Soret number in stationary convection with $\mathrm{Le} = 100$, $\mathrm{Ra_{D,S}} = 1.0$, $D_{\mathrm{f}} = 0.0005$, and $\Phi = 10$.

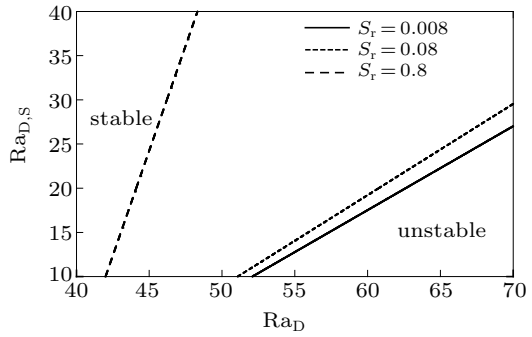


Fig. 3. Effects of the Soret number on the boundary of monotonic or stationary instability in the $\mathrm{Ra_D}$–$\mathrm{Ra_{D,S}}$ plane with $\mathrm{Le} = 100$, and $D_{\mathrm{f}} = 0.0005$.

Next, we shall study the influences of Dufour effects on the double-diffusive convection. In this case, we also observe the same result in the absence of Soret parameter or with its fixed value. It is observed in Fig. 4 that the critical value of the Darcy–Rayleigh number for stationary convection will increase when the Dufour number increases, provided that the value of Dufour number $D_{\mathrm{f}} < (1/\mathrm{Le})$ depends on the Lewis number, so that the Dufour effect serves to delay the onset of convection, i.e., it stabilizes the flow. As the Dufour number increases, the region of stability increases in $\mathrm{Ra_D}$–$\mathrm{Ra_{D,S}}$ plane, which is justified in Fig. 5. Hence, the Dufour effect tends to stabilize the double-diffusive system, resulting in the delay of convection.

The critical value of the Darcy–Rayleigh number for the oscillatory convection at the marginal state depends on the porosity parameter Φ, as shown by the relation (43). Figure 6 exhibits that the oscillatory convection prevails earlier with higher value of Φ. Hence, the increasing porosity parameter has tendency to destabilize the fluid flow. It is also to be noted that the frequency of oscillation decreases for increasing Soret number, while increases with increasing Dufour number, as depicted in Figs. 7 and 8, respectively.

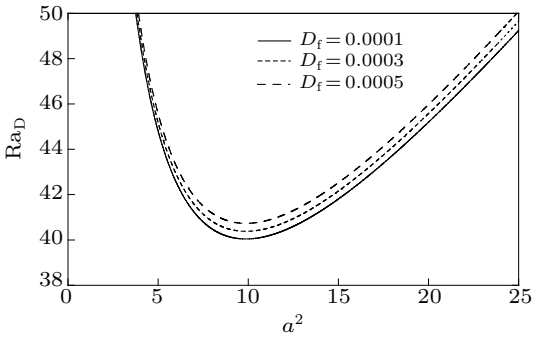


Fig. 4. Variations of the Darcy–Rayleigh number with the square of the wave number for different values of the Dufour number in stationary convection with Le = 100, $\mathrm{Ra_{D,S}} = 1.0$, $S_r = 0.6$, and $\Phi = 10.0$.

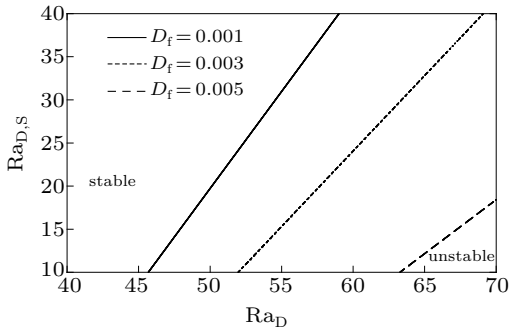


Fig. 5. Effects of the Dufour number on the boundary of monotonic or stationary instability in the $\mathrm{Ra_D}$–$\mathrm{Ra_{D,S}}$ plane with Le = 100, and $S_r = 0.6$.

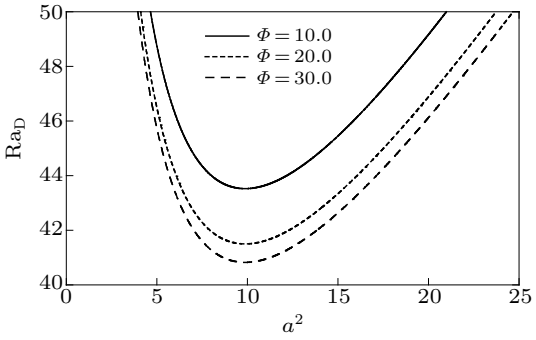


Fig. 6. Variations of the Darcy–Rayleigh number with the square of the wave number for different values of the porosity parameter Φ in oscillatory convection with Le = 100, and $\mathrm{Ra_{D,S}} = 1.0$.

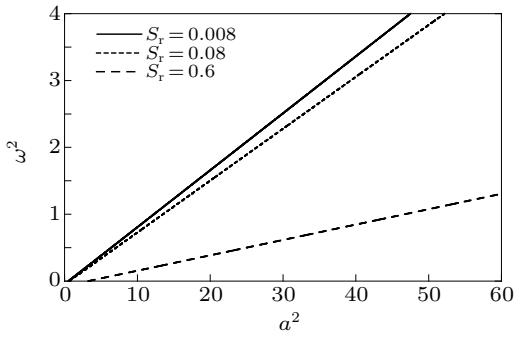


Fig. 7. Variations of the square of frequency of the periodic convection with the square of the wave number for different values of the Soret number with Le = 100, $\mathrm{Ra_{D,S}} = 1.0$, $D_f = 0.0005$, and $\Phi = 10.0$.

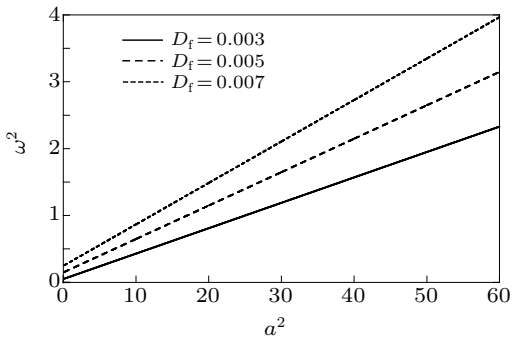


Fig. 8. Variations of the square of the frequency of the periodic convection with the square of the wave number for different values of the Dufour number with Le = 100, $\mathrm{Ra_{D,S}} = 1.0$, $S_r = 0.6$, and $\Phi = 10.0$.

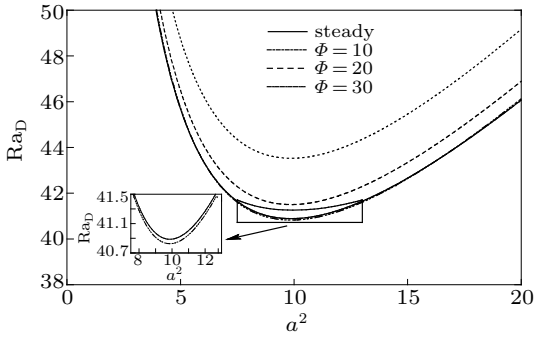


Fig. 9. Variations of the Rayleigh number with the square of the wave number for steady and oscillatory convections with Le = 100, $\mathrm{Ra_{D,S}} = 1.0$, $S_r = 0.0$, and $D_f = 0.0001$.

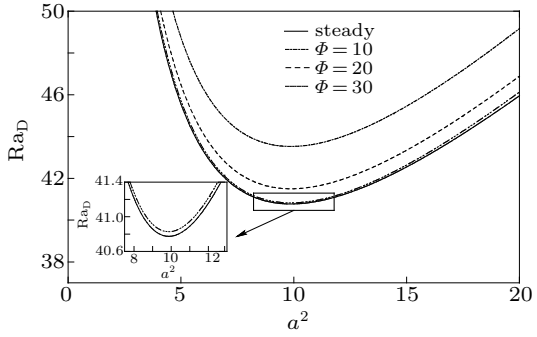


Fig. 10. Variations of Rayleigh number with the square of the wave number for steady and oscillatory convections with Le = 100, $\mathrm{Ra_{D,S}} = 1.0$, $S_r = 0.08$, and $D_f = 0.0001$.

It is not possible to conclude analytically the type of instability at the onset when both the cross-diffusion terms are present in the relations (34) and (38). We investigate the nature of instability at the onset numerically with the help of graphs. It is observed from Fig. 9 that the entire curve representing the oscillatory convection lies below the curve representing the steady convection for $S_r = 0.0$, $D_f = 0.0001$, $\Phi = 30$, and thus $\mathrm{Ra}_{D_{cr}}^{(osc)} < \mathrm{Ra}_{D_{cr}}^{(steady)}$. Hence, for these values of the Soret, Dufour and porosity parameters, the onset of instability is manifested by oscillatory convection. However, in Fig. 10, with $S_r = 0.08$, $D_f = 0.0001$, and $\Phi = 10$, 20, 30, it is observed that the entire curve representing the steady convection lies below the curves representing the oscillatory convection, so the steady convection prevails at the onset. It is observed in

Fig. 11 that with $S_r = 0.08$, $D_f = 0.0005$, and $\Phi = 20$, 30, the onset is manifested by the oscillatory convection. It is shown in Fig. 12 that for $S_r = 0.6$, $D_f = 0.0005$, and $\Phi = 10$, 20, 30, the steady convection prevails at the onset. Thus, analyzing the graphs in Figs. 9–12, we find that the increasing Dufour number with higher value of the porosity parameter provokes oscillatory instability at the onset, whereas the increasing Soret number instigates stationary instability at the onset. Hence, the type of instability at the onset is influenced by the Soret and Dufour effects along with the porosity parameter.

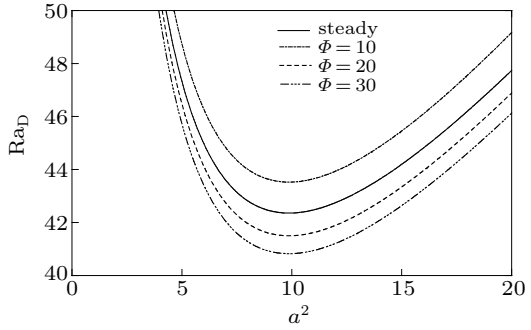


Fig. 11. Variations of the Rayleigh number with the square of the wave number for steady and oscillatory convection with Le = 100, $\mathrm{Ra_{D,S}} = 1.0$, $S_r = 0.08$, and $D_f = 0.0005$.

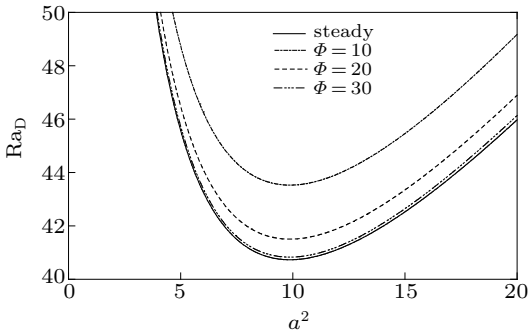


Fig. 12. Variations of the Rayleigh number with the square of the wave number for steady and oscillatory convections with Le = 100, $\mathrm{Ra_{D,S}} = 1.0$, $S_r = 0.6$, and $D_f = 0.0005$.

6. Conclusions

Linear stability analysis for the onset of convection is employed to investigate the effects of cross-diffusion terms on the double-diffusive convection in a horizontal viscous porous fluid layer heated and salted from above with stress-free boundaries. The principal aim of this work is to find the critical value of the Darcy–Rayleigh number at the onset of double-diffusive convection taking into account Soret and Dufour effects and their roles on the onset of instabilities. It was found that in the case of stationary instability, the critical values of the Darcy–Rayleigh number will decrease when the Soret number increases, so the Soret parameter has a destabilizing effect. On the other hand, it will increase as the Dufour number increases, so the Dufour parameter has a stabilizing effect in this double-diffusive system. The critical value of Darcy–Rayleigh number at the onset of the oscillatory convection does not depend on the cross-diffusion coefficients, but depends on the porosity parameter. The porosity parameter has a destabilizing effect on the oscillatory convection. The effects of cross-diffusion parameters on the frequency of the oscillatory convection are also obtained. The analysis also reveals that the type of instability at the onset of double-diffusive convection is influenced by the cross-diffusion terms and porosity parameter.

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