Consensus of second-order multi-agent systems with nonuniform time delays*

Zhang Wen-Guang(张文广)[a)b)†], Liu Ji-Zhen(刘吉臻)[a)], Zeng De-Liang(曾德良)[a)], and Hu Yong(胡 勇)[b)]

[a)] State Key Laboratory of Alternate Electrical Power System with Renewable Energy Sources, North China Electric Power University, Beijing 102206, China
[b)] School of Control & Computer Engineering, North China Electric Power University, Beijing 102206, China

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In this study, the consensus problem for a class of second-order multi-agent systems with nonuniform time delays is investigated. A linear consensus protocol is used to make all agents reach consensus and move with a constant velocity. By a frequency-domain analysis, a simplified sufficient condition is given to guarantee the consensus stability of the dynamic system. Finally, the effectiveness of the obtained theoretical results is illustrated through numerical simulations.

Keywords: consensus, nonuniform time delay, multi-agent system, constant velocity

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1. Introduction

Distributed coordination control of multi-agent systems has attracted a great number of researchers with different backgrounds, such as physics, biology, control theory, robotics, and computer science. One of the fundamental problems in this area is called a consensus control problem, i.e., to design appropriate network protocols such that all agents asymptotically reach an agreement.

In the past few years, numerous results have been obtained for distributed consensus problems of multi-agent systems.[1–20] In Ref. [1], Jadbabaie *et al.* provided a theoretical explanation for the observed behavior reported in Ref. [2] and analyzed the alignment of an undirected network of agents with switching topologies that are periodically connected. In Refs. [3]–[8], some protocols were presented for multi-agent systems, which make all agents track a leader or reach consensus on a static value. Protocols were introduced in Refs. [9]–[11], which enable agents to reach consensus and move together with a constant velocity in the absence and presence of time delay. Bliman *et al.*[12] and Tian *et al.*[13,14] used the frequency-domain analysis approach to investigate nonuniform delays for first-order multi-agent systems with fixed topology. Moreover, consensus protocols for first- and second-order multi-agent systems were formally introduced in Refs. [15]–[20], in which consensus problems were addressed with time delays. Furthermore, in Refs. [17]–[20], consensus protocols were discussed not only with time delays, but also with dynamically switching topologies.

Motivated by the above review, we study the consensus problem for second-order multi-agent systems with nonuniform time delays in this paper. Firstly, the original system is changed into an equivalent system by transforming the model. Then, based on this equivalent system, a frequency-domain method is employed to give a sufficient condition which can guarantee the consensus stability of the system. In Ref. [11], a sufficient and necessary condition is given for the case of all equal time delays. However, we investigate the nonuniform time delay case, which is more general and simple.

The rest of this paper is organized as follows. In Section 2, some basic notations and some concepts in graph theory are introduced. In Section 3, the second-order model concerned is formulated and a consensus protocol is proposed. The main results are shown in Section 4. In Section 5, numerical simulations are provided to demonstrate the effectiveness of the obtained theoretical results. We conclude this paper in Section 6.

2. Notations and preliminaries

The following notations will be used throughout this paper. The $\mathbb{R}^m$ denotes the set of all m-dimensional real column vectors; $\mathbb{C}^m$ denotes the set of all m-dimensional complex column vectors; I_m denotes the m-dimensional unit matrix; $\mathbf{1}$ represents $[1,1,\ldots,1]^{\mathrm{T}}$ with compatible dimensions (sometimes, $\mathbf{1}_n$ is used to denote $\mathbf{1}$ with dimension n); 0 denotes zero value or zero matrix with appropriate dimensions; the symbol $*$ denotes the symmetric term of a symmetric matrix.

Let $G(\mathscr{V},\varepsilon,\mathscr{A})$ be a weighted undirected graph of order n, where $\mathscr{V}=[v_1,\ldots,v_n]$ is the set of nodes with node i representing the i-th agent, $\varepsilon\subseteq\mathscr{V}\times\mathscr{V}$ is the set of edges, and $\mathscr{A}=[a_{ij}]\in\mathbb{R}^{n\times n}$ is a symmetric weighted adjacency matrix. The node indexes belong to a finite index set $\mathscr{I}=[1,2,\ldots,n]$. An directed edge of G is denoted by $e_{ij}=(v_i,v_j)$, where the

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†Corresponding author. E-mail: zwgbuaa@126.com

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first element v_i of the e_{ij} is said to be the tail of the edge and v_j to be the head. The symmetric adjacency matrix is defined as $a_{ii}=0$ and $a_{ij}=a_{ji}\geq 0$. $a_{ij}>0$ if and only if there exists an edge between node v_i and node v_j. The set of neighbors of node v_i is denoted by $N_i=\{v_j\in\mathscr{V}:(v_i,v_j)\in\varepsilon\}$. The in-degree and out-degree of node v_i are defined respectively as $\deg_{\text{in}}(v_i)=\sum_{j=1}^{n}a_{ji}$ and $\deg_{\text{out}}(v_i)=\sum_{j=1}^{n}a_{ij}$. Then, the Laplacian matrix corresponding to the weighted undirected graph is defined as $\boldsymbol{L}=[l_{ij}]\in\mathbb{R}^{n\times n}$, where $l_{ii}=\deg_{\text{out}}(v_i)$ and $l_{ij}=-a_{ij}$ $(i\neq j)$. Obviously, $\boldsymbol{L}=\boldsymbol{L}^{\text{T}}$ for any undirected graph. A path is a sequence of ordered edges of the form $(v_{i_1},v_{i_2}),(v_{i_2},v_{i_3}),\ldots$, where $v_{i_j}\in\mathscr{V}$. If there exists a path from every node to every other node, the graph is said to be connected. The G is called a balanced digraph, if $\deg_{\text{in}}(v_i)=\deg_{\text{out}}(v_i)$ $(i=1,2,\ldots,n)$. The G is called an undirected graph, if $\mathscr{A}$ is a symmetric matrix.

Lemma 1[21] If the graph G is connected, then its Laplacian $\boldsymbol{L}$ satisfies: (i) zero is a simple eigenvalue of $\boldsymbol{L}$, and $\boldsymbol{1}$ is the corresponding eigenvector, i.e., $\boldsymbol{L}\boldsymbol{1}=0$ and $\boldsymbol{1}^{\text{T}}\boldsymbol{L}=0$; (ii) the rest $(n-1)$ eigenvalues are all positive and real numbers.

3. Problem formulation

Suppose that the multi-agent system under consideration consists of n agents. Each agent is regarded as a node in an undirected graph $G_{\sigma(t)}$, where $\sigma(t):[0,+\infty)\to P=\{1,2,\ldots,N\}$ ($N\in\mathbb{Z}^+$ denotes the total number of all possible graphs) is a switching signal that determines the communication topology, and the time interval between every two switching times is larger than a specified constant T. Each edge $(v_j,v_i)\in\varepsilon(G_{\sigma(t)})$ or $(v_i,v_j)\in\varepsilon(G_{\sigma(t)})$ corresponds to an available information channel between agent v_i and agent v_j at time t. The set of the neighbors of the i-th agent associated to the graph $G_{\sigma(t)}$ is denoted by $N_{\sigma(t),i}$. The Laplacian of the graph $G_{\sigma(t)}$ is denoted by $\boldsymbol{L}_{\sigma(t)}$. Moreover, each agent updates its current state based on the information received from its neighbors. To simplify the notation, $\sigma(t)$ will be written as σ.

Precisely, the i-th second-order agent dynamics at time t is given as follows:

$$\dot{x}_i(t)=v_i(t),\quad \dot{v}_i(t)=u_i(t), \tag{1}$$

where $x_i(t)\in\mathbb{R}$ is the position state with the initial condition $x_i(s)=x_i(0)$ $(s\in(-\infty,0])$, $v_i(t)\in\mathbb{R}$ is the velocity state, and $u_i(t)\in\mathbb{R}$ is the control input at time t $(i=1,2,\ldots,n)$.

The control input (or protocol) $u_i(t)$ can solve the consensus problem asymptotically, if and only if the states of agents satisfy

$$\lim_{t\to+\infty}[x_i(t)-x_j(t)]=0,$$

$$\lim_{t\to+\infty}[v_i(t)-v_j(t)]=0,\quad (i,j\in\mathscr{I}).$$

In this paper, we consider the consensus problem for networks of agents with nonuniform time delays. To solve this problem, we propose the following linear consensus protocol:

$$\begin{aligned}u_i(t)\;=\;&-kv_i(t)+\sum_{v_j\in N_{\sigma,i}}a_{ij}(t)\\&\times\left(x_j(t-\tau_{ij}(t))-x_i(t-\tau_{ij}(t))\right)\\&+\frac{2}{k}\sum_{v_j\in N_{\sigma,i}}a_{ij}(t)\left(v_j(t-\tau_{ij}(t))-v_i(t-\tau_{ij}(t))\right),\end{aligned}\tag{2}$$

where $k>0$ denotes the feedback gains, $\tau_{ij}(t)=\tau_{ji}(t)$ is the time-varying delay for the i-th agent to gain the state information of the j-th agent. Suppose that the system has M different time delays, denoted by $\tau_m(t)\in\{\tau_{ij}(t),i,j\in\mathscr{I}\}$ $(m=1,2,\ldots,M)$.

Let

$$\begin{aligned}&\bar{v}_i(t)=\frac{2}{k}v_i(t)+x_i(t),\\&\boldsymbol{\phi}(t)=[x_1(t),\bar{v}_1(t),\ldots,x_n(t),\bar{v}_n(t)]^{\text{T}},\\&\boldsymbol{A}=\begin{bmatrix}-\frac{k}{2}&\frac{k}{2}\\\frac{k}{2}&-\frac{k}{2}\end{bmatrix},\quad \boldsymbol{B}=\begin{bmatrix}0&0\\0&\frac{2}{k}\end{bmatrix}.\end{aligned}$$

Using consensus protocol (2), we summarize the network dynamics as

$$\begin{aligned}\dot{\boldsymbol{\phi}}(t)\;=\;&(\boldsymbol{I}_n\otimes\boldsymbol{A})\boldsymbol{\phi}(t)\\&-\sum_{m=1}^{M}(\boldsymbol{L}_{\sigma m}\otimes\boldsymbol{B})\boldsymbol{\phi}(t-\tau_m(t)),\end{aligned}\tag{3}$$

with the initial condition $\boldsymbol{\phi}(s)=\boldsymbol{\phi}(0)$, $s\in(-\infty,0]$, where $(\boldsymbol{L}_{\sigma m}\otimes\boldsymbol{B})$ is the coefficient matrix of the variable $\boldsymbol{\phi}(t-\tau_m(t))$ $(m=1,2,\ldots,M)$. Clearly, $\boldsymbol{L}_\sigma=\sum_{m=1}^{M}\boldsymbol{L}_{\sigma m}$.

If all time delays are equal to zero, then system (3) can be rewritten as

$$\dot{\boldsymbol{\phi}}(t)=(\boldsymbol{I}_n\otimes\boldsymbol{A}-\boldsymbol{L}_\sigma\otimes\boldsymbol{B})\boldsymbol{\phi}(t). \tag{4}$$

4. Analysis and results

In this section, we will provide the convergence analysis of the consensus problem for second-order multi-agent systems with nonuniform time delays.

Lemma 2 Suppose that the communication graph is fixed and connected. The matrix $(\boldsymbol{I}_n\otimes\boldsymbol{A}-\boldsymbol{L}_\sigma\otimes\boldsymbol{B})$ has a simple zero eigenvalue, and all other eigenvalues have negative real parts.

Proof From Lemma 1, $\boldsymbol{L}_\sigma=\boldsymbol{L}_\sigma^{\text{T}}$, so there is an orthogonal matrix $\boldsymbol{W}$, such that $\boldsymbol{W}^{\text{T}}\boldsymbol{L}_\sigma\boldsymbol{W}=\text{diag}\{0,\lambda_2,\ldots,\lambda_n\}$, where 0 and $\lambda_j>0$ $(j=2,3,\ldots,n)$ are the eigenvalues of $\boldsymbol{L}_\sigma$. It follows that

$$\begin{aligned}&(\boldsymbol{W}\otimes\boldsymbol{I}_2)^{\text{T}}(\boldsymbol{I}_n\otimes\boldsymbol{A}-\boldsymbol{L}_\sigma\otimes\boldsymbol{B})(\boldsymbol{W}\otimes\boldsymbol{I}_2)\\=\;&\text{diag}\{\boldsymbol{A},\boldsymbol{A}-\lambda_2\boldsymbol{B},\ldots,\boldsymbol{A}-\lambda_n\boldsymbol{B}\}.\end{aligned}\tag{5}$$

Through simple calculations, we can see that the eigenvalues of $\boldsymbol{A}$ are 0, and the eigenvalues of $(\boldsymbol{A}-\lambda_j\boldsymbol{B})$ have negative real parts. Hence, the matrix $(\boldsymbol{I}_n\otimes\boldsymbol{A}-\boldsymbol{L}_\sigma\otimes\boldsymbol{B})$ has a simple zero eigenvalue, and all its other eigenvalues have negative real parts.

Theorem 1 Consider the network of multi-agents with nonuniform time delays. Suppose that the communication graph is fixed and connected. If $0\le\tau_m<k\pi/(2\lambda_{\max})$, the protocol (2) can solve the consensus problem of the system (3).

Proof Consider the system (3) in the frequency domain. Let $\phi(s)=\boldsymbol{G}_\tau^{-1}(s)\phi(0)$, where $\boldsymbol{G}_\tau(s)=s\boldsymbol{I}_{2n}-\boldsymbol{I}_n\otimes\boldsymbol{A}+\sum_{m=1}^{M}(\boldsymbol{L}_{\sigma m}\otimes\boldsymbol{B})\,\mathrm{e}^{-\tau_m s}$. Suppose that $s=jw\neq 0$ is an imaginary root of $\boldsymbol{G}_\tau(s)$, and $\boldsymbol{u}=\boldsymbol{u}_1\otimes[1,0]^{\mathrm{T}}+\boldsymbol{u}_2\otimes[0,1]^{\mathrm{T}}$ is a corresponding eigenvector, where $\|\boldsymbol{u}\|=1$, and $\boldsymbol{u}_1,\boldsymbol{u}_2\in\mathbb{C}^n$.

To study the stability of the system (3), we only need to consider the values of time delays to guarantee $\boldsymbol{G}_\tau(s)$ has nonzero roots on imaginary axis. Then, we have

$$\left[\mathrm{j}w\boldsymbol{I}_{2n}-\boldsymbol{I}_n\otimes\boldsymbol{A}+\sum_{m=1}^{M}(\boldsymbol{L}_{\sigma m}\otimes\boldsymbol{B})\,\mathrm{e}^{-\tau_m\mathrm{j}w}\right]\boldsymbol{u}=0, \tag{6}$$

where u is the eigenvector associated with the eigenvalue $\mathrm{j}w$.

Let $\alpha_m\doteq\boldsymbol{u}^*L_{\sigma m}\boldsymbol{u}/(\boldsymbol{u}^*\boldsymbol{u})$, then it follows that

$$\begin{aligned}&w\boldsymbol{u}_1^*\boldsymbol{u}_1+w\boldsymbol{u}_2^*\boldsymbol{u}_2\\&+\frac{2}{k}\sum_{m=1}^{M}\alpha_m\boldsymbol{u}_2^*\boldsymbol{u}_2(-\sin w\tau_m)=0,\\&\frac{k}{2}(\boldsymbol{u}_1^*\boldsymbol{u}_1+\boldsymbol{u}_2^*\boldsymbol{u}_2)-\frac{k}{2}(\boldsymbol{u}_1^*\boldsymbol{u}_2+\boldsymbol{u}_2^*\boldsymbol{u}_1)\\&+\frac{2}{k}\sum_{m=1}^{M}\alpha_m\boldsymbol{u}_2^*\boldsymbol{u}_2\cos w\tau_m=0.\end{aligned} \tag{7}$$

Thus,

$$\begin{aligned}&\sum_{m=1}^{M}\alpha_m\boldsymbol{u}_2^*\boldsymbol{u}_2\sin w\tau_m=\frac{k}{2}w,\\&\sum_{m=1}^{M}\alpha_m\boldsymbol{u}_2^*\boldsymbol{u}_2\cos w\tau_m\\&=\left(\frac{k}{2}\right)^2(\boldsymbol{u}_1^*\boldsymbol{u}_2+\boldsymbol{u}_2^*\boldsymbol{u}_1-\boldsymbol{u}_1^*\boldsymbol{u}_1-\boldsymbol{u}_2^*\boldsymbol{u}_2)\le 0.\end{aligned} \tag{8}$$

It is easy to see that

$$|w|\le\frac{2}{k}\sum_{m=1}^{M}\alpha_m=\frac{2}{k}\frac{\boldsymbol{u}^*\boldsymbol{L}_\sigma\boldsymbol{u}}{\boldsymbol{u}^*\boldsymbol{u}}\le\frac{2}{k}\lambda_{\max},$$

where $\lambda_{\max}$ is the largest eigenvalue of L_σ.

By adjusting τ_m to satisfy $w\tau_m\le\pi/2$ (for $m=1,2,\ldots,M$), we have

$$0\le\sum_{m=1}^{M}\alpha_m\boldsymbol{u}_2^*\boldsymbol{u}_2\cos w\tau_m\le 0.$$

Then, it is easy to see that $w\tau_m=\pi/2$, namely $\tau_m=\pi/(2w)$, and

$$|w|=\frac{2}{k}\sum_{m=1}^{M}\alpha_m\boldsymbol{u}_2^*\boldsymbol{u}_2=\frac{1}{k}\frac{\boldsymbol{u}^*L_\sigma\boldsymbol{u}}{\boldsymbol{u}^*\boldsymbol{u}}\le\frac{1}{k}\lambda_{\max}.$$

Therefore, if $0<\tau_m<k\pi/(2\lambda_{\max})$, there is no nonzero imaginary eigenvalue in the system.

From Lemma 2, when $\tau_m=0$ for all $m=1,2,\ldots,M$, the system (3) has a simple zero eigenvalue and all other eigenvalues have negative real parts. From the continuity of function, the real part of each eigenvalue of the system does not change the sign when τ_m varies in the interval $[0,k\pi/(2\lambda_{\max}))$. Moreover, it is easily obtained that the eigenvector associated with zero eigenvalue are $\mathbf{1}_{2n}$ from system (6) with $w=0$. Thus, the system (3) finally converges to the space spanned by the vectors $\mathbf{1}_{2n}$. Namely, $\lim_{t\to+\infty}[\phi_i(t)-\phi_j(t)]=0$. Thus,

$$\lim_{t\to+\infty}[x_i(t)-x_j(t)]=0,\quad\lim_{t\to+\infty}[v_i(t)-v_j(t)]=0,$$

for all i and j from the definition of $\phi(t)$. We complete the proof of the theorem.

In the protocol (2), the local feedback velocity gain and distributed velocity feedback gain have the same form of k, which makes the protocol not have a general form. The future work will consider the general form of the protocol (2) to eliminate such a constraint. On the other hand, the protocol (2) contains the velocity feedback damping part that makes all agents' velocity finally vanish. The next work will be directed to the case where the velocity feedback damping part is removed in the protocol (2), which is more complicated to deal with following the line of the proof of Theorem 1.

5. Simulations

In this section, numerical simulations are given to demonstrate the effectiveness of the theoretical results obtained. The undirected graph with 4 nodes is shown in Fig. 1. The graph is connected and the weight of each edge is 1. The parameter k is set as 1.

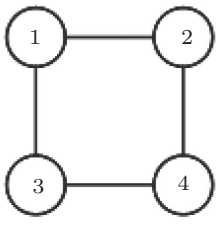


Fig. 1. The undirected graph with 4 nodes.

From Theorem 1, it can be derived that when $\tau_{\max}\le 0.7854$, the system (3) reaches the consensus. Figures 2 and 3 show the corresponding simulation results with $\tau_{12}=\tau_{21}=\tau_{13}=\tau_{31}=0.1$ and $\tau_{24}=\tau_{42}=\tau_{34}=\tau_{43}=0.2$. It is clear

that all agents finally reach consensus, which is consistent with Theorem 1.

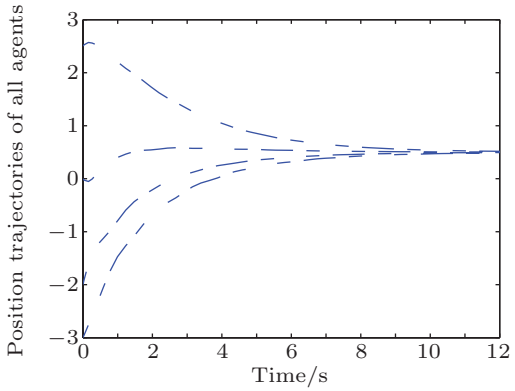


Fig. 2. (color online) Position trajectories of all agents. From top to bottom: agent 1, agent 2, agent 3, and agent 4.

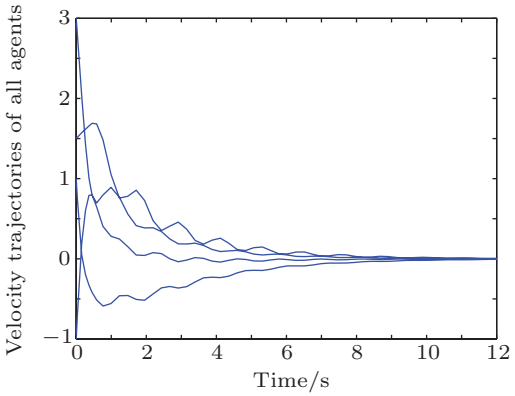


Fig. 3. (color online) Velocity trajectories of all agents.

6. Conclusion

In this paper, we present a linear consensus protocol with nonuniform time delays about the consensus problem of a class of second-order multi-agent systems, which makes all agents reach consensus and move with a constant velocity. Transforming the original system into an equivalent system, then basing on this equivalent system, we employ a frequency-domain approach to give a sufficient condition simplified to guarantee the consensus stability of the system. Finally, the effectiveness of the obtained theoretical results is illustrated through numerical simulations.

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