New exact solutions of a (3+1)-dimensional Jimbo–Miwa system*

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By using the (G'/G)-expansion method and the variable separation method, a new family of exact solutions of the (3+1)-dimensional Jimbo–Miwa system is obtained. Based on the derived solitary wave solutions, we obtain some special localized excitations and study the interactions between two solitary waves of the system.

Keywords: (3+1)-dimensional Jimbo–Miwa system, (G'/G)-expansion method, exact solutions, interactions between two solitary waves

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1. Introduction

In the past few decades, the construction of exact solutions to nonlinear partial differential equations (NLPDEs) has made a tremendous amount of progress in mathematics and physics.[1–4] According to the previous studies, a vast variety of methods have been developed to obtain the exact solutions to a given NLPDE, such as the Lie group method of infinitesimal transformations, the Clarkson and Kruskal (CK) direct method,[5,6] the conditional similarity reduction method,[7–9] the improved mapping approach,[10–16] the Hirota method,[17] and the (G'/G)-expansion method.[18–25] The (G'/G)-expansion method is often used in finding the exact traveling wave solutions of nonlinear evolution equations.[18–22] In this paper, the (G'/G)-expansion method is used to obtain the exact non-traveling wave solutions of (3+1)-dimensional Jimbo–Miwa system.

The following (3+1)-dimensional Jimbo–Miwa (JM) system:

$$u_{xxxy}+3u_{xx}u_y+3u_xu_{xy}+2u_{yt}-3u_{xz}=0, \tag{1}$$

has been studied by means of the extended mapping approach.[26–28] It has been shown that the (3+1)-dimensional JM Eq. (1) introduced by Jimbo and Miwa[29] does not have the Painlevé property[30] defined by Weiss *et al.*[31] Equation (1) has also been discussed via an extended homogeneous balance method by Bai *et al.*[32]

The organization of this paper is as follows. In Section 2, the (G'/G)-expansion method which simplifies the (3+1)-dimensional JM equation is introduced. In Section 3, we will study special soliton-structure excitations (one of the new exact solutions) by selecting appropriately arbitrary functions in the solutions. The conclusions will be given in Section 4.

2. The (G'/G)-expansion method and new exact solutions

Before using the (G'/G)-expansion method, we will give a simple description of the method. For doing so, we consider a four-variable general form of nonlinear PDE

$$F(u,u_t,u_x,u_y,u_z,u_{tt},u_{xt},u_{yt},u_{tz},u_{xy},u_{xx},u_{yy},u_{zz},\cdots)=0. \tag{2}$$

The fundamental idea of the (G'/G)-expansion method is that the solution of Eq. (2) can be expressed by a polynomial of (G'/G) as follows:[19,23–25]

$$u=\sum_{i=1}^{m}a_i\left(\frac{G'}{G}\right)^i+a_0, \tag{3}$$

where $G=G(\xi)$, $\xi=\alpha x+\beta y+\gamma z+\omega t$ is the traveling wave transformation, and $\alpha,\beta,\gamma,\omega,a_i$ $(i=1,2,\ldots,m)$ are constants to be determined later. The G satisfies the second-order linear ordinary differential equation (LODE)

$$G''+\lambda G'+\mu G=0. \tag{4}$$

In order to construct exact solutions with arbitrary function $\xi(x,y,z,t)$ for the (3+1)-dimensional JM system (1), we suppose that its solutions can be express as

$$u=\sum_{i=0}^{m}a_i\left(\frac{G'}{G}\right)^i, \tag{5}$$

where a_i $(i=0,1,2,\ldots,m)$ are functions of x,y,z,t to be determined later, $\xi=\xi(x,y,z,t)$ is an arbitrary function of x,y,z,t, and $G(\xi)$ satisfies the second-order LODE (4).

Applying the homogenous balance principle, we obtain $m=1$. Thus, equation (5) can be converted into

$$u=a_0+a_1\left(\frac{G'}{G}\right). \tag{6}$$

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For simplifying the computation, we seek for the variable separation solutions of the JM system (1) by taking $\xi(x,y,z,t) = f(x+cz)+g(y)+h(t)$.

Substituting Eq. (6) into system (1) yields a set of algebra for a_0, a_1, f, g, and h. These systems are

$$\left(\frac{G'}{G}\right)^5 : a_1 + 2f_x = 0, \tag{7}$$

$$\begin{aligned}\left(\frac{G'}{G}\right)^4 :\ & 20\lambda a_1 f_x^3 g' + 5a_1 a_{1x} f_x g' + a_1^2 f_{xx} g' - 2a_{1y} f_x^3 \\ & - 10\lambda a_1^2 f_x^2 g' - 6a_{1x} f_x^2 g' - 6a_1 f_x f_{xx} g' \\ & + 3a_1 a_{1y} f_x^2 = 0,\end{aligned} \tag{8}$$

$$\begin{aligned}\left(\frac{G'}{G}\right)^3 :\ & 6a_{0y} a_1 f_x^2 - 3a_{1x}^2 g' - 12\lambda a_{1y} f_x^3 \\ & - 6ca_1 f_x f_z + 4a_1 g' h' + 30\lambda^2 a_1 f_x^3 g' \\ & - 30\mu a_1^2 f_x g' + 60\mu a_1 f_x^3 g' - 36\lambda a_{1x} f_x^2 g' \\ & - 6(\lambda^2 - \mu) a_1^2 f_x^2 g' + 20(\lambda^2 - \mu) a_1 f_x^3 g' \\ & + 15\lambda a_1 a_{1y} f_x^2 + 6a_{0x} a_1 f_x g' - 18\lambda^2 a_1^2 f_x^2 g' \\ & + 27\lambda a_1 a_{1x} f_x g' + 6a_{1y} f_x^2 - 9a_{1x} a_{1y} f_x - 3a_1 a_{1xx} g' \\ & + 2a_1 f_{xxx} g' - 3a_1 a_{1xy} f_x + 6a_{1y} f_x f_{xx} - 3a_1 a_{1y} f_{xx} \\ & + 6a_{1xx} f_x g' + 6\lambda a_1^2 f_{xx} g' - 36\lambda a_1 f_x f_{xx} g' \\ & + 6a_{1x} f_{xx} g' = 0,\end{aligned} \tag{9}$$

$$\begin{aligned}\left(\frac{G'}{G}\right)^2 :\ & -6a_{0y} a_{1x} f_x - 3a_{0xy} a_1 f_x - 3a_{1xy} f_{xx} - 3a_{1xxy} f_x \\ & - 2a_{1y} h' + 9\lambda a_{0x} a_1 f_x g' + 3a_{1z} f_x - 2a_{1t} g' \\ & - 3\lambda^2 a_{1y} f_x^3 - 4(\lambda^2 - \mu) a_{1y} f_x^3 + 15\mu a_1 a_{1y} f_x^2 \\ & - 9\lambda^2 a_{1x} f_x^2 g' - 12(\lambda^2 - \mu) a_{1x} f_x^2 g' + 3\lambda^2 a_1 a_{1y} f_x \\ & - 36\mu a_{1x} f_x^2 g' + 9\lambda a_{0y} a_1 f_x^2 + 3(\lambda^2 - \mu) a_1 a_{1y} f_x^2 \\ & - 5(2\lambda\mu - \lambda^3) a_1 f_x^3 g' - 9\lambda a_{1x} a_{1y} f_x - 9c\lambda a_1 f_x f_z \\ & + 80\lambda\mu a_1 f_x^3 g' + 10\lambda(\lambda^2 - \mu) a_1 f_x^3 g' + 6\lambda a_1 g' h' \\ & + 9\lambda^2 a_1 a_{1x} f_x g' - 6\lambda(\lambda^2 - \mu) a_1^2 f_x^2 g' + 42\lambda\mu a_1^2 f_x^2 g' \\ & + 3ca_{1x} f_z + 3(\lambda^2 - \mu) a_1 a_{1x} f_x g' - 3a_{0x} a_{1y} f_x \\ & + 27\mu a_1 a_{1x} f_x g' - 3a_{0x} a_{1x} g' - 12\mu a_{1y} f_x^3 - 3a_{1xx} a_{1y} \\ & - 3a_{0y} a_1 f_{xx} + 9\lambda a_{1xy} f_x - 3a_{0xx} a_1 g' - a_{1y} f_{xxx} \\ & + 3ca_1 f_{xz} - 3a_{1xxx} g' + 3a_{1x} a_{1xy} - 9\lambda^2 a_1 f_x f_{xx} g' \\ & - 36\mu a_1 f_x f_{xx} g' + 3\lambda a_1 f_{xxx} g' + 9\lambda a_{1y} f_x f_{xx} \\ & + 9\lambda a_{1xx} f_x g' + 6\mu a_1^2 f_{xx} g' + 12(\lambda^2 - \mu) a_1 f_x f_{xx} g' \\ & - 3\lambda a_1 a_{1xy} f_x - 3\lambda a_1 a_{1xx} g' + 3\lambda^2 a_1^2 f_{xx} g' \\ & - 3\lambda a_1 a_{1y} f_{xx} + 9\lambda a_1 f_{xx} g' = 0,\end{aligned} \tag{10}$$

$$\begin{aligned}\left(\frac{G'}{G}\right)^1 :\ & -2\lambda a_{1y} h' + 9\mu a_{0y} a_1 f_x^2 - 10\lambda\mu a_{1y} f_x^3 \\ & + 30\mu^2 a_1 f_x^3 g' - 6\lambda a_{0y} a_{1x} f_x - 3\lambda a_{0x} a_{1y} f_x \\ & - 3\lambda a_{0x} a_{1x} g' - 6\mu(2\lambda^2 - \mu) a_1^2 f_x^2 g' \\ & + 3\lambda a_{1z} f_x + 3c\lambda a_{1x} f_z - 3\mu a_{1x}^2 g' \\ & + (2\lambda\mu - \lambda^3) a_{1y} f_x^3 - 3c(\lambda^2 - \mu) a_1 f_x f_z + 2a_{1yt} \\ & + 6\mu a_1 g' h' + (\mu^2 - 3\mu\lambda^2 + \lambda^4) f_x^3 g' \\ & + 5(\lambda^2\mu - \mu^2) a_1 f_x^3 g' + 3(\lambda^2 - \mu) a_{0y} a_1 f_x^2 \\ & + 3(2\mu\lambda - \lambda^3) a_{1x} f_x^2 g' \\ & - 9c\mu a_1 f_x f_z - 18\mu^2 a_1^2 f_x^2 g' + 10\mu(\lambda^2 - \mu) a_1 f_x^3 g' \\ & - 30\lambda\mu a_{1x} f_x^2 g' + 10\lambda^2 \mu a_1 f_x^3 g' + 21\lambda\mu a_1 a_{1x} f_x g' \\ & + 3(\lambda^2 - \mu) a_{0x} a_1 f_x g' + 9\lambda\mu a_1 a_{1y} f_x^2 \\ & + 9\mu a_{0x} a_1 f_x g' - 3a_{1xz} + a_{1xxxy} \\ & - 3\mu a_1 a_{1y} f_{xx} + 3(\lambda^2 - \mu) a_{1y} f_x f_{xx} \\ & - 3\lambda a_{0xx} a_1 g' + 3a_{0xx} a_{1y} - 3\mu a_1 a_{1xy} f_x \\ & + 9\mu a_{1x} f_{xx} g' + (\lambda^2 - \mu) a_1 f_{xxx} g' \\ & - 3\lambda a_{0y} a_1 f_{xx} - 30\lambda\mu a_1 f_x f_{xx} g' \\ & - 3\mu a_1 a_{1xx} g' + 6\lambda\mu a_1^2 f_{xx} g' + 3(2\lambda\mu - \lambda^3) a_1 f_x f_{xx} g' \\ & + 99\mu a_{1xx} f_x g' - 3\lambda a_{0xy} a_1 f_x + 3(\lambda^2 - \mu) a_{1xx} f_x g' \\ & + 3\mu a_1 f_{xxx} g' + 3(\lambda^2 - \mu) a_{1x} f_{xx} g' \\ & + 9\mu a_{1y} f_x f_{xx} - \lambda a_{1y} f_{xxx} - 3\lambda a_{1xxy} f_x \\ & + 3\lambda a_1 f_{xy} + 3(\lambda^2 - \mu) a_{1xy} f_x^2 \\ & - 3\lambda a_{1xy} f_{xx} - \lambda a_{1xxx} g' + 9\mu a_{1xy} f_x^2 \\ & + 3a_{0x} a_{1xy} + 3a_{0xy} a_{1x} + 3a_{0y} a_{1xx} \\ & + 2(\lambda^2 - \mu) a_1 g' h' - 2\lambda a_{1t} g' - 9\mu a_{1x} a_{1y} g' = 0,\end{aligned} \tag{11}$$

$$\begin{aligned}\left(\frac{G'}{G}\right)^0 :\ & 3\mu a_{1z} f_x + 3c\mu a_1 f_{xz} - \mu a_{1xxx} g' - \mu a_{1y} f_{xxx} \\ & + 3a_{0x} a_{0xy} + 10\lambda\mu^2 a_1 f_x^3 g' - 3\mu a_{1y} f_x^3 \\ & + 3\mu^2 a_1 a_{1y} f_x^2 + (\mu^2 - \lambda^2\mu) a_{1y} f_x^3 - 3a_{0xz} \\ & + 3(\mu^2 - \lambda^2\mu) a_{1x} f_x^2 g' - \lambda\mu^2(2\mu - \lambda^2) a_1 f_x^3 g' \\ & - 6\mu a_{0y} a_{1x} f_x - 3\mu a_{0x} a_{1y} f_x - 3\mu a_{0x} a_{1x} g' \\ & - 6\lambda\mu^2 a_1^2 f_x^2 g' + 9\mu^2 a_1 a_{1x} f_x g' + 3\lambda\mu a_{0x} a_1 f_x g' \\ & + 2\mu\lambda a_1 g' h' - 3c\mu\lambda a_1 f_x f_z + 3\mu\lambda a_{0y} a_1 f_x^2 \\ & + 3a_{0xx} a_{0y} + 3\lambda\mu a_{1xy} f_x^2 - 3\mu a_{1xxy} f_x + 2a_{0yt} \\ & - 9\mu^2 a_{1x} f_x^2 g' + 3(\mu^2 - \lambda^2\mu) a_1 f_x f_{xx} g' \\ & + \lambda\mu a_1 f_{xxx} g' + a_{0xxy} - 9\mu^2 a_1 f_x f_{xx} g' \\ & + 3\lambda\mu a_{1y} f_x f_{xx} + 3\lambda\mu a_{1xx} f_x g' \\ & + 3\lambda\mu a_{1x} f_{xx} g' + 3\mu^2 a_1^2 f_{xx} g' - 3\mu a_{0xy} a_1 f_x \\ & - 3\mu a_{0y} a_1 f_{xx} - 3\mu a_{0xx} a_1 g' - 2\mu a_{1y} h' \\ & - 2\mu a_{1t} g' - 3\mu a_{1xy} f_{xx} + 3c\mu a_{1x} f_z = 0.\end{aligned} \tag{12}$$

Solving Eqs. (7)–(12) yields

$$a_0 = ay + F(x,z,t), \quad a_1 = 2f_x, \tag{13}$$

where $F(x,z,t)$ is an arbitrary function of x, z, and t, and a is an arbitrary constant.

Substituting Eq. (13) and the general solutions of Eq. (4) into Eq. (5), we can obtain new exact solutions for the (3+1)-dimensional JM system (1).

Case 1 When $\lambda^2 - 4\mu > 0$, by using the general solutions of Eq. (4), we can derive

$$\frac{G'}{G} = -\frac{\lambda}{2} + \Delta_1 \frac{C_1 \cosh \Delta_1(f+g+h) + C_2 \sinh \Delta_1(f+g+h)}{C_1 \sinh \Delta_1(f+g+h) + C_2 \cosh \Delta_1(f+g+h)}. \tag{14}$$

Thus, the hyperbolic solutions of system (1) are expressed as

$$u_1 = ay + F(x,z,t) - \lambda f_x + 2 f_x \Delta_1 \frac{C_1 \cosh \Delta_1(f+g+h) + C_2 \sinh \Delta_1(f+g+h)}{C_1 \sinh \Delta_1(f+g+h) + C_2 \cosh \Delta_1(f+g+h)}, \tag{15}$$

where $\Delta_1 = \sqrt{\lambda^2 - 4\mu}/2$, f_x exists, and a, C_1, and C_2 are arbitrary constants.

Case 2 When $\lambda^2 - 4\mu < 0$, by using the general solutions of Eq. (4), we can derive

$$\frac{G'}{G} = -\frac{\lambda}{2} + \Delta_2 \frac{-C_1 \sin \Delta_2(f+g+h) + C_2 \cos \Delta_2(f+g+h)}{C_1 \cos \Delta_2(f+g+h) + C_2 \sin \Delta_2(f+g+h)}. \tag{16}$$

So the trigonometric solutions of system (1) are expressed as

$$u_2 = ay + F(x,z,t) - \lambda f_x + 2 f_x \Delta_2 \frac{-C_1 \sin \Delta_2(f+g+h) + C_2 \cos \Delta_2(f+g+h)}{C_1 \cos \Delta_2(f+g+h) + C_2 \sin \Delta_2(f+g+h)}, \tag{17}$$

where $\Delta_2 = \sqrt{4\mu - \lambda^2}/2$, f_x exists, and a, C_1, and C_2 are arbitrary constants.

Case 3 When $\lambda^2 - 4\mu = 0$, by using the general solutions of Eq. (4), we can derive

$$\frac{G'}{G} = -\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2(f+g+h)}. \tag{18}$$

In this case, the rational solutions of system (1) are shown as

$$u_3 = ay + F(x,z,t) - \lambda f_x + \frac{2C_2 f_x}{C_1 + C_2(f+g+h)}, \tag{19}$$

where f_x exists, and a, C_1, and C_2 are arbitrary constants.

3. Soliton structure

Due to the arbitrary functions $f(x+cz)$, $g(y)$, and $h(t)$ in the solutions (15)–(19), it is convenient to excite abundant soliton structures. We take solution (15) as an example to study the soliton excitations for the (3+1)-dimensional JM system (1).

3.1. New dromion soliton

According to solution (15), we first discuss its dromion excitations.

Case 1 We choose

$$f = 1 + \operatorname{sech}(x+cz)^2, \quad g = my, \quad h = nt,$$
$$F(x,z,t) = \operatorname{sech}(x^2 + z^2 + t^2). \tag{20}$$

Then we can obtain a dromion soliton structure, as shown in Fig. 1(a), where $m = 0.15$, $n = 0.1$, $C_1 = 1$, $C_2 = 2$, $\lambda = 1$, $\mu = 0.1$, $a = 0$, $c = 1$, $y = 1$, $t = 0$.

Case 2 We choose

$$f = \frac{1}{\coth(x+cz)^2 + \operatorname{csch}(x+cz)^2}, \quad g = my, \quad h = nt,$$
$$F(x,z,t) = \cos(\sqrt{x^2 + z^2 - t^2}). \tag{21}$$

We can get a dromion soliton structure, as shown in Fig. 1(b), where $m = 0.15$, $n = 0.1$, $C_1 = 1$, $C_2 = 2$, $\lambda = 1$, $\mu = 0.1$, $a = 1$, $c = 1$, $y = 1$, $t = 0$.

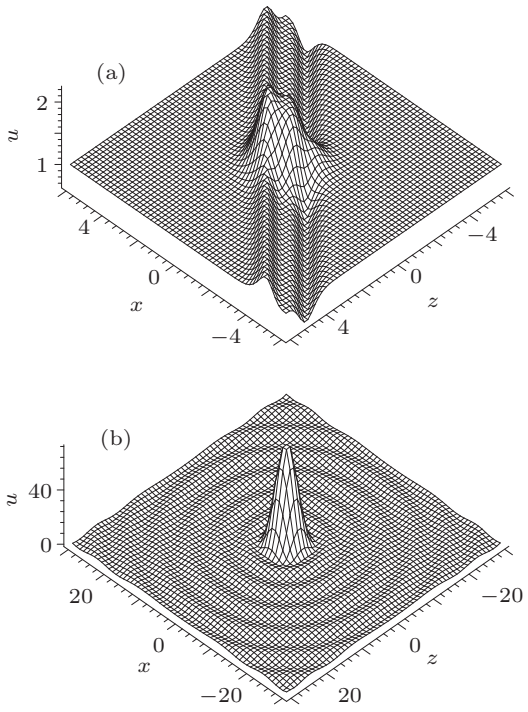


Fig. 1. Special types of dromion structure for solution (15) with choices of functions in (a) Eq. (20) and (b) Eq. (21).

3.2. Periodic vibration of the dromion soliton

In Eq. (20), if we choose different time, such as (a) $t = -3$, (b) $t = -1.5$, (c) $t = 0$, (d) $t = 1.5$, and (e) $t = 3$, we can get the evolutions of the dromion soliton as shown in Figs. 2(a)–2(e).

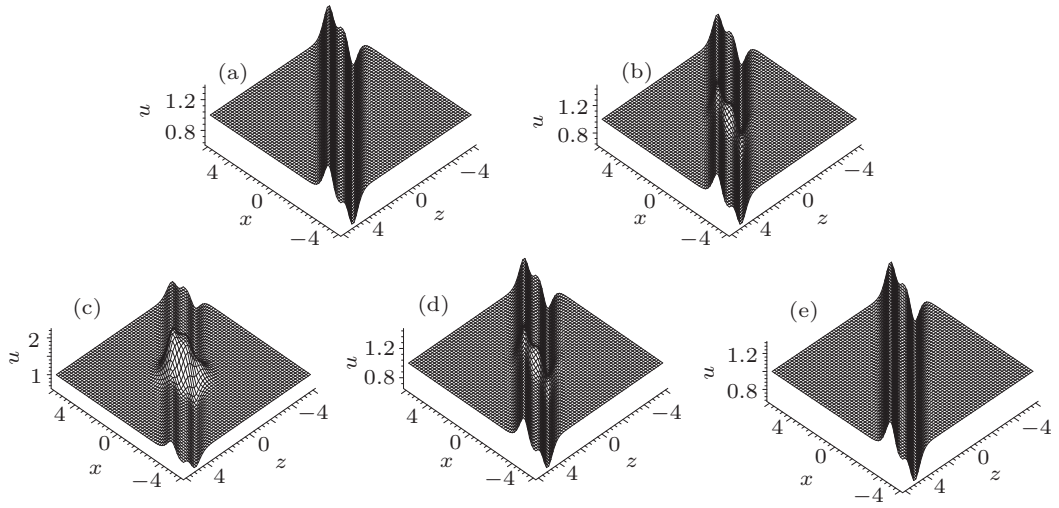


Fig. 2 Evolution of the dromion soliton for solution (15) with Eq. (20), the solitons at (a) $t=-3$, (b) $t=-1.5$, (c) $t=0$, (d) $t=1.5$, (e) $t=3$ are shown.

3.3. Interactions between two solitary waves

Now we focus our attention on the interactions between two solitary waves. In the usual cases, the interactions between solitary waves are regarded as completely elastic. For instance, when f, g, h, and $F(x,z,t)$ in Eq. (15) are

$$f = \frac{1}{1+0.2\mathrm{sech}(\mathrm{x}+\mathrm{cz})^2+0.2\mathrm{tanh}(x+cz)^2} - \mathrm{sech}(x+cz)^2, \quad (22)$$

$$g = my, \quad h = nt^2,$$

$$F(x,z,t) = \mathrm{sech}(x^2-z^2+t^2)+\mathrm{sech}(x^2+z^2+t^2), \quad (23)$$

we can obtain the interactions between two solitary waves. Figure 3 shows an evolutional profile corresponding to u expressed by Eq. (15) with $m=0.15$, $n=0.1$, $C_1=1$, $C_2=2$, $\lambda=1$, $\mu=0.1$, $a=1$, $c=1$, $y=1$. From Fig. 3 and through a detailed analysis, we find that the shapes, amplitudes, and velocities of the two solitary waves are completely conserved after their interactions.

Above we show the excitation process of a special dromion soliton structure of solution (15) for the (3+1)-dimensional JM system (1), and we discuss the variation of solution (15) with time and the interactions between two solitary waves. It is clear that other selections of $f(x+cz), g(y)$, and $h(t)$ in Eq. (15) may generate rich localized soliton structures. On the other hand, solutions (17) and (19) may also be used to excite abundant soliton structures.

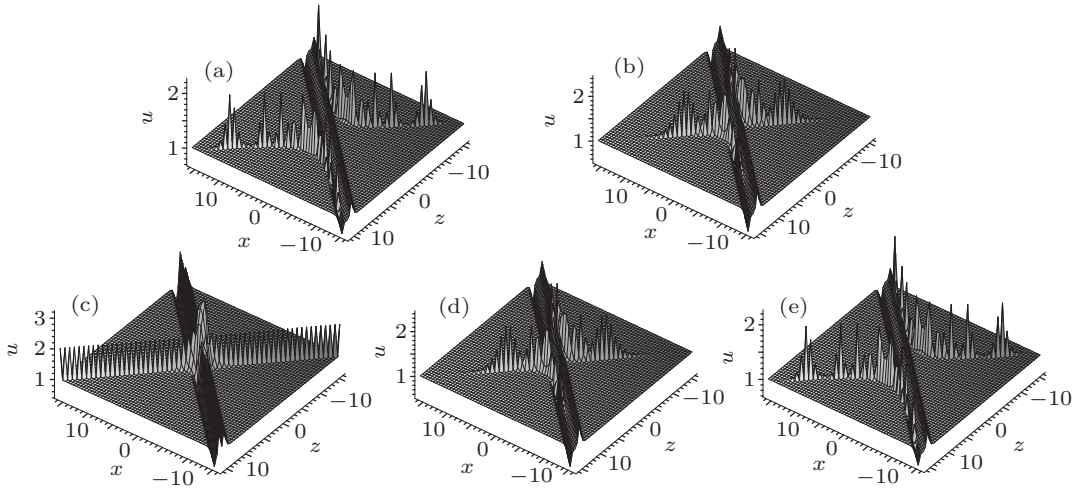


Fig. 3. Interactions between two solitary waves. The physical quantity u with Eqs. (22) and (23) at (a) $t=-5$, (b) $t=-2.5$, (c) $t=0$, (d) $t=2.5$, and (e) $t=5$ is shown.

4. Discussion and summary

In summary, via extending the (G'/G)-expansion method, more rich types of explicit and exact non-traveling wave solutions of the (3+1)-dimensional JM system (1) are found, and the traveling wave solutions are included in these exact solutions. So the exact solutions are more general.

Furthermore, by choosing an appropriately arbitrary function $\xi(x,y,z,t)$ in the solutions, we can study various interesting localized soliton excitations. Due to the wide applications of the soliton theory, learning more about the localized excitations and their applications in reality is worthy of further study.

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