Complexity analyses of multi-wing chaotic systems*

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The complexities of multi-wing chaotic systems based on the modified Chen system and a multi-segment quadratic function are investigated by employing the statistical complexity measure (SCM) and the spectral entropy (SE) algorithm. How to choose the parameters of the SCM and SE algorithms is discussed. The results show that the complexity of the multi-wing chaotic system does not increase as the number of wings increases, and it is consistent with the results of the Grassberger–Procaccia (GP) algorithm and the largest Lyapunov exponent (LLE) of the multi-wing chaotic system.

Keywords: complexity, multi-wing chaotic system, statistical complexity measure (SCM), spectral entropy (SE)

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1. Introduction

Chaos has been widely studied for nearly four decades, and it shows the potential applications in the field of secure communication and information encryption.[1–4] Although research of attractors with more complicated topological structures has received much attention,[5,6] the complexity analysis of chaotic signals is still a new topic at present.[7,8]

Compared with multi-scroll attractors, generation of multi-wing chaotic attractors seems to be a more challenging topic. Multi-wing chaotic systems were proposed by designing particular nonlinear functions, such as the multi-segment quadratic function,[9] the sawtooth function,[10] the Julia process fractal,[11] or multi-level-logic pulse exciting sources.[12] In addition, control approaches are also used to generate multi-wing attractors.[13] A natural question to follow is whether these multi-wing chaotic systems are more complex than the usual ones. As far as we know, this issue has not been reported in literatures yet. The major task of this paper is to deal with this issue.

At present, there are several methods of measuring complexity of time series, including the electroencephalography (EEG), magnetoencephalograph (MEG), and electrocardiograph (ECG) signals. Fractal dimension (FD) complexity studied in Refs. [14]–[16] is an indicator which estimates the fractal complexity. Lempel–Ziv complexity (LZC)[17] is also one of the most common complexity measure approaches. Entropy measures, such as permutation entropy (PE),[18] spectral entropy (SE),[19] approximate entropy,[20] and sample entropy[21] are widely used to analyze chaotic time sequences.[22,23] As new methods, SCM[24,25] and fuzzy entropy[26] were also used to compute the complexities of chaotic systems. Among those approaches, SCM and SE have appealing advantages. The SCM method is qualified to grasp essential details of the dynamical systems, has an intensive quantity, and is capable of discerning different degrees of periodicity and chaos. While the SE is based on the power spectrum of the chaotic signal and describes the irregularity of the signal spectrum. On the other hand, the computation cost of these two methods is low. Hence, in this paper, we utilize the SCM and SE methods to analyze the complexity of a multi-wing Chen system,[9] which is a typical representative among multi-wing chaotic systems. The complexities of another two systems are also calculated to support the conclusions. We should point out that the complexities in this paper refer to the SCM and SE complexities which are respectively defined in time domain and frequency domain. Although this complexity does not show each aspect of the complexity about the chaotic system, it does provide some valuable information about the system.

The rest of the paper is organized as follows. In Section 2, SCM and SE algorithms are introduced and their parameters are discussed. In Section 3, multi-wing chaotic systems are presented, and the complexities of multi-wing chaotic systems are calculated and analyzed. In Section 4, we summarize the results and indicate future directions.

2. Complexity algorithms

In this section, the SCM and SE algorithms are presented in detail, and their parameters are discussed carefully.

2.1. Statistical complexity measure algorithm

The statistical complexity measure algorithm was proposed by Lopez-Ruiz *et al.*,[24] and reflects the interplay between the amount of information stored in the system and its disequilibrium. It was improved in Ref. [25], and now has

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become a popular algorithm. The steps of this algorithm are described as follows.

Step 1 Following the method proposed by Brandt and Pompe,[18] the probability distribution P associated with the time series (dynamical system) is defined. For the time series $\{x(n),\ n=0,1,\ldots,N-1\}$ and the order d, a d-dimensional space of the i-th point is generated by

$$(i)\rightarrow(x_i,x_{i+1},\cdots,x_{i+d-1}). \tag{1}$$

Clearly, the greater the value of d is, the more information the vector (i) stores. In view of the "ordinal pattern" related to the i-th point, the permutation $\pi=(r_0,r_1,\ldots,r_{d-1})$ of $(0,1,\ldots,d-1)$ is defined as

$$x_{i-r_d}\leq x_{i-r_{d-1}}\leq\cdots\leq x_{i-r_1}\leq x_{i-r_0}. \tag{2}$$

Generally, we set $r_i<r_{i-1}$ if $x_{r_i}=x_{r_{i-1}}$. Then, for all the $d!$ possible permutation π of order d, the probability distribution $P=p(\pi)$ is defined as

$$p(\pi)=\#\{i|i\leq T-d+1\}/(T-d+1), \tag{3}$$

where the symbol # denotes "number". The normalized entropy and the statistical complexity are then evaluated for this "permutation" probability distribution.

Step 2 For $P=\{p_i;i=1,2,\ldots,M\}$, the normalized Shannon entropy H is denoted as

$$H[P]=S[P]/S_{\max}, \tag{4}$$

and the Jensen–Shannon divergence Q is calculated by

$$Q_{\mathrm{J}}[P,P_{\mathrm{e}}]=Q_0\{S[(P+P_{\mathrm{e}})/2]-S[P]/2-S[P_{\mathrm{e}}]/2\}, \tag{5}$$

where $S_{\max}=\log M$, Q_0 is the normalization constant, and $S[\cdot]=-\sum p_i\ln p_i$.

Step 3 Define intensive SCM as

$$C[P]=Q[P,P_{\mathrm{e}}]H[P]. \tag{6}$$

Obviously, the larger the value of C is, the lower the complexity is.

For this algorithm, it should be noted that the sampling time τ of chaotic sequence, dimension d, and sequence length N should be chosen carefully. Let X_{Chen} represent the time series generated by x variable in the Chen system ($a=35$, $b=3$, $c=28$).[9] Here, X_{Chen} is used to calculate the SCM complexity under different conditions. Figure 1 shows the SCM complexity with different values of sampling time τ. It shows that when $\tau>0.1$, the SCM values become stable. In this paper, sampling time $\tau=0.1$ is chosen to calculate SCM complexity. In Fig. 2(a), when d takes values of 5–7, the SCM has the largest value. Therefore, $d=7$ is chosen to compute the SCM complexity of the time series with 7! possible kinds of patterns. In Fig. 2(b), T is the length of time sequence. It shows that as the length of the sequence increases, the complexity value approaches 0.2. Taking into account the computation time, T is chosen to be 100 s for a window.

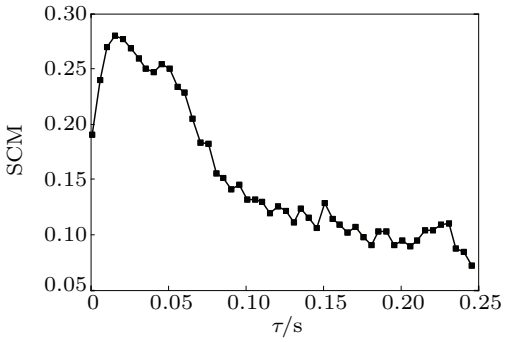


Fig. 1. SCM complexity versus the sampling time τ.

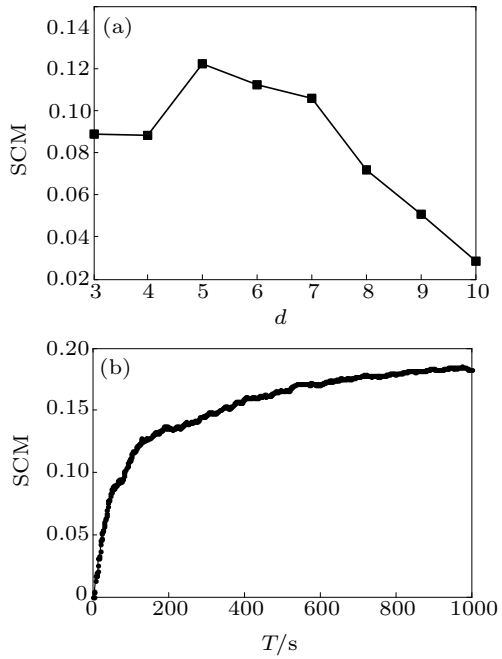


Fig. 2. SCM complexity versus (a) d and (b) T.

2.2. Spectral entropy algorithm

The SE is a method based on discrete Fourier transformation (DFT).[19] The original meaning of Shannon's entropy refers to the uncertainty of information in the sense of disorder. Associated with DFT, the SE reflects the disorder in the transform domain. A flatter spectrum means a larger value of SE, which shows a higher complexity of the time series. The steps of the SE algorithm are described as follows.

Step 1 Suppose the time series $\{x(n),\ n=0,1,\ldots,N-1\}$ with a length of N, and its corresponding DFT is defined as

$$X(k)=\sum_{n=0}^{N-1}x(n)\,\mathrm{e}^{-2\pi\mathrm{j}nk/N}=\sum_{n=0}^{N-1}x(n)W_N^{nk}, \tag{7}$$

where $k=0,1,\ldots,N-1$, and $j=\sqrt{-1}$ is the unit imaginary.

Step 2 If the power of a discrete power spectrum with the k-th frequency is $|X(k)|^2$, then the "probability" of this frequency is defined as

$$P_k=|X_k|^2\Big/\sum_{k=0}^{N/2-1}|X_k|^2. \tag{8}$$

When the DFT is employed, the summation runs from $k = 0$ to $k = N/2 - 1$.

Step 3 The entropy of this probability distribution with the same summation limits as that in Eq. (8) is

$$\mathrm{SE} = \sum_{k=0}^{N/2-1} \frac{1}{\mathrm{SE}_{\max}} P_k \ln P_k, \tag{9}$$

where $\mathrm{SE}_{\max}$ is the maximum value of the spectral entropy. Generally, a real white noise signal with power at all frequencies leads to this result, which is expressed as

$$\mathrm{SE}_{\max} = \ln(N/2). \tag{10}$$

From the above three steps, the SE of a given time series with length N can be computed, without choosing any other parameters. The SE complexity of a sequence length N based on X_{Chen} is shown in Fig. 3. Also, $T = 100$ s is chosen as the time of a window to calculate the SE value.

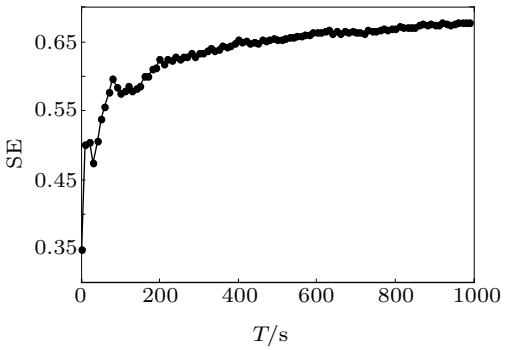


Fig. 3. SE complexity versus the sequence length T.

3. Complexity analyses

3.1. Multi-wing chaotic system models

In this study, we choose a kind of multi-wing attractor to study its complexity, i.e., to see how the complexity changes as the number of wings increases. The chaotic systems are simulated by Runge–Kutta method with a time step of 0.01 s, and x sequence is used to compute the complexity. The time length of a window is 100 s, and the complexity of 50 windows is calculated. In Lorenz-like systems, $2N+2$ wings ($N = 0, 1, \ldots, 5$) are generated by using a multi-segment quadratic function. In Ref. [14], these systems include a modified Chen system, a modified Lü system, a modified Rucklidge system, a modified Sprott-1 system, and a modified Sprott-2 system. Here, the modified Chen system is chosen to be the representative of the Lorenz-like systems to analyse the complexity. The equations of the modified Chen system are[9]

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = (c - a)x + cy - (1/P)xz, \\ \dot{z} = f(x) - bz, \end{cases} \tag{11}$$

where $a = 35$, $b = 3$, $c = 28$, and $P = 0.05$. When $P = 1$ and $f(x) = xy$, the system is the Chen system. The multi-segment quadratic function is

$$f(x) = F_0 x^2 - \sum_{i=1}^{N} F_i[1 + 0.5\mathrm{sgn}(x + E_i) - 0.5\mathrm{sgn}(x + E_i)]. \tag{12}$$

With parameters given in Ref. [9], $2N + 2$ ($N = 0, 1, \ldots, 5$) wings are generated. Figure 4 shows the phase diagram of the multiple wings generated by the modified Chen system.

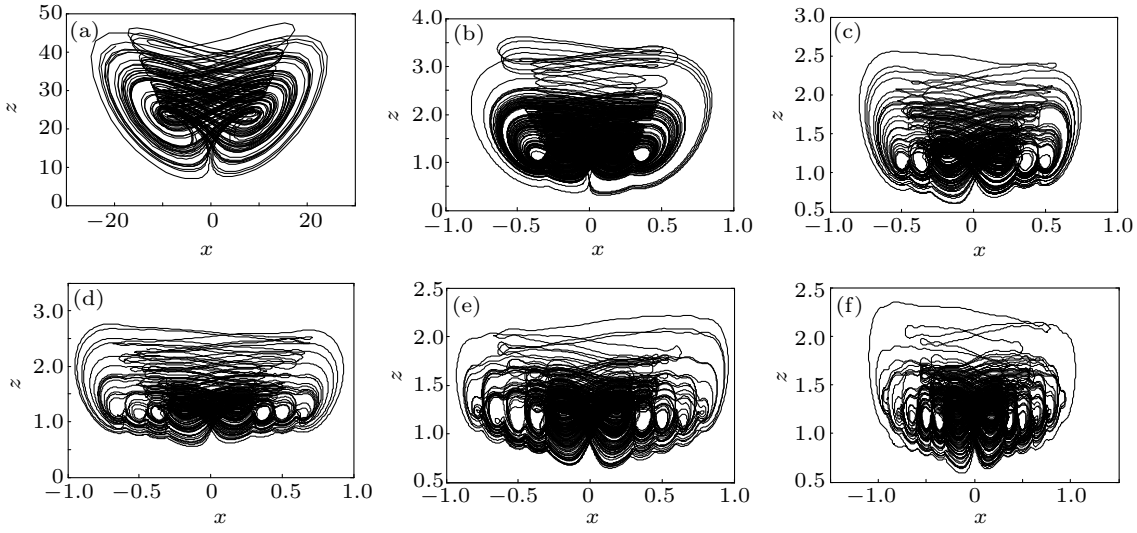


Fig. 4. Multi-wing attractors based on the modified Chen system. (a) 2-wing attractor (Chen system), (b) 4-wing attractor, (c) 6-wing attractor, (d) 8-wing attractor, (e) 10-wing attractor, and (f) 12-wing attractor.

3.2. Complex analyses of multi-wing systems

In this section, SCM and SE are used to estimate the fifty 100-s windows of variable x took from the modified Chen system. The main results are shown in Figs. 5 and 6, as well as Tables 1 and 2.

Figure 5 shows the SCM and SE complexity values changing with time. It can be seen that the complexity of Chen system fluctuates within a small range. Hence, for this chaotic

system, the complexity has certain stability.

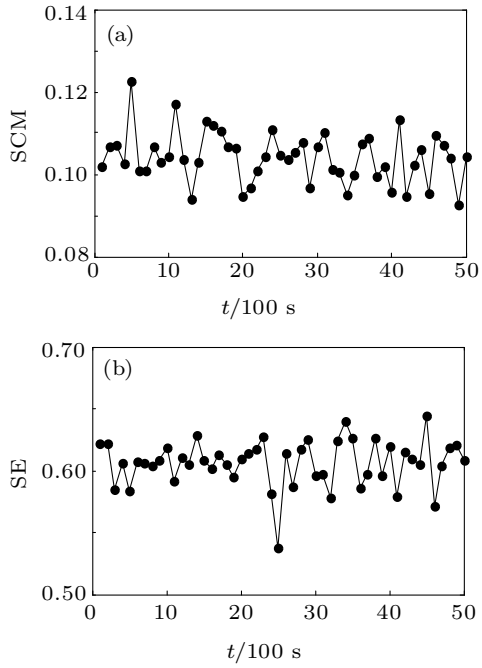


Fig. 5. Time evolution characteristics of the Chen system. (a) SCM complexity and (b) SE complexity.

The boxplots of complexity of multi-wing system are shown in Fig. 6. The median in the box is shown by the solid circle, which extends from 25% to 75%. The hollow circles represent the outliers. Figure 6 shows the complexities of different systems. According to Fig. 6(a), the SCM complexities of two-wing chaotic system and four-wing chaotic system are about at the same level, while the complexities of 8-, 10-, and 12-wing systems are all about at another level. There is a downward trend in Fig. 6(b). To compare their complexities, further statistical analysis is still needed.

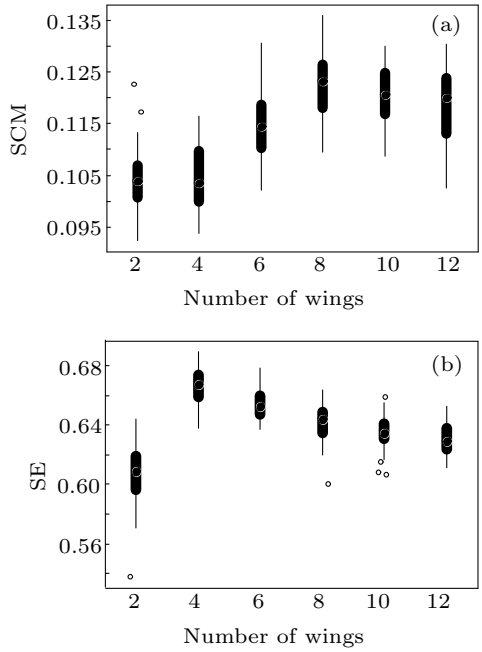


Fig. 6. Boxplots of complexity of modified Chen system. (a) SCM complexity and (b) SE complexity.

Statistical analysis of the data is made with the SPSS for windows, version 13.0. For the complexity measure, one-way analysis of variance (ANOVA) and post hoc multiple comparison (least significant difference (LSD)) are carried out. The dependent variables are SE and SCM complexities, and the factor is the 'Group'. 'Group I' represents the 50 complexity values of I-wing system calculated by SE or SCM. Here, the null hypothesis means that the multi-wing chaotic systems have the same complexity. We take $p = 0.05$ as the level of statistical significance for the tests. Tables 1 and 2 show the p values of post hoc multiple comparison (i.e., LSD) for SCM complexity and SE complexity, respectively.

Table 1. Values of p of LSD for SCM complexity.

SCM	2	4	6	8	10	12
2	N	0.851	0.000	0.000	0.000	0.000
4	0.851	N	0.000	0.000	0.000	0.000
6	0.000	0.000	N	0.000	0.000	0.001
8	0.000	0.000	0.000	N	0.224	0.012
10	0.000	0.000	0.000	0.224	N	0.084
12	0.000	0.000	0.001	0.012	0.084	N

Table 2. Values of p of LSD for SE complexity.

SE	2	4	6	8	10	12
2	N	0.000	0.000	0.00	0.000	0.000
4	0.000	N	0.000	0.000	0.000	0.000
6	0.000	0.000	N	0.000	0.000	0.001
8	0.000	0.000	0.000	N	0.004	0.000
10	0.000	0.000	0.000	0.004	N	0.033
12	0.000	0.000	0.001	0.012	0.033	N

The one-way ANOVA results of SCM are $F = 87.61$ and $P = 0$, and results of SE are $F = 142.9$, $P = 0$. Obviously, the p value of ANOVA is 0, which is less than 0.05, so we can reject the null hypothesis. That is to say, the complexities of different systems are all statistically significant. According to the p values of LSD in Table 1, the p value between 2- and 4-wing system is 0.851, which is greater than 0.05, which means the SCM complexities of 2- and 4-wing chaotic systems are the same. The SCM complexity of 6-wing system is less than those of 2- and 4-wing systems. The p value between 8- and 10-wing systems is 0.224, and between 10- and 12-wing systems is 0.084, which are greater than 0.05. It means that SCM complexities of these three groups are at the same level. According to Fig. 6(a), these three systems have the minimal complexities. Figure 6(b) shows that when the wing number of the system is greater than 2, the SE complexity presents a decreasing trend. According to the p value in Table 1 and the p value of LSD in Table 2, the trend is verified. Therefore, we come to a conclusion that the complexity of the multi-wing system does not increase with the number of wings.

3.3. LLE and GP complexity

As an important characteristics of chaotic system, LLE[27] and GP also reflect the complexity. LLE is a parameter that characterizes the rate of convergence or divergence. For a chaotic system, the LLE is greater than 0, and larger LLE causes more complex system. GP is a measure of strange attractors, and describes the complexity of the chaotic system quantitatively. To compare with the results of SCM and SE, LLE and GP of chaotic systems with different wings are calculated. The results are shown in Fig. 7, which shows that the systems are chaotic, and the LLE does not increase as the number of wings increases. In Fig. 7(b), values of GP show a decreasing trend as the number of wings increases. It shows once again that the complexity of the multi-wing chaotic systems does not increase with the number of wings.

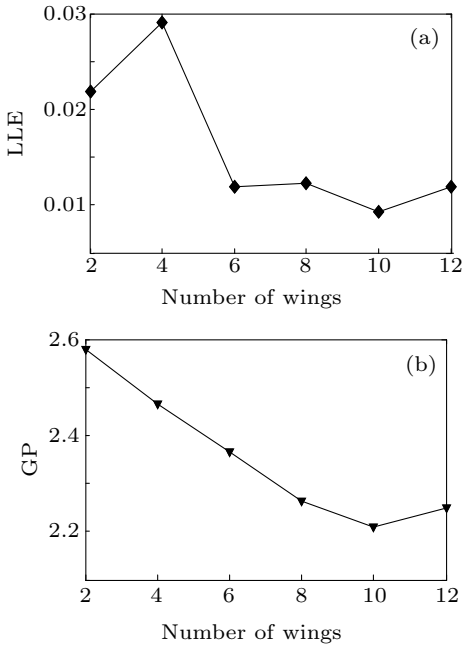


Fig. 7. (a) LLE complexity and (b) GP complexity veusus the number of wings.

3.4. Complex analyses of other multi-wing systems

Here, the complexities of other two types of multi-wing chaotic systems are calculated. These two kinds of wings were described in Refs. [10] and [12], respectively. Compared with the multi-wing systems analysed above, systems in Ref. [10] are based on sawtooth function, and the technique used to generate multiple wings in Ref. [12] is basically a nonautonomous technique utilizing the multi-level-logic pulse. The two kinds of wings were also realized physically. For convenience, multiple wings in Ref. [10] are called A-type wings, and those in Ref. [12] are called B-type wings. The SCM and SE complexities of the two types of multi-wing chaotic systems are shown in Figs. 8 and 10, respectively. Values of LLE and GP versus the number of wings are shown in Figs. 9 and 11, respectively.

As shown in Fig. 8(a), the values of SCM complexity have a decreasing trend, but the results overlap with each other seriously. It means that the SCM complexities of systems with different wings are about at the same level. However, the SE complexity declines as the number of wings increases. Figure 9(a) shows that the LLE values first increase, then decrease. Figure 9(b) indicates that the GP values decrease with increasing number of wings. Hence, it is concluded that the complexity of the A-type wings does not increase with the number of wings increasing either.

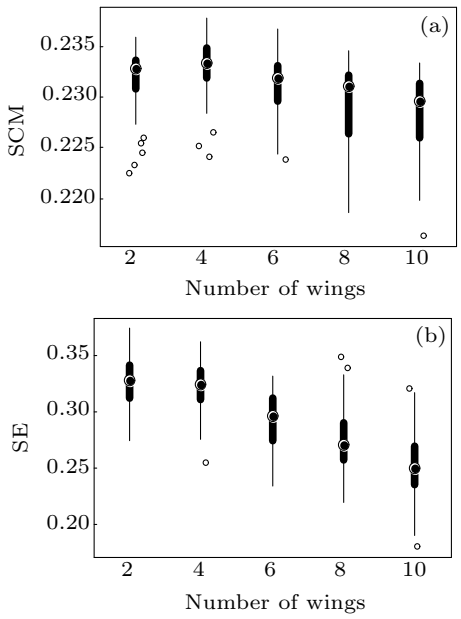


Fig. 8. Boxplots of complexity of A-type wings. (a) SCM complexity and (b) SE complexity.

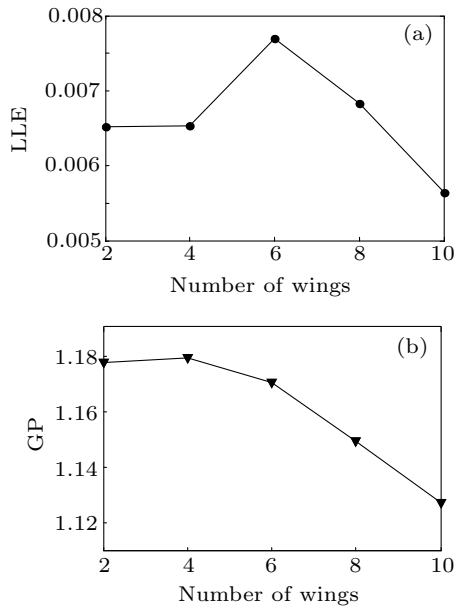


Fig. 9. (a) LLE complexity and (b) GP complexity varying with the number of wings.

As shown in Fig. 10(a), the values of SCM complexity of the B-type wings have an increasing trend, and the results

overlap with each other. The values of SE complexity of different systems are about at the same level obviously. Figure 11(a) shows that the LLE values fluctuate, and Fig. 11(b) indicates that the GP values decrease. Thus, from the above analyses, it is clear that the complexity does not increase as the number of wings increases either.

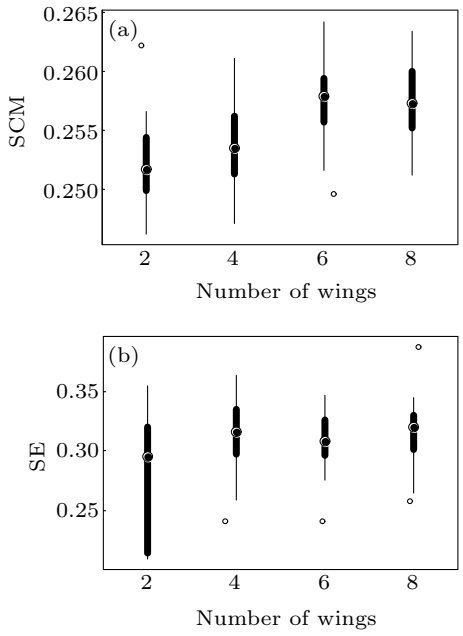


Fig. 10. Boxplots of complexity of B-type wings. (a) SCM complexity and (b) SE complexity.

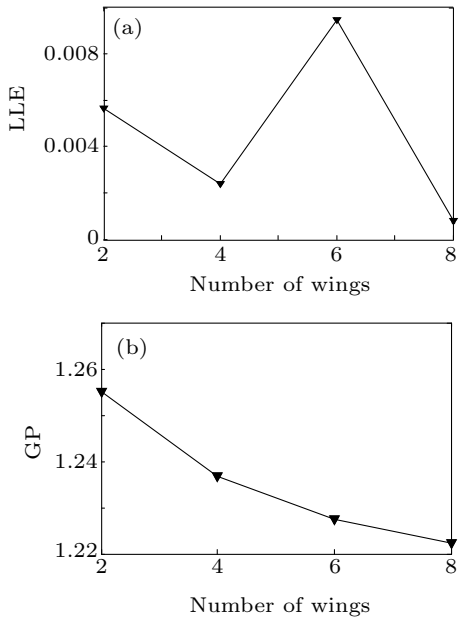


Fig. 11. (a) LLE complexity and (b) GP complexity varying with the number of wings.

Why does the complexity not increase with the number of wings increasing? Does it mean that the multi-wing modeling is useless? In our opinion, firstly, compared with the original system, the multi-wing system usually is the system in which one nonlinear term in the original system is replaced with a nonlinear function, but this change may make system performance down. Secondly, although the complexity of multi-wing chaotic system does not increase, it just declines a little. This kind of chaotic system still provides a good model choice for its practical applications.

4. Conclusions

In this paper, we investigate the SCM and SE complexities of multi-wing chaotic systems based on the modified Chen system and the other two kinds of multi-wing systems. The complexities of multi-wing chaotic systems are shown in the boxplots, and one-way ANOVA and LSD are applied to test the statistical characteristics. LLE and GP are also used to analyse the complexity, and the results show the same trends as those computed by SCM and SE. After analysing the results, we find that the complexity of a multi-wing chaotic system does not increase as the number of wings increases. Here, we should point out that the fact that the complexity does not increase does not mean that multi-wing system modeling is meaningless. We believe that it still provides a good choice for its applications in the field of information security.

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