New prediction of chaotic time series based on local Lyapunov exponent*

Zhang Yong(张 勇)†

Department of Mathematics and Computer, Wuhan Polytechnic University, Wuhan 430024, *China*

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A new method of predicting chaotic time series is presented based on a local Lyapunov exponent, by quantitatively measuring the exponential rate of separation or attraction of two infinitely close trajectories in state space. After reconstructing state space from one-dimensional chaotic time series, neighboring multiple-state vectors of the predicting point are selected to deduce the prediction formula by using the definition of the local Lyapunov exponent. Numerical simulations are carried out to test its effectiveness and verify its higher precision over two older methods. The effects of the number of referential state vectors and added noise on forecasting accuracy are also studied numerically.

Keywords: chaotic time series, prediction of chaotic time series, local Lyapunov exponent, least squares method

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1. Introduction

Chaotic behaviors extensively exist in various natural and social systems, such as the atmosphere, traffic, acoustics, economics, and biomedicine. They are an evolutionary pattern standing between randomness and certainty. The sensitivity to initial conditions is the most essential characteristic of chaos, and it can be quantified by calculating the Lyapunov exponent (LE).[1–4] The Lyapunov exponent is a macroscopic statistical quantity that measures the average exponential speed of separation or attraction of adjacent states, and its value depends only on the related systems because of the ergodic property of chaos.[4] To describe partially the characteristics of a chaotic trajectory, Abarbanel *et al.*[5,6] proposed the concept of the local Lyapunov exponent (LLE), which is used to depict how nearby states diverge in space. Its value depends on the position of the state vector in space and the diverging step.[4–8]

Prediction of chaotic time series plays an important role in all studies on chaos. Although chaotic systems contain random elements, short-term prediction of chaotic time series is possible because of certain factors in chaotic systems. The first step of forecasting chaotic time series is to reconstruct state space.[3] Reconstructed systems from chaotic time series with certain embedding dimensions and delaying time has the identical topological structure of the original system. Methods of forecasting chaotic time series can be classified into three main types. (i) Partial domain methods use partial history data in the vicinity of the predicting point to approximate the partial trajectory.[9–17] (ii) Universal domain methods adopt all history data to deal with dynamic equations.[18–20] (iii) Self-adaptive forecasting methods are integrated, self-adaptive, partial- or universal-domain arithmetics.[21–25] It is also worth noticing that no method is superior to all the other methods under every evaluating criterion.

Forecasting chaotic time series based on LE is a simple partial-domain method, which is derived from the definition of LE.[14] An average quantity results in inaccurate partial values.[15–17] In Ref. [15], LLE was substituted for LE in the forecasting formula, and numerical results demonstrated that the chaotic time series was predicted better by using LLE than LE. However, in the forecasting method based on either LE[14] or LLE,[15] the nearest state vector of the predicting point is required as a referential state vector to evaluate LE or LLE. Contained noise may cause the unsuitable choice of the nearby referential state vector and thus wrong evaluation of LE or LLE, leading to false results. In addition, each method has two final results at one point, and how to choose the right one is not known yet. In Ref. [16], Li *et al.* evaluated LLE by using a neuro-network at the predicting point to forecast, and to approach the system equations. Some nearby points would be discarded if the LLE is larger during the estimation, but the computation is time-consuming because of the training of the neuro-network. In Ref. [17], Zhang and Guan deduced a new predication approach on the basis of the definition of LE by adopting many nearby points to the predicting point. This method yields only one final result, but still has to calculate LE.

In this study, a new forecasting method is presented based on LLE by adopting many nearby state vectors of the predicting point in reconstruction space. It is not necessary to calculate LE or LLE, and only one result exists. The effect of the wrong choice of nearby points may be reduced because of the multiple referential vectors adopted. Numerical sim-

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†Corresponding author. E-mail: ballack-13@163.com

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ulation demonstrates its capability and higher predicting precision compared with two older methods. The influences of related parameters on forecasting accuracy are also demonstrated by numerical studies.

The rest of this paper is organized as follows. Section 2 introduces two methods for forecasting chaotic time series, which are based on LE and LLE. We outline our method and deduce detailed procedures in Section 3. In Section 4, numerical simulations of predicting two classic chaotic time series and discussion about the influence of related parameters are performed. The results originating from our method are inconsistent with those derived from other methods. Finally, some conclusions are summarized.

2. Prediction of chaotic time series based on LE and LLE

Let $\{x_i\}$ $(i = 1,2,\ldots,N)$ be a single scalar data stream of a chaotic dynamic system $x_i = x(t_0 + \mathrm{i}\Delta t)$, so-called chaotic time series, where t_0 and Δt are the initial time and the sample time, respectively. If the embedding dimension and the delaying time are decided, according to Takens' reconstruction theory, a reconstructed vector series can be written as follows:

$$\boldsymbol{Y}_i = (x_i, x_{i-\tau}, \ldots, x_{i-(m-1)\tau}), \tag{1}$$

where $(m-1)\tau + 1 \leq i \leq N$, m and τ are the embedding dimension and the delaying time, respectively. We call them the reconstructing parameters of $\{x_i\}$. There are many ways to decide the reconstructing parameters,[26–33] and methods in terms of self-correlation and average mutual information are usually used to choose τ. For m, it is adequate to reconstruct attractors if $m \geq 2D+1$, where D is the correlation dimension and can be calculated by using G–P arithmetic.[27] Large m means a huge calculation load and is not necessary in many situations. Methods using false neighborhood vector and minimizing forecast error are more suitable for deciding the proper embedding dimension m.

According to Eq. (1), each element of $\boldsymbol{Y}_N$ is known at the N-th time point. The first step of forecasting the next value x_{N+1} based on LE is to choose the nearest state vector $\boldsymbol{Y}_k$ of $\boldsymbol{Y}_N$. We denote it as

$$\boldsymbol{Y}_k = \arg\min_{j \neq N} \left\| \boldsymbol{Y}_j - \boldsymbol{Y}_N \right\|. \tag{2}$$

From the definition of LE,[1,21] $\boldsymbol{Y}_N$ and $\boldsymbol{Y}_k$ separate each other at exponential speed λ. Then,

$$\|\boldsymbol{Y}_{N+1} - \boldsymbol{Y}_{k+1}\| = \|\boldsymbol{Y}_N - \boldsymbol{Y}_k\| \, \mathrm{e}^{\lambda \Delta t}, \tag{3}$$

where λ denotes LE, and Δt is the sampling time. After a simple computation, we have

$$\hat{x}_{N+1} = x_{k+1} \pm \sqrt{\|\boldsymbol{Y}_N - \boldsymbol{Y}_k\|^2 \, \mathrm{e}^{2\lambda\Delta t} - \sum_{i=1}^{m-1} (x_{N+1-i\tau} - x_{k+1-i\tau})^2}. \tag{4}$$

Because LE is an average quantity along the whole trajectory in state space,[1,21] forecasting a partial value from it may not be proper. In order to increase precision, Dominique and Justin used LLE instead of LE,[12] namely,

$$\hat{x}_{N+1} = x_{k+1} \pm \sqrt{\|\boldsymbol{Y}_N - \boldsymbol{Y}_k\|^2 \, \mathrm{e}^{2\lambda(\boldsymbol{Y}_k)\Delta t} - \sum_{i=1}^{m-1} (x_{N+1-i\tau} - x_{k+1-i\tau})^2}, \tag{5}$$

where $\lambda(\boldsymbol{Y}_k)$, the LLE of trajectory at the $\boldsymbol{Y}_k$ position, can be evaluated by

$$\lambda(\boldsymbol{Y}_k) = \frac{1}{\Delta t} \ln \frac{\left\| \boldsymbol{Y}_{k+1} - \boldsymbol{Y}_{j+1} \right\|}{\left\| \boldsymbol{Y}_k - \boldsymbol{Y}_j \right\|}, \quad \boldsymbol{Y}_j = \arg\min_{q \neq k,N} \left\| \boldsymbol{Y}_k - \boldsymbol{Y}_q \right\|. \tag{6}$$

The local Lypunov exponent represents the local exponential detaching or approaching rate between two infinitely-closing state vectors at a given point. Methods based on LLE are more reasonable and have a higher accuracy than methods based on LE.[12]

Methods based on LE and LLE have some common disadvantages. Firstly, equations (4) and (5) both have two forecasting results, and $\hat{x}_{N+1}$ may be a non-real number if λ or $\lambda(\boldsymbol{Y}_k)$ is underestimated. Secondly, picking up the nearest state vector $\boldsymbol{Y}_k$ of $\boldsymbol{Y}_N$ on the basis of Eq. (2) may be wrong because of the contained noise in time series. Finally, precisely evaluating LE is difficult in many situations, and LLE obtained on the basis of Eq. (6) may be wrong because of a wrong choice of $\boldsymbol{Y}_j$.

Some partial forecasting methods use multiple nearby state vectors of the predicting time point to fit the partial trajectory of chaotic system.[5–8] In the next section, we aim to present a new method of predicting chaotic time series based on LLE in the same way of adopting multiple referential vectors.

3. New method

For reconstructed m-dimensional vector series $\boldsymbol{Y}_i$ $((m-1)\tau + 1 \leq i \leq N)$ from time series $\{x_i\}$ $(i = 1,2,\ldots,N)$ by using Eq. (2), we suppose that orbits $\boldsymbol{Y}_i$ in $\mathbb{R}^m$ are generated by means of a function $F : \mathbb{R}^m \to \mathbb{R}^m$ as follows:

$$\boldsymbol{Y}_{i+1} = F(\boldsymbol{Y}_i), \tag{7}$$

with an initial condition $\boldsymbol{Y}_{(m-1)\tau+1}$.

A small perturbation $\delta\boldsymbol{Y}_i$ to this orbit at the point $\boldsymbol{Y}_i$ evolves according to the following linearized function of F:

$$\delta\boldsymbol{Y}_{i+1} = \boldsymbol{J}(\boldsymbol{Y}_i) \cdot \delta\boldsymbol{Y}_i, \tag{8}$$

where $\boldsymbol{J}(\cdot)$ is the Jacobian matrix of F. From Eq. (8), the p time steps forward perturbation from the point $\boldsymbol{Y}_i$ is

$$\delta\boldsymbol{Y}_{i+p} = \boldsymbol{J}(\boldsymbol{Y}_{i+p-1}) \cdots \boldsymbol{J}(\boldsymbol{Y}_{i+1}) \cdot \boldsymbol{J}(\boldsymbol{Y}_i) \cdot \delta\boldsymbol{Y}_i$$

$$= \boldsymbol{J}^p(\boldsymbol{Y}_i)\cdot\delta\boldsymbol{Y}_i, \tag{9}$$

where $\boldsymbol{J}^p(\boldsymbol{Y}_i)$ is an abbreviation for the product of p Jacobian matrices initialed at $\boldsymbol{Y}_i$. The error propagation equation is given by

$$\|\delta\boldsymbol{Y}_{i+p}\| = \|\boldsymbol{J}^p(\boldsymbol{Y}_i)\|\|\delta\boldsymbol{Y}_i\|. \tag{10}$$

The local Lyapunov exponent reads

$$\lambda(\boldsymbol{Y}_i,p) = \frac{1}{p\Delta t}\ln\frac{\|\delta\boldsymbol{Y}_{i+p}\|}{\|\delta\boldsymbol{Y}_i\|} \approx \frac{1}{p\Delta t}\ln\|\boldsymbol{J}_i^p\|. \tag{11}$$

From Eq. (11), LLE $\lambda(\boldsymbol{Y}_i,p)$ depicts the local separation or attraction of two infinitely close state vectors. Its value depends on the position of $\boldsymbol{Y}_i$ and the forecasting step p.

Let $\hat{x}_{N+p}$ be the p-step forecasting result, and $\boldsymbol{Y}_{k(1)}$, $\boldsymbol{Y}_{k(2)}$, …, $\boldsymbol{Y}_{k(L)}$ be L nearby vectors of $\boldsymbol{Y}_N$ in reconstruction space. Each one of $\boldsymbol{Y}_{k(l)}$ $(l=1,2,\ldots,L)$ and $\boldsymbol{Y}_N$ diverge from each other at exponential speed. From Eq. (11), we have

$$\|\boldsymbol{Y}_{N+p}-\boldsymbol{Y}_{k(l)+p}\| = \|\boldsymbol{Y}_N-\boldsymbol{Y}_{k(l)}\|\,\mathrm{e}^{\lambda(\boldsymbol{Y}_N,p)p\Delta t}. \tag{12}$$

Squaring each side of Eq. (12) and considering Eq. (1), we have

$$\sum_{i=0}^{m-1}(x_{N+p-i\tau}-x_{k(l)+p-i\tau})^2 = \|\boldsymbol{Y}_N-\boldsymbol{Y}_{k(l)}\|^2\,\mathrm{e}^{2\lambda(\boldsymbol{Y}_N,p)p\Delta t}. \tag{13}$$

For another referential vector $\boldsymbol{Y}_{k(l')}$ $(l\neq l')$, similar equation to Eq. (13) reads

$$\sum_{i=0}^{m-1}(x_{N+p-i\tau}-x_{k(l')+p-i\tau})^2 = \|\boldsymbol{Y}_N-\boldsymbol{Y}_{k(l')}\|^2\,\mathrm{e}^{2\lambda(\boldsymbol{Y}_N,p)p\Delta t}. \tag{14}$$

Subtracting each side of Eq. (13) by the corresponding side of Eq. (14), we obtain the following equation:

$$\begin{aligned}&\sum_{i=0}^{m-1}2[x_{k(l)+p-i\tau}-x_{k(l')+p-i\tau}]\cdot x_{N+p-i\tau}\\&+\left(\|\boldsymbol{Y}_N-\boldsymbol{Y}_{k(l)}\|^2-\|\boldsymbol{Y}_N-\boldsymbol{Y}_{k(l')}\|^2\right)\mathrm{e}^{2\lambda(Y_N,p)p\Delta t}\\&=\sum_{i=0}^{m-1}[x_{k(l)+p-i\tau}^2-x_{k(l')+p-i\tau}^2].\end{aligned} \tag{15}$$

In Eq. (15), $x_{N+p-i\tau}$ $(i=0,1,\ldots,m-1)$ is unknown at the N-th time point if and only if $N+p-i\tau>N$. Let m_0 be an integer that satisfies

$$\begin{aligned}&N+p-(m_0-1)\tau>N,\\&N+p-m_0\tau\le N.\end{aligned} \tag{16}$$

Equation (16) means $x_{N+p-i\tau}$ $(i=m_0,m_0+1,\ldots,m-1)$ are known and $x_{N+p-i\tau}$ $(i=0,1,\ldots,m_0-1)$ are unknown at the N-th time point. Equation (16) can be changed into

$$p\le m_0\tau<p+\tau, \tag{17}$$

then

$$m_0 = \lfloor p/\tau\rfloor, \tag{18}$$

where $\lfloor p/\tau\rfloor$ denotes the largest one of the integral numbers which are smaller than p/τ, for example, $\lfloor 1.2\rfloor=1$.

For Eq. (15), removing known items, including $x_{N+p-i\tau}$ $(i=m_0,m_0+1,\ldots,m-1)$, from the left side to the right side, we have

$$\begin{aligned}&\sum_{i=0}^{m_0-1}2[x_{k(l)+p-i\tau}-x_{k(l')+p-i\tau}]x_{N+p-i\tau}\\&+\left(\|\boldsymbol{Y}_N-\boldsymbol{Y}_{k(l)}\|^2-\|\boldsymbol{Y}_N-\boldsymbol{Y}_{k(l')}\|^2\right)\mathrm{e}^{2\lambda(Y_N,p)p\Delta t}\\&=\sum_{i=0}^{m-1}[x_{k(l)+p-i\tau}^2-x_{k(l')+p-i\tau}^2]\\&-\sum_{i=m_0}^{m-1}2[x_{k(l)+p-i\tau}-x_{k(l')+p-i\tau}]x_{N+p-i\tau}.\end{aligned} \tag{19}$$

All elements on the right side of Eq. (19) are known, $x_{N+p-i\tau}$ $(i=0,1,\ldots,m_0-1)$ and $\lambda(Y_N,p)$ on the left side are unknown at the N-th time point.

For any two different referential vectors $\boldsymbol{Y}_{k(l)}$ and $\boldsymbol{Y}_{k(l')}$ $(1\le l,l'\le L)$, we have a similar equation to Eq. (19). For all selected different vectors pairs in $\{\boldsymbol{Y}_{k(1)},\boldsymbol{Y}_{k(2)},\ldots,\boldsymbol{Y}_{k(L)}\}$, $L(L-1)$ equations similar to Eq. (19) are obtained. We use the least squares method to fit the common unknown elements in these equations. Defining

$$\boldsymbol{X} = \left[\begin{array}{cccc}x_{N+p} & \cdots & x_{N+p-(m_0-1)\tau} & \mathrm{e}^{2\lambda(\boldsymbol{Y}_N,p)p\Delta t}\end{array}\right]^{\mathrm{T}}, \tag{20}$$

$$\boldsymbol{A} = \left[\begin{array}{cccc}
2x_{k(1)+p}-2x_{k(2)+p} & \cdots & 2x_{k(1)+p-(m_0-1)\tau}- & \|\boldsymbol{Y}_N-\boldsymbol{Y}_{k(1)}\|^2-\\ & & 2x_{k(2)+p-(m_0-1)\tau} & \|\boldsymbol{Y}_N-\boldsymbol{Y}_{k(2)}\|^2\\
\vdots & \ddots & \vdots & \vdots\\
2x_{k(1)+p}-2x_{k(L)+p} & \cdots & 2x_{k(1)+p-(m_0-1)\tau}- & \|\boldsymbol{Y}_N-\boldsymbol{Y}_{k(1)}\|^2-\\ & & 2x_{k(L)+p-(m_0-1)\tau} & \|\boldsymbol{Y}_N-\boldsymbol{Y}_{k(L)}\|^2\\
2x_{k(2)+p}-2x_{k(3)+p} & \cdots & 2x_{k(2)+p-(m_0-1)\tau}- & \|\boldsymbol{Y}_N-\boldsymbol{Y}_{k(2)}\|^2-\\ & & 2x_{k(3)+p-(m_0-1)\tau} & \|\boldsymbol{Y}_N-\boldsymbol{Y}_{k(3)}\|^2\\
\vdots & \ddots & \vdots & \vdots\\
2x_{k(L-1)+p}-2x_{k(L)+p} & \cdots & 2x_{k(L-1)+p-(m_0-1)\tau}- & \|\boldsymbol{Y}_N-\boldsymbol{Y}_{k(L-1)}\|^2-\\ & & 2x_{k(L)+p-(m_0-1)\tau} & \|\boldsymbol{Y}_N-\boldsymbol{Y}_{k(L)}\|^2
\end{array}\right], \tag{21}$$

$$\boldsymbol{B}=\begin{bmatrix}\sum_{i=0}^{m-1}[x^2_{k(1)+p-i\tau}-x^2_{k(2)+p-i\tau}]-\sum_{i=m_0}^{m-1}2[x_{k(1)+p-i\tau}-x_{k(2)+p-i\tau}]x_{N+p-i\tau}\\ \vdots\\ \sum_{i=0}^{m-1}[x^2_{k(1)+p-i\tau}-x^2_{k(L)+p-i\tau}]-\sum_{i=m_0}^{m-1}2[x_{k(1)+p-i\tau}-x_{k(L)+p-i\tau}]x_{N+p-i\tau}\\ \sum_{i=0}^{m-1}[x^2_{k(2)+p-i\tau}-x^2_{k(3)+p-i\tau}]-\sum_{i=m_0}^{m-1}2[x_{k(2)+p-i\tau}-x_{k(3)+p-i\tau}]x_{N+p-i\tau}\\ \vdots\\ \sum_{i=0}^{m-1}[x^2_{k(L-1)+p-i\tau}-x^2_{k(L)+p-i\tau}]-\sum_{i=m_0}^{m-1}2[x_{k(L-1)+p-i\tau}-x_{k(L)+p-i\tau}]x_{N+p-i\tau}\end{bmatrix}, \tag{22}$$

then we have

$$\boldsymbol{AX}=\boldsymbol{B}, \tag{23}$$

hence

$$\boldsymbol{X}=(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{B}. \tag{24}$$

The first component of $\boldsymbol{X}$ is the p-step forecasting result $\hat{x}_{N+p}$ at the N-th time point.

4. Numerical simulations and discussion

To test the feasibility of our method, time series of two classic chaotic systems, Henon and Lorenz systems, are adopted as experimental objects. Equations of Henon reads

$$\begin{cases}x(n+1)=1-1.4x(n)^2+y(n),\\ y(n+1)=0.3x(n).\end{cases} \tag{25}$$

The starting value is (0,0), 5000-step values are generated from Eq. (25), and the previous 1000-step values are abandoned to diminish the influence of the starting value. The x-variable sequence is adopted as the experimental chaotic time series. Its embedding dimension $m=2$ and delaying time $\tau=1$ are determined by using the method of forecast error and the method of mutual information, respectively. Reconstructed vectors from the 2500-step values are regarded as referential vectors, and the following 1000-step values are used to test the forecasting capacity.

Lorenz system with the initial value (1, −1, 1) reads

$$\begin{cases}\dot{x}=10(y-x),\\ \dot{y}=28x-y-xz,\\ \dot{z}=-8/3z+xy.\end{cases} \tag{26}$$

Time series of all variables are obtained on the basis of Eq. (26) via 4-order Runge–Kutta method. The integral step is 0.01, the previous 10000-step values are abandoned, and the following 10000-step values of x-variable are adopted. The embedding dimension is $m=3$, and the delay time is $\tau=16$ for this time series. The 5000-step values are used to reconstruct the referential vectors, and the following 1000-step values are forecast with respect to different prediction steps.

In order to quantitatively assess and compare methods in this paper, two evaluation criterions, i.e., average absolute error (AAE)

$$e_{\mathrm{AAE}}=\frac{1}{N}\sum_{n=1}^{N}|\hat{x}_n-x_n|, \tag{27}$$

and average relative error (ARE)

$$e_{\mathrm{ARE}}=\frac{1}{N}\sum_{n=1}^{N}\frac{|\hat{x}_n-x_n|}{x_n} \tag{28}$$

are employed, where x_n and $\hat{x}_n$ denote actual values and predicted values, respectively, and N is the counting number of estimation.

4.1. One-step prediction

One-step prediction of Henon and Lorenz chaotic time series are conducted by a new method. In other words, we set $p=1$ in Eqs. (20)–(24). The number of referential state vectors of forecasting Henon and Lorenz time series are $L=10$ and $L=20$, respectively. Comparison between prediction values and real values of two time series are displayed in Figs. 1 and 2, respectively. High coherence between real values and predicted values in Figs. 1 and 2 indicates the validity of our method for one-step prediction.

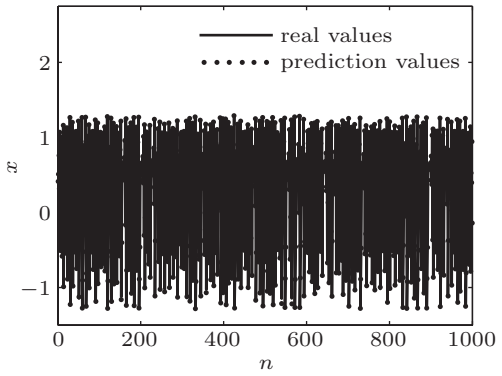


Fig. 1. Result of one-step predicted Henon time series.

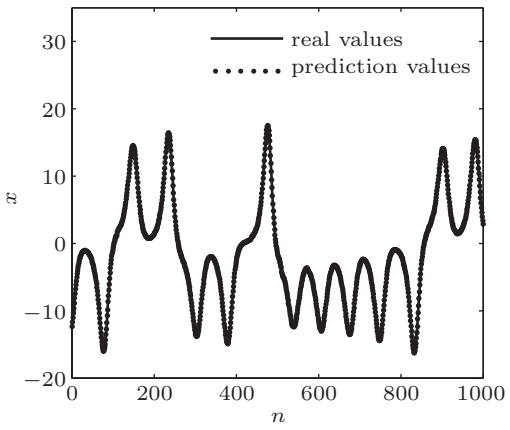


Fig. 2. Result of one-step predicted Lorenz time series.

Since there exist two results of the methods based on LE and LLE shown by Eq. (4) and Eq. (5), to compare our method with them, we choose the positive result in Eq. (4) and Eq. (5), and the real part be taken as the final result if it is a plural in the case that LE or LLE is underestimated. Comparison of forecasted Henon and Lorenz time series resulting from three methods is made, as shown in Tables 1 and 2, respectively.

Table 1. Evaluation of forecasting Henon time series by using three methods.

Error	Method based on LE	Method based on LLE	New method
AAE	0.0066	0.0060	0.0046
ARE	0.0412	0.0461	0.0177

Seen from Tables 1 and 2, the new method has smaller AAE and ARE than those of other methods. For Lorenz time series, AAE and ARE obtained by using the new method are approximately a quarter of that of the other two methods. Our new method outperforms the two older methods under two evaluation criterions.

Table 2. Evaluation of forecasting Lorenz time series by using three methods.

Error	Method based on LE	Method based on LLE	New method
AAE	0.1837	0.1799	0.0440
ARE	0.0659	0.0676	0.0177

From the above results, the new method has two main advantages compared with the two older methods. First, it does not need LE or LLE that is usually hard to obtain accurately for many systems, and ends with only one certain final result, while each of the two older methods has two forecasting results. Second, its forecasted result has higher accuracy.

4.2. Multi-step prediction

Results of the predicted Lorenz chaotic time series in the case of steps $p = 5, 10, 15, 20$ by using the new method, where the number of referential vectors is $L = 20$, are presented in Fig. 3. From Fig. 3(c), our method is effective when the forecasting step is $p = 15$, but the real values and the predicted values lack coherence if the forecasting step is $p = 20$, as shown in Fig. 3(d). Hence, this method is more suitable when the forecasting step p is smaller than the delaying time τ, because the delaying time of Lorenz time series is $\tau = 16$. In the case of $p > \tau$, $N + p - \tau > N$, then the number of columns of X in Eq. (20) is smaller than that of $p \geq \tau$.

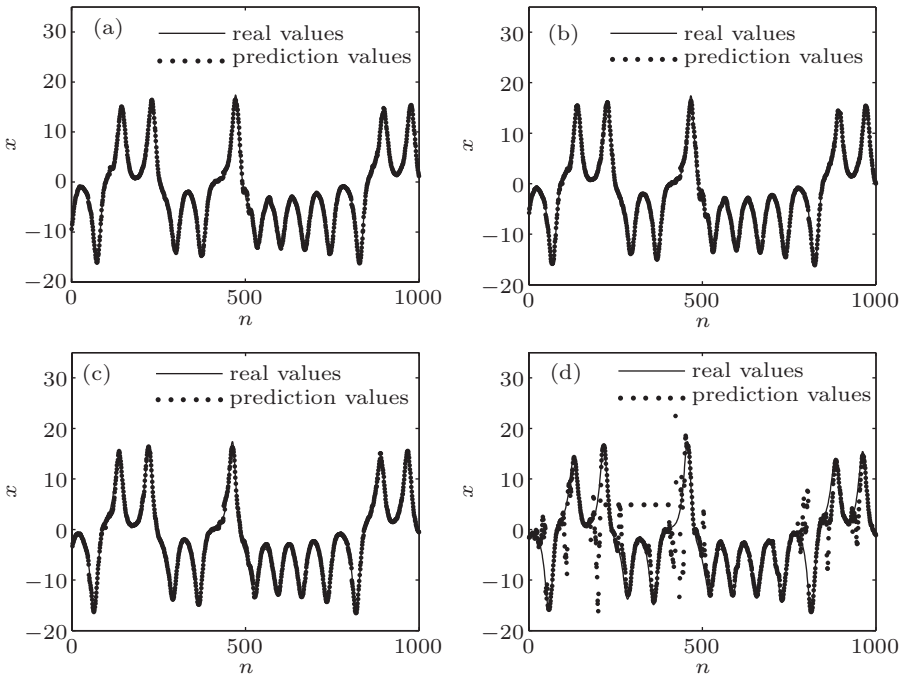


Fig. 3. Results of multi-step predicted Lorenz time series. (a) $p = 5$, (b) $p = 10$, (c) $p = 15$, (d) $p = 20$.

Except for the forecasting step p, the number of referential vectors L is another crucial parameter for this method. The AAE of forecasted Lorenz time series with different parameters L and p are depicted in Fig. 4. Figure 4 shows the influence of the number of referential vectors on the forecasting effect. There is an optimum number of selected referential vectors corresponding to the minimal AAE for $p = 1, 5, 10, 15, 20$. Either fewer or more referential state vectors will lead to larger AAE.

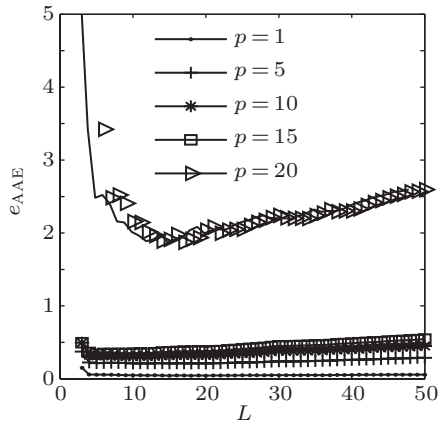


Fig. 4. AAE of predicted Lorenz time series in the case of using different numbers of referential vectors L and forecasting steps p.

The optimum numbers for different step numbers are shown in Fig. 5. For $1 \le p \le \tau$, it can be seen that the optimum number of referential state vectors nearly decreases linearly as the step number p increases. Thus, a relatively small L is more suitable if the forecasting step p is larger. For $p > \tau$, the tendency of L corresponding p is also the same as that of $p \le \tau$. We suggest using the parameter L in the following way:

$$L = L_0 \frac{N}{T}(\tau + 1 - p - m_0\tau), \tag{29}$$

where T is the cycle of chaotic time series, and L_0 is a coefficient satisfying $0.2 < L_0 < 0.5$. In Eq. (29), for larger N and smaller cycle of chaotic time series, more referential vectors can be adopted, and the factor $(\tau + 1 - p - m_0\tau)$ is obtained (see Fig. 5).

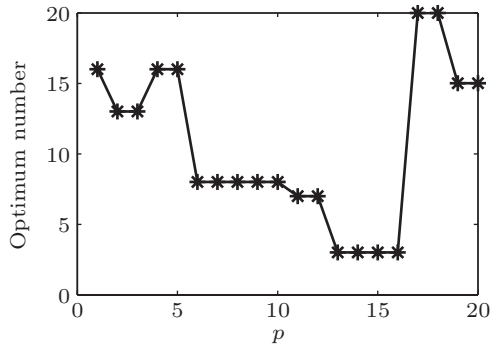


Fig. 5. Optimum numbers of selected referential vectors for different forecasting steps for predicting Lorenz time series.

4.3. Effect of adding noise

Forecasts of the Henon chaotic time series when adding white noise are carried out to test the influence of noise on forecasting precision and the optimum number of referential vectors. Figure 6 shows AAE of the predicted Henon time series when adding different white noise with different variances. Of course, larger variance results in a larger AAE and a larger optimum number of referential vectors.

The optimum numbers are 5, 22, 29, and 31 corresponding to 0, 0.01, 0.05, and 0.1 variance of added white noise, respectively. Therefore, vectors closer to the predicting point should be selected as referential vectors in the computation when the chaotic time series contains stronger noise.

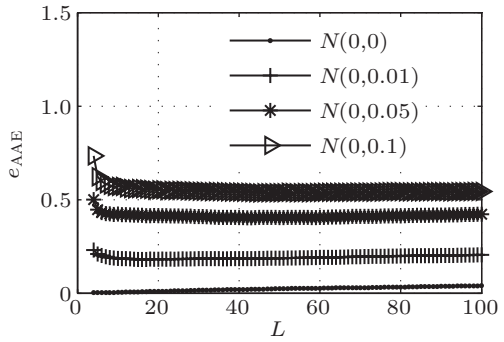


Fig. 6. AAE of forecasted Henon time series when adding different white noise.

5. Conclusion

We proposed a new method for predicating chaotic time series based on LLE. Many nearby state vectors are selected to deduce the predicting formulas based on the definition of LLE. The new method works without calculating LLE and ends with only one final result, while the method based on LE or LLE leads to two results. Numerical experiments of one- and multi-step predictions are carried out to test and verify its higher precision compared with two older methods. From the numerical experiments, the forecasting step and the number of referential state vectors are two influential parameters of the new method. An empirical formula for determining the optimum number of referential vectors is presented. It is better adopting more referential nearby vectors if stronger noise is contained in time series.

Two research tasks will be considered in the future. The first is to extend our method to predict multi-variable chaotic time series in the case of multi-variable reconstruction. From Eqs. (20) and (24), local Lyapunov exponent can be evaluated by using the last component of X, whether the estimation is accurate and whether its value has an influence on the forecasting precision will be studied.

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