Feedback control of wave segments in an excitable medium*

Wu Ning-Jie(吴宁杰)[a)†], Gao Hong-Jun(高洪俊)[a)], and Ying He-Ping(应和平)[b)]

[a)] *Jiangsu Key Laboratory for Numerical Simulation of Large Scale Complex Systems, School of Mathematical Sciences, Nanjing Normal University, Nanjing* 210046, *China*

[b)] *Institute of Modern Physics and Department of Physics, Zhejiang University, Hangzhou* 310027, *China*

(Received 7 September 2012; revised manuscript received 5 November 2012)

Depending on the excitability of the medium, a propagating wave segment will either contract or expand to fill the medium with spiral waves. This paper aims to introduce a simple mechanism of feedback control to stabilize such an expansion or contraction. To do this, we lay out a feedback control system in a block diagram and reduce it into a bare, universal formula. Analytical and experimental findings are compared through a series of numerical simulations of the Barkley model.

Keywords: feedback control, wave segments, excitable medium

PACS: 05.45.–a, 05.65.+b, 47.54.–r, 82.40.Bj **DOI:** 10.1088/1674-1056/22/5/050501

1. Introduction

A great deal of experimental and theoretical approaches have been pursued to control wave patterns in excitable medium,[1] due to their great potential for applications in chemical reactions and biological systems. For example, the Belousov–Zhabotinsky reaction can show a spiral-like pattern with macroscopic characteristic dimensions and sufficiently slow evolution,[2,3] while in cardiac tissue spiral waves and spatiotemporal chaos are generally recognized as the main cause of cardiac arrhythmias.[4,5] It is also believed that the excitability of the medium plays a significant role in the wave instability.[6–10] Therefore, many efforts have been made to manipulate the excitability by adding external signals, such as a periodical variation of distribution[11–14] in time and space. Recently, researchers have shown an increasing interest in feedback control (FD), although it was discovered many years ago.[15] However, despite their achievement, applications of FD to excitation waves have not been completely developed yet.[16–21]

Natural and artificial feedback control share certain features in common. They both compare the current output "Y" with some desired reference value "R" in order to generate a feedback signal, named an error "E". The error can be functionally adjusted "$C(E)$" (incorporating with an initially fixed offset "$I_{\rm ini}$") to produce an updated output. For example, control of spiral-wave dynamics usually uses the core location of a spiral wave as the feedback signal,[21,22] which can lead to a synchronized drift of the spiral wave. Considering that spiral waves originate from preexisting spirals or broken wave segments, the advantage of feedback control for stabilizing wave segments[23–27] has also been discussed recently. Various formulae have been deduced in these studies, however, most of them were attached to final quantity relations between the shape of the stabilized wave and the excitability after complicated procedures.

In this paper, we demonstrate a simple mechanism of stabilization of wave segments under feedback control. A formula is given by using a block diagram where we classify feedback signals into negative or positive. As for the output, Zykov *et al*.[26] chose the tip position of the wave along the axis orthogonal to the propagation direction, and Sakurai *et al*.[27] chose the integration over the entire domain of the field. However, in this paper we choose the total length of the wave segment instead. We will show that feedback control works well only because there exists a excitability limit of wave propagation. Beyond this limit, a wave segment would either expend or contract. Therefore, the excitability of the medium is chosen to be the input in this paper.

2. Feedback and dynamics

The first step of analyzing a feedback control system is to lay it out in a block diagram (see Fig. 1). Conventionally, signals along the main path flow from left to right, so the final output "Y" is on the right, i.e., the variable being controlled (e.g., the shape of propagation wave). The reference "R" is on the left: the compared variable. The total input sinal "I" that drives the whole system is introduced right before the system by incorporating an initial offset $I_{\rm ini}$ into the adjusted error (hereafter called the controller "$C(E)$"). With this setup, a control system can be fed with

$$I = I_{\rm ini} + C\left(\sigma(Y-R)\right), \tag{1}$$

*Project supported by the National Natural Science Foundation of China (Grant Nos. 11105074 and 11005026), the Natural Science Foundation of the Higher Education Institutions of Jiangsu Province, China (Grant Nos. 11KJB140004 and 11KJA110001), and the Qing Lan Project of Jiangsu Province, China.

†Corresponding author. E-mail: njwu@njnu.edu.cn

© 2013 Chinese Physical Society and IOP Publishing Ltd

where the sign of the controller C is the same as the sign of the error $E = \sigma(R - Y)$. Depending on the choice of system parameter, the output may change in the same (or opposite) direction as we increase the value of the input. The simple mechanism of stabilization of wave segments can be discussed in the light of two cases of feedback control: $\sigma = -1$ and $\sigma = +1$ (hereafter referred to as "FD−" and "FD+" control, respectively).

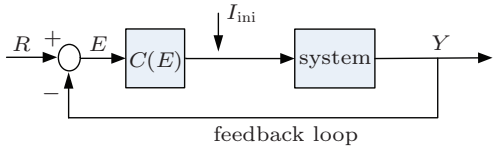


Fig. 1. A feedback control system in a block diagram with signal flowing from left to right.

Next, we go into more detail. The successful application of "FD−" control requires that an increase/decrease in the input I from a critical point $I_{\rm crit}$ results in an increase/decrease in the output Y.

For instance, let us initially set an offset to satisfy $I_{\rm ini} > I_{\rm crit}$. Suppose the control system starts at time t_0 and $Y(t_0) < R$, the controller C comparing the current output with the reference, generates an positive error $E = R - Y > 0$. Since the total input I is still larger than the critical point $I_{\rm ini} + C(E) > I_{\rm crit}$, the output continuously increases until it reaches the reference and conquers it. Then, the controller becomes negative and gets stronger with increasing output, which is favored as long as the condition $I_{\rm ini} + C(E) > I_{\rm crit}$ is satisfied. Eventually, subtracting over-valued $I_{\rm ini}$ from the negative error makes $I_{\rm ini} + C(E)$ equal to $I_{\rm crit}$. The output Y keeps constant. We hereafter refer to $I_{\rm ini} + C(E) = I_{\rm crit}$ as a balance condition.

If the control system starts with $Y(t_0) > R$ and $C(E) < 0$, there are two situations. If $I_{\rm ini} + C(E) < I_{\rm crit}$, the output decreases so that the controller increases accordingly, but is still negative until the balance condition $I_{\rm ini} + C(E) = I_{\rm crit}$ is satisfied. The output stops shrinking. If $I_{\rm ini} + C(E) > I_{\rm crit}$, the output increases, causing the controller to decrease until the balance condition is satisfied. Then, the output stops growing.

For $I_{\rm ini} < I_{\rm crit}$, the same analysis can be done as above. We conclude that whatever the situation is, the output will eventually satisfy the balance condition.

Different from "FD−" control, "FD+" control requires that an increase/decrease in the input from the critical value will result in an opposite change (decrease/increase) in the output. Although the discussion about "FD+" is neglected for simplification, the same result is obtained, predicting that any deviation from the critical point would ultimately revert to meet the balance condition $I_{\rm ini} + C(\sigma(Y - R)) = I_{\rm crit}$.

Finally, as a result of the "FD−" and "FD+" controls, the balance condition plays a key role in the stabilization of the output. According to this condition, the initial input has to be a critical one if we want to gain a "desirable" segment size of $Y = R$. This prediction also confirms the experimental findings that the output approaches infinity when a system excitability gets close to the limit, as shown by Fig. 3 in Ref. [25].

3. Simulations

We aim to show feedback control in excitable medium where a wave segment is initially generated. There are many numerical field equations of excitable medium which exhibit diverse phenomena. The choice of those equations depends on the different numerical procedures and kinetic schemes characterized by different behaviors. The disagreement partly reflects diverse uses of a small parameter "ε". To confirm our theory and simplify numerical procedures, we pick the Barkley model that allows us to picture a wave segment as an oriented curve by neglecting the narrow excitation zone. Therefore, the shape of a wave segment can be detected by isolating the excitation boundary.

Numerical experiments are carried out within the two-variable Barkley model describing the light sensitive BZ reaction

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{\varepsilon}\left[u(1-u)(u-\frac{v+b}{a})\right] + \nabla^2 u, \\ \frac{\partial v}{\partial t} &= u - v. \end{aligned} \tag{2}$$

Here, $\varepsilon \ll 1$ is a small positive parameter, reflecting the disparate time scales of the fast activator variable u and the slow inhibitor variable v; the v-dependent threshold $u_{\rm th} = (b+v)/a$ defines the excitability of the medium; $\varepsilon = 0.02$ is fixed in the following simulations. The particular parameters in this model (2) do not have a clear physiological meaning, but are adjusted to reproduce many experimentally determined features of spiral waves. Numerical simulations are performed within 256×256 grids of grad spacing $\Delta x = \Delta y = 0.39$ and time step $\Delta t = 0.02$, non-flux boundary condition is used.

Assuming that the thickness of the wavefront and the waveback is small, we define an iso-concentration curve of $u(\boldsymbol{X}, t) = u_{\rm th}$ $(u_0 < u_{\rm th} < u_e)$ to be the boundary of the excitable zone. The exact value of the threshold $u_{\rm th}$ is not important. The length of the boundary is determined by counting the pixels with an edge detection technique.

The (a,b)-phase diagram is shown in Fig. 2. The dotted line (hereafter referred to as the bifurcation line of instability) is composed of critical excitabilities. At each couple of the excitability, there is a wave segment in a particular shape. Any small perturbation below this line will cause the wave segment to grow until it forms a spiral wave, and that above this line will make the wave segment contract until they disappear. The (a,b)-phase diagram just meets the requirements for both the

"FD−" and "FD+" control, where parameter a is better to be an input variable for "FD−" and b for "FD+".

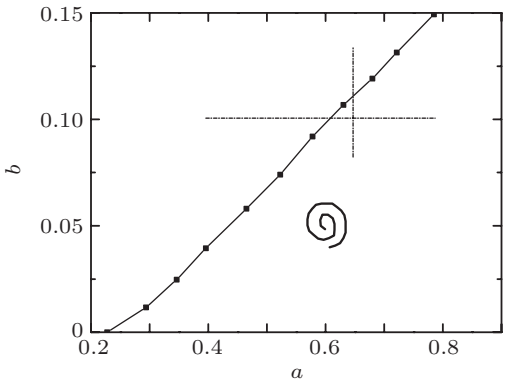


Fig. 2. Parameter phase diagram of the Barkley model. Dotted line: the points at which wave segment keeps stable, disappears in the above region, and grows to a spiral in the below.

We firstly illustrate the "FD−" control by drawing a horizontal dashed line in the phase diagram with $b = 0.1$ and choosing $a_{\rm crit}$ as the critical point of intersection of this line and the bifurcation line. Performing the same simulation three times: parameter a is located at the right side of $a_{\rm crit}$ for the first time, let $a = a_{\rm crit}$ for the second time, and finally a is located at the left side. We show in Fig. 3 that three corresponding cases of wave evolution occur: one stretches its two free arms in panel (a), one keeps the shape stable in panel (b), and one shrinks to virtual extinction in panel (c). In the numerical calculation, the curve length of wave segment L serves as the output "Y" (as defined by Eq. (1)). Wave stabilization can be easily realized by linearly adjusting the parameter

$$a = a_{\rm ini} + k_{\rm fd}(L_{\rm R} - L(t)). \tag{3}$$

The value of $a_{\rm ini}$ is picked within a small interval of point $a_{\rm crit}$, and $k_{\rm fd}$ is the amplitude of a feedback signal. We chooses the referenced length $L_{\rm R}$ to be the instantaneous length $L(t_0)$ of the wave segment at time t_0, denoted by an arrow in Fig. 4. Both kinds of wave segments (i.e., the expending case in Fig. 3(a) and the contracting case in Fig. 3(c)) are stabilized with their final wave lengths being equal to $L_{\rm R} - (a_{\rm crit} - a_{\rm ini})/k_{\rm fd}$, which is consistent with the prediction of the balance condition in the previous section.

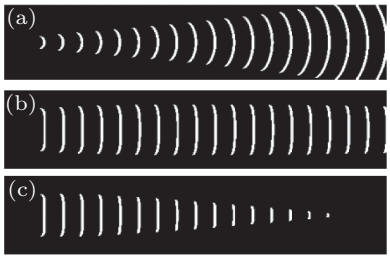


Fig. 3. Simulation of wave segment evolution for three different values of parameter a: (a) 0.623, (b) 0.22117, (c) 0.619 at fixed $b = 0.1$. The images in each panel represent an overlay of snapshots taken every $200\delta t$ time.

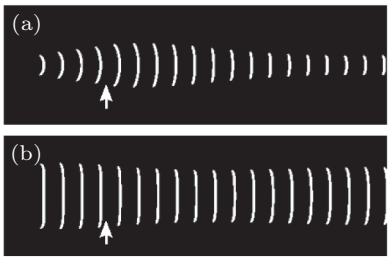


Fig. 4. Two types of wave segments corresponding to (a) Fig. 2(a) and (b) Fig. 2(c) are stabilized for $a = a_{\rm ini} + k_{\rm fd}(L_{\rm R} - L(t))$. Here, $k_{\rm fd} = 0.0002\Delta x$ and $L_{\rm R} = L(200\Delta t)$.

In the case of the "FD+" control, the parameter b is now set to be the input variable. Simulations are performed under the vertical dashed line (see the (a,b)-phase diagram), where $b_{\rm crit} = 0.1086$ is the critical point. As shown in Fig. 5, there are three corresponding cases of wave evolutions: expansion, stabilization, and contraction. The stabilization of wave segments is now realized by adjusting the parameter b to

$$b = b_{\rm ini} + k_{\rm fd}(L_{\rm t} - L_{\rm R}). \tag{4}$$

As we can see in Fig. 6, both the expending and the contracting wave segment are stabilized under the "FD+" control.

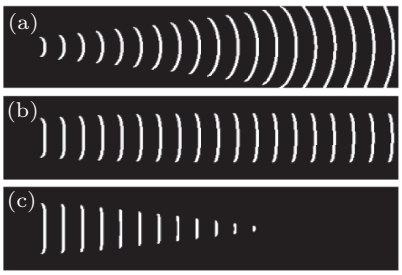


Fig. 5. Simulation of wave segment evolution for three different values of parameter b: (a) 0.108, (b) 0.1086, (c) 0.11 at fixed $a = 0.65$.

The balance condition of a feedback control system is a bare and universal form. It holds true for other types of the output: the size of the excitable area[24] and the location $Y_{\rm max}(t)$.[26] When performing the numerical simulations, the size of the excitable area corresponding to the activator u, is measured by using $S = \sum_{x,y}(u(x,y) - u_{\rm th})$, and the free tip of the wave segment is calculated to be the point with maximum curvature. Simulations by feeding back these different types of output variables tell that they all exhibit the same scenes by feeding back the wave length.

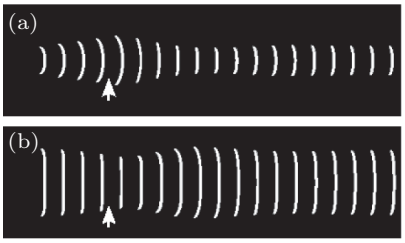


Fig. 6. Two types of wave segments corresponding to (a) Fig. 5(a) and (b) Fig. 5(c) are stabilized for $b = b_{\rm ini} + k_{\rm fd}(L_{\rm R} - L(t))$. Here, $k_{\rm fd} = 0.0002\Delta x$ and $L_{\rm R} = L(200\Delta t)$.

Balance conditions of the parameter a in the "FD−" control and b in the "FD+" control can be unified and reformed into $I_{\rm crit} - I_{\rm ini} = k_{\rm fd}(\sigma(L(t_{\rm u}) - L_{\rm R}))$, and $t_{\rm u}$ denotes the ultimate time when the wave segment is stabilized. A small perturbation of $I_{\rm ini}$ from the critical value $I_{\rm ini}$ results in a $k_{\rm fd}$-timed variation of the ultimate difference $L_{\rm tu} - L_{\rm R}$. Figures 4 and 6 confirm this prediction that the last pieces of wave segments change a little, compared to the size of those denoted with arrows. Moreover, in this paper we aim to stabilize wave segments instead of spirals. Therefore, the variation of $I_{\rm ini}$ should be limited to a small region of $I_{\rm crit}$. Otherwise, the wave segment would either shrink or grow so fast that the feedback control has no time to stop it. If a spiral wave is generated (as is commonly believed that only the region of the excitability that the spiral's tip visits matters for its motion), then the feedback signal may be proportional to a system variable averaged over a certain integration domain (i.e., a portion of a spiral wave instead of the whole). In this case, a resonant spiral wave drift occurs.

4. Conclusion

In conclusion, we have presented a simple block diagram for a feedback control system and reduce it into a bare universal form. For further analysis, we have obtained a stabilization condition. Besides, a feedback control system is classified into different systems under the "FD−" and the "FD+" control, respectively. Increasing input results in increasing output in the case of the "FD−" control, but leads to an opposite direction for the "FD+" control. Feedback control is applied to the dynamics of a propagating wave segment where there exits a certain critical point of the excitability. With the excitability beyond the critical point, wave segments either grow to fill the medium with spiral or contract until they disappear. Simulation results show that both the "FD−" and "FD+" control can stabilize an expending or contracting wave in order to meet the same balance condition as that obtained in the theoretical section. We expect more applications in feedback control in the future.

References

[1] Mikhailov A and Showalter K 2006 *Phys. Rep.* **425** 79
[2] Winfree A T 1972 *Science* **175** 634
[3] Zhabotinsky A M and Zaikin A N 1973 *J. Theor. Bio.* **40** 45
[4] Winfree A T 1994 *Science* **266** 1003
[5] Alonso S, Sagués F and Mikhailov A S 2003 *Science* **299** 1722
[6] Ma J, Wang C N, Jin W Y and Wu Y 2010 *Appl. Math. Comput.* **217** 3844
[7] Panfilov A V, Keldermann R H and Nash M P 2007 *Proc. Natl. Acad. Sci.* **104** 7922
[8] Faridon A 2010 *Appl. Math. Comput.* **217** 3385
[9] Henry H 2004 *Phys. Rev. E* **70** 026204
[10] Lenka S, Elena S and Hana S 2005 *Biophys. Chem.* **113** 269
[11] Zhang H, Bambi H and Hu G 2003 *Phys. Rev. E* **68** 026134
[12] Zykov V S, Steinbock O and Muller S C 1994 *Chaos* **4** 509
[13] Braune M, Schrader A and Engel H 1994 *Chem. Phys. Lett.* **222** 358
[14] Jia X, Zhou L Q and Ouyang Q 2004 *Chin. Phys. Lett.* **21** 435
[15] Franklin G, Powell J D and Emami-Naeini A 2005 *Feedback Control of Dynamic Systems* (6th edn.) (Prentice Hall)
[16] Luo X S, Gao Y, Wang B H, Feng J and Yuan G 2001 *Chin. Phys.* **10** 17
[17] Schlesner J, Zykov V, Engel H and Schöll E 2006 *Phys. Rev. E* **74** 046215
[18] Grill S, Zykov V S and Müller S C 1995 *Phys. Rev. Lett.* **75** 3368
[19] Zykov V S and Engel H 2004 *Physica D* **199** 243
[20] Guo W Q, Qiao C, Zhang Z M, Ouyang Q and Wang H L 2010 *Phys. Rev. E* **81** 056214
[21] Schlesner J, Zykov V S, Brandtstädter H, Gerdes I and Engel H 2008 *New J. Phys.* **10** 015003
[22] Wu N J, Gao H J and Ying H P 2010 *Phys. Rev. E* **82** 066206
[23] Mihaliuk E, Sakurai T, Chirila F and Showalter K 2002 *Faraday Discuss.* **120** 383
[24] Sakurai T, Osaki K and Tsujikawa T 2008 *Physica D* **237** 3165
[25] Mihaliuk E, Sakurai T, Chirila F and Showalter K 2002 *Phys. Rev. E* **65** 065602
[26] Zykov V S 2008 *Eur. Phys. J. Spec. Top.* **157** 209
[27] Sakurai T and Osaki K 2008 *Commun. Nonlinear Sci. Numer. Simul.* **13** 1067