Superposition solitons in two-component Bose–Einstein condensates*

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We develop the Hirota bilinear method and obtain the exact one and two superposition soliton solutions for two-component Bose–Einstein condensates. The conversion of three kinds of solitons including the superposition solitons, bright–bright solitons, and dark–bright solitons is discussed. With the energy analysis, we find that the superposition soliton state is an excitation state for this system. Moreover, the collision of two superposition solitons is found to be elastic.

Keywords: superposition solitons, excitation states, Hirota bilinear method

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1. Introduction

Owing to the different propagating velocities of different wave components, waves tend to expand in dispersive linear media. However, in many nonlinear media, certain wave envelopes (solitons) propagate undistorted because of the balance between dispersion and nonlinearity.[1–3] Solitons as nonlinear wave phenomena are observed in many physical systems, such as the surface of water, nonlinear transmission lines, and optical fibers.[4] Particularly, the realization of a solitary wave in Bose–Einstein condensate (BEC) strongly stimulates a tremendous number of explorations. At zero temperature, the dynamics of BEC is well described by the time-dependent Gross–Pitaevskii (GP) equation, and the nonlinearity results from the interatomic interactions. Either dark[5–15] or bright soliton[12–15] solutions are admitted in BEC depending on the repulsive or attractive nature of the interatomic interactions. Dark solitons have been researched in condensates, and they can only present in the condensates. The bright solitons can travel over much longer distances, and they are condensates themselves.[16,17] Naturally, there are growing interests in the bright and the dark soliton train dynamics of the BEC with time varying scattering lengths.[18,19]

As a result of controlling the intra- and inter-species interaction strengths via the Feshbach resonances and the state-dependent potentials,[20,21] many investigations are no longer confined to the study of the single-component BEC. Multi-component BECs possess complicated quantum phases[22] and combined soliton solutions.[23–25] It is far from being an insignificant extension of the single-component one, and presents originally and fundamentally different scenarios about its ground-state wavefunctions and excitations.[26–29] For example, a dark soliton and a bright soliton may coexist in a two-component BEC. In general, the two-component BECs are realized on binary mixtures of alkalis, such as different alkalis ^{87}Rb–^{23}Na, different isotopes ^{87}Rb–^{85}Rb, and different hyperfine states of the same alkali $|F=1,m_F=1\rangle$ and $|F=2,m_F=-1\rangle$ of ^{87}Rb.[30–32] The so-called dark–bright (DB) solitons exist owing to that the dark-soliton component creates a trapping mechanism, which allows the bright-soliton component to be formed through the inter-species interaction.[33–35] Such structures have recently been observed in experiments carried out in optics and BECs.[36–39] Nevertheless, the problem of the superposition solitons (SPSs), i.e., dark and bright solitons in different component of a two-component BEC, has not been addressed in detail so far. In this paper, we aim to study this problem by using the Hirota bilinear method.

Consider a quasi-one-dimensional two-component BEC with attractive interactions between atoms, which can be described by the dimensionless coupled GP equations[15]

$$\begin{aligned} \mathrm{i}\frac{\partial\phi_1}{\partial t} &= -\frac{1}{2}\frac{\partial^2\phi_1}{\partial x^2} - g\left(\frac{a_{11}}{a_{12}}|\phi_1|^2 + |\phi_2|^2\right)\phi_1, \\ \mathrm{i}\frac{\partial\phi_2}{\partial t} &= -\frac{1}{2}\frac{\partial^2\phi_2}{\partial x^2} - g\left(\frac{a_{22}}{a_{12}}|\phi_2|^2 + |\phi_1|^2\right)\phi_2, \end{aligned} \tag{1}$$

where ϕ_j $(j=1,2)$ is measured in units of $\sqrt{n_0}$, with n_0 being the maximum density for the initial distribution of the condensate. Parameters a_{jj} $(j=1,2)$ and a_{12} are the scattering lengths of intra- and inter-species atomic collisions, respectively, and $g=4\pi n_0 l_0^2|a_{12}|$ is the interaction constant. Generally, Eq. (1) is not integrable. However, under the condition $a_{11}=a_{22}=a_{12}$, the coupled equations are equivalent to an integrable Manakov system.[35] In this case, we will get exact one and two SPSs. The one SPS means the superposition of one dark soliton and one bright soliton in each component, and the two SPSs mean the superposition of two dark solitons

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and two bright solitons. The SPS solutions can be easily transformed to bright–bright (BB) soliton solutions. In addition, we find that the BB soliton state is the ground state through comparing its energy with that of two other kinds of solitons. We get a phase separation in the process that the ground state jumps to the excitation state. A detailed analysis of the asymptotic behavior of the solutions shows that the collision of two SPSs is elastic.

2. The developed Hirota method and superposition solitons

One of the most famous methods to construct soliton solutions in two-component BECs is the Hirota bilinear method. In order to clarify the derivation of solutions, we introduce its main idea briefly.[40] The first step of this method is to make a direct transformation to the nonlinear equations. Then in terms of some bilinear operators, the nonlinear equations can be decoupled into a set of equations, from which the SPS can be obtained effectively. Performing the normal procedure, we consider the complex functions $G(x,t)$, $H(x,t)$ and the real function $F(x,t)$ forming the transformation

$$\phi_1 = \frac{G}{F}, \quad \phi_2 = \frac{H}{F}. \tag{2}$$

Substituting Eq. (2) into Eq. (1), we have

$$\begin{aligned} 0 &= F\left(\mathrm{i}\mathrm{D}_t + \frac{1}{2}\mathrm{D}_x^2\right) G\cdot F \\ &\quad - G\left[\frac{1}{2}\mathrm{D}_x^2 F\cdot F - g\left(|G|^2+|H|^2\right)\right], \end{aligned} \tag{3}$$

$$\begin{aligned} 0 &= F\left(\mathrm{i}\mathrm{D}_t + \frac{1}{2}\mathrm{D}_x^2\right) H\cdot F \\ &\quad - H\left[\frac{1}{2}\mathrm{D}_x^2 F\cdot F - g\left(|G|^2+|H|^2\right)\right], \end{aligned} \tag{4}$$

where D_t and D_x^2 are called the Hirota bilinear operators and defined by

$$\begin{aligned} &\mathrm{D}_x^m \mathrm{D}_t^n G(x,t)\cdot F\left(x',t'\right) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m, \\ &\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n G(x,t) F\left(x',t'\right)|_{x=x',t=t'}. \end{aligned} \tag{5}$$

Equations (3) and (4) can be decoupled easily into two equations respectively. In order to get the exact SPS for Eq. (1), we determine a real parameter λ, which is added to the Hirota bilinear operators, and then many attempts show that equations (3) and (4) can be decoupled into

$$\begin{aligned} \hat{\mathscr{A}_1} G\cdot F &= 0, \quad \hat{\mathscr{A}_1} H\cdot F = 0, \\ \hat{\mathscr{A}_2} F\cdot F &= g\left(|G|^2+|H|^2\right), \end{aligned} \tag{6}$$

where the Hirota bilinear operators $\hat{\mathscr{A}_1}$ and $\hat{\mathscr{A}_2}$ are given by

$$\hat{\mathscr{A}_1} = \mathrm{i}\mathrm{D}_t + \frac{1}{2}\mathrm{D}_x^2 - \lambda, \tag{7}$$

$$\hat{\mathscr{A}_2} = \frac{1}{2}\mathrm{D}_x^2 - \lambda. \tag{8}$$

In the following sections, we will obtain the SPS by constructing the expressions of $G(x,t)$, $H(x,t)$, and $F(x,t)$ in Eq. (6) skillfully.

2.1. One superposition soliton

Different form the classical procedure of the Hirota method, the expressions of $G(x,t)$, $H(x,t)$, and $F(x,t)$ are respectively assumed to be

$$G = Q_0\left(1+\chi^2 Q_2\right) + \chi Q_1, \tag{9}$$

$$H = Q_0\left(1+\chi^2 Q_2\right) - \chi Q_1, \tag{10}$$

$$F = 1 + \chi^2 F_2, \tag{11}$$

which admit the novel superposition soliton solutions of Eq. (1). In Eqs. (9)–(11), χ is an arbitrary auxiliary parameter, which will be absorbed in the expressions of solutions. Substituting Eqs. (9)–(11) into Eq. (6) and collecting the coefficients with the same powers of χ, we have 1) for the coefficient of χ^0,

$$0 = \hat{\mathscr{A}_1} Q_0\cdot 1, \tag{12}$$

$$0 = \hat{\mathscr{A}_2} 1\cdot 1 - 2g|Q_0|^2, \tag{13}$$

2) for the coefficient of χ,

$$0 = \hat{\mathscr{A}_1} Q_1\cdot 1, \tag{14}$$

3) for the coefficient of χ^2,

$$0 = \hat{\mathscr{A}_1}\left(Q_0 Q_2\cdot 1 + Q_0\cdot F_2\right), \tag{15}$$

$$\begin{aligned} 0 &= \hat{\mathscr{A}_2}\left(F_2\cdot 1 + 1\cdot F_2\right) \\ &\quad - 2g\left[|Q_0|^2\left(Q_2+\bar{Q}_2\right) + |Q_1|^2\right], \end{aligned} \tag{16}$$

4) for the coefficient of χ^3,

$$0 = \hat{\mathscr{A}_1} Q_1\cdot F_2, \tag{17}$$

5) for the coefficient of χ^4,

$$0 = \hat{\mathscr{A}_1} Q_0 Q_2\cdot F_2, \tag{18}$$

$$0 = \hat{\mathscr{A}_2} F_2\cdot F_2 - 2g|Q_0|^2|Q_2|^2, \tag{19}$$

where the overbar denotes the complex conjugate.

Using the definition of Hirota bilinear operator (5), we can express the above equations in detail. Solving Eqs. (12) and (13), we get

$$Q_0 = \gamma_0 \mathrm{e}^{\mathrm{i}\eta_0}, \tag{20}$$

where $\eta_0 = k_0 x - \left(k_0^2/2 + \lambda\right)t$ with a real k_0 and the complex parameter γ_0 satisfying $|\gamma_0|^2 = -\lambda/(2g)$, which shows that the

existing of the SPS demands $\lambda < 0$. From Eq. (14), we obtain the expression of Q_1

$$Q_1 = \alpha_1 \mathrm{e}^{\eta_1}, \tag{21}$$

where $\eta_1 = k_1 x + \mathrm{i}\left(k_1^2/2 - \lambda\right)t$, with k_1 being complex. The parameter α_1 to be determined is an arbitrary complex parameter. Expanding Eqs. (15) and (16) with the definition of Eq. (5), we can see that Q_2 and F_2 admit the expressions

$$Q_2 = Z_1 \mathrm{e}^{\eta_1 + \bar{\eta}_1 + R_1}, \tag{22}$$

$$F_2 = \mathrm{e}^{\eta_1 + \bar{\eta}_1 + R_1}, \tag{23}$$

where the complex parameter Z_1 is to be determined, and R_1 is a real parameter. Substituting Eqs. (20)–(23) into Eqs. (15) and (16), we have

$$\begin{aligned} 0 = {} & \frac{1}{2}\left(\bar{k}_1^2 - k_1^2\right)(Z_1 - 1) + \mathrm{i}k_0\left(k_1 + \bar{k}_1\right)(Z_1 - 1) \\ & + \frac{1}{2}\left(k_1 + \bar{k}_1\right)^2 (Z_1 + 1), \end{aligned} \tag{24}$$

$$\begin{aligned} 0 = {} & \mathrm{e}^{R_1}\left(k_1 + \bar{k}_1\right)^2 - 2\lambda\,\mathrm{e}^{R_1} - 2g|\alpha_1|^2 \\ & - 2g|\gamma_0|^2\left[Z_1 \mathrm{e}^{R_1} + \bar{Z}_1 \mathrm{e}^{R_1}\right]. \end{aligned} \tag{25}$$

In the case of $Z_1 \neq \pm 1$ (the case of $Z_1 = \pm 1$ will be discussed below), we obtain

$$Z_1 = \frac{\mathrm{i}k_0 - k_1}{\mathrm{i}k_0 + \bar{k}_1}, \tag{26}$$

$$\mathrm{e}^{R_1} = \frac{2g|\alpha_1|^2}{\left(k_1 + \bar{k}_1\right)^2}\left(1 + \frac{2g|\gamma_0|^2}{|k_1 - \mathrm{i}k_0|^2}\right)^{-1}, \tag{27}$$

which implies $|Z_1|^2 = 1$. In terms of Eqs. (20)–(23), we can find that equations (17)–(19) are satisfied to the moment.

With the help of Eqs. (2) and (20)–(23), by absorbing parameter χ, the one SPS of Eq. (1) can be derived as

$$\begin{aligned} \phi_j = {} & \frac{1}{2}\gamma_0 \mathrm{e}^{\mathrm{i}\eta_0}\left[(1 + Z_1) - (1 - Z_1)\tanh\left(\theta_1 + \frac{R_1}{2}\right)\right] \\ & + (-1)^{3-j}\frac{\alpha_1}{2\mathrm{e}^{\frac{R_1}{2}}}\mathrm{e}^{\mathrm{i}\varphi_1}\,\mathrm{sech}\left(\theta_1 + \frac{R_1}{2}\right), \end{aligned} \tag{28}$$

where $j = 1, 2$ and

$$\varphi_1 = k_{1\mathrm{I}} x + \frac{1}{2}\left(k_{1\mathrm{R}}^2 - k_{1\mathrm{I}}^2 - 2\lambda\right)t, \tag{29}$$

$$\theta_1 = k_{1\mathrm{R}}\left(x - k_{1\mathrm{I}} t\right). \tag{30}$$

In the above equations, suffixes R and I denote the real and the imaginary parts, respectively. From the above solution (28), we can see two special cases, i.e., $Z_1 = \pm 1$. When $Z_1 = 1$, parameter k_1 will become a pure imaginary number, and solution (28) degenerates to a plane-wave solution $\phi_{1,2} = \gamma_0 \exp(\mathrm{i}\eta_0)$. It implies that the exact solution (28) describes a vector soliton embedded in a plane-wave background in the two-component BECs. When $Z_1 = -1$, equation (28) reduces to superposition solutions of black and bright solitons in the form

$$\begin{aligned} \phi_j = {} & -\gamma_0 \mathrm{e}^{\mathrm{i}\eta_0}\tanh\left(\theta_1 + \frac{R_1}{2}\right) \\ & + (-1)^{3-j}\frac{\alpha_1}{2\mathrm{e}^{\frac{R_1}{2}}}\mathrm{e}^{\mathrm{i}\varphi_1}\,\mathrm{sech}\left(\theta_1 + \frac{R_1}{2}\right), \end{aligned}$$

where $j = 1, 2$, $\varphi_1 = k_0 x + \left(k_{1\mathrm{R}}^2 - k_0^2 - 2\lambda\right)t/2$, and $\mathrm{e}^{R_1} = g|\alpha_1|^2 / \left[2(k_{1\mathrm{R}}^2 - \lambda)\right]$. The θ_1 and other parameters are given before. In particular, we find that when $\gamma_0 = 0$, the one SPS for Eq. (28) can be simplified to a special one-BB soliton solution of the two-component BECs in the form of $\phi_j = (-1)^{3-j}\left[\alpha_1 \mathrm{e}^{\mathrm{i}\varphi_1}\,\mathrm{sech}\left(\theta_1 + R_1/2\right)\right] / \left(2\mathrm{e}^{R_1/2}\right)$, $j = 1, 2$. Here φ_1 and e^{R_1} are reduced to $\varphi_1 = k_{1\mathrm{I}} x + \left(k_{1\mathrm{R}}^2 - k_{1\mathrm{I}}^2\right)t/2$ and $\mathrm{e}^{R_1} = g|\alpha_1|^2 / \left(2k_{1\mathrm{R}}^2\right)$, respectively, and θ_1 is given in Eq. (30). It implies that we can conveniently get another form of the solution (BB soliton) of Eq. (1) from Eq. (28).

As discussed above, there are one BB soliton solution, DB soliton solution, and SPS in the two-component BECs with attractive interactions between atoms. The BB soliton state can be viewed as the ground state of this system, and the other two kinds of solitons represent the excited states. For a better understanding, their energies will be studied in detail. The energy is expressed as

$$\begin{aligned} E = {} & \int_{-\infty}^{+\infty}\left[-\frac{1}{2}\bar{\phi}_1\frac{\mathrm{d}^2\phi_1}{\mathrm{d}x^2} - \frac{1}{2}\bar{\phi}_2\frac{\mathrm{d}^2\phi_2}{\mathrm{d}x^2}\right. \\ & \left. - \frac{g}{2}\left(|\phi_1|^2 + |\phi_2|^2\right)^2\right]\mathrm{d}x. \end{aligned} \tag{31}$$

The BB soliton energy can be easily obtained as $E_{\mathrm{BB}} = -k_{1\mathrm{R}}^3/(3g) + k_{1\mathrm{I}}^2 k_{1\mathrm{R}}/g$ from the above expression. However, the integrals of the other two kinds of soliton energies are infinite, so we modify the integrals to remove the corresponding contributions of the continuous wave background. We get the absolute energies of the DB soliton and the SPS

$$E_{\mathrm{DB}} = -\frac{k_{1\mathrm{R}}^3}{3g} + \frac{k_{1\mathrm{I}}^2 k_{1\mathrm{R}}}{g} + \frac{n_0 k_{1\mathrm{R}}}{N} + \frac{k_0^2}{2} - \frac{3n_0 g}{2N},$$

$$E_{\mathrm{SPS}} = -\frac{k_{1\mathrm{R}}^3}{3g} + \frac{k_{1\mathrm{I}}^2 k_{1\mathrm{R}}}{g} + \frac{2k_{1\mathrm{R}} n_0}{N} + k_0^2 - \frac{2gn_0}{N},$$

where the soliton velocity and width are defined by $k_{1\mathrm{I}}$ and $W = 1/k_{1\mathrm{R}}$, respectively, and N is the number of particles for each component of the system. The expressions show that the energy will change with the velocity and the width of the soliton when the soliton phase and scattering length are both given. It is clear that the energy increases when the velocity and the width increase (see Fig. 1). From the energy expressions, we find that the ground state (BB soliton state) has the lowest energy for this system, and the energies of the first (DB soliton state) and the second (SPS state) excited states increase

as $n_0(2k_{1R}-3g)/(2N)+k_0^2/2$ and $2n_0(k_{1R}-g)/N+k_0^2$, respectively. The transition energy is $n_0(2k_{1R}-g)/(2N)+k_0^2/2$ from the DB soliton state to the SPS state. Rich phase separation behaviors can be observed in the two-component BECs, which are driven primarily by the nonlinear interactions between the different atomic species or states that make up the condensates. A quantum phase transition is a fundamental change in the character of a system's ground state when some parameters of its Hamiltonian pass through critical values.[41,42] From the analysis of the energy, we find that the phase separation can be obtained for this system in a simple way, even the miscibility of the two-component BECs is determined by the scattering lengths. Initially the BECs described by the ground-state solution (BB soliton solution) have a uniform phase, as shown in Fig. 2(a). We modify the phase distribution of the BECs by adding energy. During this process, the BECs will jump to the DB state shown in Fig. 2(b) and the SPS state shown in Fig. 2(c) in turn, while each single state of the two pure condensates has a slight offset in space, i.e., the two densities do not coincide with each other, so there are phase differences. This novel discovery will play a directive role for the further researching about the two-component BECs in experiments.

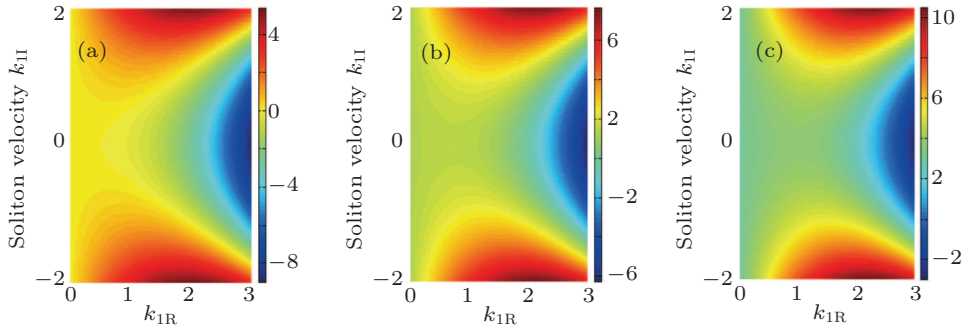


Fig. 1. (color online) Column energies of (a) the BB solitons, (b) the DB solitons, and (c) the SPS change with the soliton width and velocity. The parameters are $g=1$, $n_0/N=0.5$, and $k_0=2$.

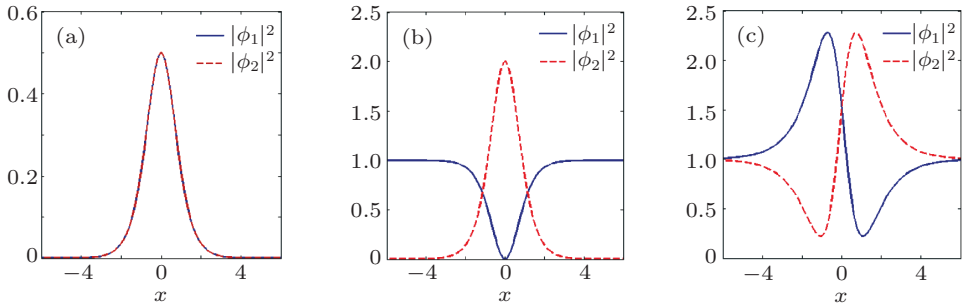


Fig. 2. (color online) Densities of components 1 and 2 are compared for $k_1=1+0.3\mathrm{i}$, $g=1$, $k_0=0.3$, and (a) $\gamma_0=0$, $\alpha_1=1+\mathrm{i}$, (b) $\gamma_0=1$, $\alpha_1=2+2\mathrm{i}$, (c) $\gamma_0=1$, $\alpha_1=2+\sqrt{2}\mathrm{i}$. Panels (a)–(c) correspond to the BB soliton state, the DB soliton state, and the SPS state at $t=0$, respectively.

2.2. Interaction of two superposition solitons

In this section, we will give the analytical expressions of two superposition solitons of Eq. (1). In this case, $G(x,t)$, $H(x,t)$, and $F(x,t)$ of Eq. (2) are assumed to be

$$G = Q_0\left(1+\chi^2Q_2+\chi^4Q_4\right)+\chi Q_1+\chi^3Q_3, \tag{32}$$

$$H = Q_0\left(1+\chi^2Q_2+\chi^4Q_4\right)-\chi Q_1-\chi^3Q_3, \tag{33}$$

$$F = 1+\chi^2F_2+\chi^4F_4, \tag{34}$$

where Q_0 has been obtained in Eq. (20). Employing a procedure similar to that in the above section, we obtain the following set of equations from Eq. (6): 1) for the coefficient of χ,

$$0=\hat{\mathscr{A}}_1Q_1\cdot 1, \tag{35}$$

2) for the coefficient of χ^2,

$$0 = \hat{\mathscr{G}}_1\left(Q_2\cdot 1+1\cdot F_2\right), \tag{36}$$

$$0 = \hat{\mathscr{A}}_2\left(F_2\cdot 1+1\cdot F_2\right) - 2g\left[|Q_1|^2+|Q_0|^2\left(Q_2+\bar{Q}_2\right)\right], \tag{37}$$

3) for the coefficient of χ^3,

$$0=\hat{\mathscr{A}}_1\left(Q_3\cdot 1+Q_1\cdot F_2\right), \tag{38}$$

4) for the coefficient of χ^4,

$$0 = \hat{\mathscr{G}}_1\left(Q_4\cdot 1+Q_2\cdot F_2+1\cdot F_4\right), \tag{39}$$

$$0 = \hat{\mathscr{A}}_2\left(F_4\cdot 1 + F_2\cdot F_2 + 1\cdot F_4\right) - 2g\left[|Q_0|^2\left(Q_4 + |Q_2|^2 + \bar{Q}_4\right) + Q_3\bar{Q}_1 + Q_1\bar{Q}_3\right], \quad (40)$$

5) for the coefficient of χ^5,

$$0 = \hat{\mathscr{A}}_1\left(Q_3\cdot F_2 + Q_1\cdot F_4\right), \quad (41)$$

6) for the coefficient of χ^6,

$$0 = \hat{\mathscr{G}}_1\left(Q_4\cdot F_2 + Q_2\cdot F_4\right), \quad (42)$$

$$0 = \hat{\mathscr{A}}_2\left(F_4\cdot F_2 + F_2\cdot F_4\right) - 2g\left[|Q_0|^2\left(Q_4\bar{Q}_2 + Q_2\bar{Q}_4\right) + Q_3\bar{Q}_3\right], \quad (43)$$

7) for the coefficient of χ^7,

$$0 = \hat{\mathscr{A}}_1 Q_3\cdot F_4, \quad (44)$$

8) for the coefficient of χ^8,

$$0 = \hat{\mathscr{G}}_1 Q_4\cdot F_4, \quad (45)$$

$$0 = \hat{\mathscr{A}}_2 F_4\cdot F_4 - 2g|Q_0|^2|Q_4|^2, \quad (46)$$

where $\hat{\mathscr{G}}_1 = \mathrm{i}\mathrm{D}_t + \frac{1}{2}\mathrm{D}_x^2 + \mathrm{i}k_0\mathrm{D}_x$, and $\hat{\mathscr{A}}_1$ and $\hat{\mathscr{A}}_2$ have been given.

It is obvious that we can solve Eqs. (35)–(46) in turn to obtain the expressions of G, H, and F. A detailed analysis shows that Q_1 admits the form

$$Q_1 = \alpha_1 \mathrm{e}^{\eta_1} + \alpha_2 \mathrm{e}^{\eta_2}, \quad (47)$$

where α_1 and α_2 are complex parameters, $\eta_j = k_j x + \mathrm{i}\left(k_j^2/2 - \lambda\right)t$, $j = 1,2$ with k_j being a complex constant. In order to satisfy Eqs. (36) and (37), we can assume that Q_2 and F_2 have the following forms:

$$Q_2 = A_{11}\mathrm{e}^{\eta_1+\bar{\eta}_1} + A_{21}\mathrm{e}^{\eta_2+\bar{\eta}_1} + A_{12}\mathrm{e}^{\eta_1+\bar{\eta}_2} + A_{22}\mathrm{e}^{\eta_2+\bar{\eta}_2}, \quad (48)$$

$$F_2 = \mu_{11}\mathrm{e}^{\eta_1+\bar{\eta}_1} + \mu_{21}\mathrm{e}^{\eta_2+\bar{\eta}_1} + \mu_{12}\mathrm{e}^{\eta_1+\bar{\eta}_2} + \mu_{22}\mathrm{e}^{\eta_2+\bar{\eta}_2}, \quad (49)$$

where

$$A_{ij} = \frac{\mathrm{i}k_0 - k_i}{\mathrm{i}k_0 + \bar{k}_j}\mu_{ij},$$

$$\mu_{ij} = \frac{2g\alpha_i\bar{\alpha}_j}{\left(k_i + \bar{k}_j\right)^2}\left(1 + \frac{2g|\gamma_0|^2}{\left(\mathrm{i}k_0 + \bar{k}_j\right)\left(-\mathrm{i}k_0 + k_i\right)}\right)^{-1},$$

in which A_{ij} $(j = 1,2)$ are complex parameters, $\mu_{21} = \bar{\mu}_{12}$, μ_{11}, and μ_{22} are real parameters. With the help of Eqs. (47) and (49), we can simplify Eq. (38) as

$$\mathrm{i}Q_{3t} + \frac{1}{2}Q_{3xx} - \lambda Q_3 = \left(k_1 - k_2\right)\left[\alpha_1\mu_{21}\left(k_2 + \bar{k}_1\right) - 2\alpha_2\mu_{11}k_{1\mathrm{R}}\right]\mathrm{e}^{\eta_1+\eta_2+\bar{\eta}_1} + \left(k_1 - k_2\right)\left[2\alpha_1\mu_{22}k_{2\mathrm{R}} - \alpha_2\mu_{12}\left(k_1 + \bar{k}_2\right)\right]\mathrm{e}^{\eta_1+\eta_2+\bar{\eta}_2}.$$

The equation implies that Q_3 has the form

$$Q_3 = \mathrm{e}^{\eta_1+\bar{\eta}_1+\eta_2+\delta_1} + \mathrm{e}^{\eta_2+\bar{\eta}_2+\eta_1+\delta_2}, \quad (50)$$

where parameters δ_j $(j = 1,2)$ are given by

$$\mathrm{e}^{\delta_1} = \frac{\left(k_1 - k_2\right)^2}{2g\bar{\alpha}_1}\left(1 - \frac{2g|\gamma_0|^2}{\left(k_1 - \mathrm{i}k_0\right)\left(k_2 - \mathrm{i}k_0\right)}\right)\mu_{21}\mu_{11},$$

$$\mathrm{e}^{\delta_2} = \frac{\left(k_1 - k_2\right)^2}{2g\bar{\alpha}_2}\left(1 - \frac{2g|\gamma_0|^2}{\left(k_1 - \mathrm{i}k_0\right)\left(k_2 - \mathrm{i}k_0\right)}\right)\mu_{22}\mu_{12}.$$

From Eqs. (41) and (42), we obtain the expressions of Q_4 and F_4

$$Q_4 = \mathrm{e}^{\eta_1+\bar{\eta}_1+\eta_2+\bar{\eta}_2+\kappa_1}, \quad (51)$$

$$F_4 = \mathrm{e}^{\eta_1+\bar{\eta}_1+\eta_2+\bar{\eta}_2+\kappa_2}, \quad (52)$$

where

$$\mathrm{e}^{\kappa_1} = \frac{\left(\mathrm{i}k_0 - k_1\right)\left(\mathrm{i}k_0 - k_2\right)}{\left(\mathrm{i}k_0 + \bar{k}_1\right)\left(\mathrm{i}k_0 + \bar{k}_2\right)}\mathrm{e}^{\kappa_2},$$

$$\mathrm{e}^{\kappa_2} = \frac{\mu_{11}\mu_{22}\mu_{12}\mu_{21}\left|k_2 - k_1\right|^4}{4g^2\left|\alpha_1\alpha_2\right|^2} \times \left|1 - \frac{2g|\gamma_0|^2}{\left(k_1 - \mathrm{i}k_0\right)\left(k_2 - \mathrm{i}k_0\right)}\right|^2,$$

in which κ_1 is a complex while κ_2 is a real parameter.

Combining Eqs. (47)–(52), we can find that the other equations gotten from Eq. (6) are satisfied to the moment. After absorbing χ, we obtain the exact superposition solutions of two-dark-soliton and two-bright-soliton for the two-component BECs

$$\phi_1 = \frac{L+S}{M}, \quad \phi_2 = \frac{L-S}{M}, \quad (53)$$

where

$$L = \gamma_0\mathrm{e}^{\mathrm{i}\eta_0}\left[1 + A_{11}\mathrm{e}^{\eta_1+\bar{\eta}_1} + A_{21}\mathrm{e}^{\eta_2+\bar{\eta}_1} + A_{12}\mathrm{e}^{\eta_1+\bar{\eta}_2} + A_{22}\mathrm{e}^{\eta_2+\bar{\eta}_2} + \mathrm{e}^{\eta_1+\bar{\eta}_1+\eta_2+\bar{\eta}_2+\kappa_1}\right],$$

$$S = \alpha_1\mathrm{e}^{\eta_1} + \alpha_2\mathrm{e}^{\eta_2} + \mathrm{e}^{\eta_1+\bar{\eta}_1+\eta_2+\delta_1} + \mathrm{e}^{\eta_2+\bar{\eta}_2+\eta_1+\delta_2}, \quad (54)$$

$$M = 1 + \mu_{11}\mathrm{e}^{\eta_1+\bar{\eta}_1} + \mu_{21}\mathrm{e}^{\eta_2+\bar{\eta}_1} + \mu_{12}\mathrm{e}^{\eta_1+\bar{\eta}_2} + \mu_{22}\mathrm{e}^{\eta_2+\bar{\eta}_2} + \mathrm{e}^{\eta_1+\bar{\eta}_1+\eta_2+\bar{\eta}_2+\kappa_2}, \quad (55)$$

with $\eta_j = \theta_j + \mathrm{i}\varphi_j$, $\theta_j = k_{j\mathrm{R}}\left(x - k_{j\mathrm{I}}t\right)$, and $\varphi_j = k_{j\mathrm{I}}x + \left(k_{j\mathrm{R}}^2 - k_{j\mathrm{I}}^2 - 2\lambda\right)t/2$, $j = 1,2$. The solution Eq. (53) describes the movement of two SPSs. If we choose proper parameters, the two solitons can move toward each other. To clearly understand the nature of the SPS interaction, we analyze the asymptotic behavior of the solution Eq. (53). Asymptotically, the solution can be written as a combination of two soliton solutions expressed by Eq. (28). The asymptotic form of two soliton solutions in the limits $t \to -\infty$ and $t \to \infty$ is similar to that of one

soliton solution Eq. (28). (i) Before collision (limit $t \to -\infty$): 1) superposition soliton 1 ($\theta_1 \approx 0$, $\theta_2 \to -\infty$)

$$\phi_{j,1} \to \frac{1}{2}\gamma_0 \mathrm{e}^{\mathrm{i}\eta_0}\left[(1+Z_1)-(1-Z_1)\tanh\left(\theta_1+\ln\sqrt{\mu_{11}}\right)\right] + (-1)^{3-j}\frac{\alpha_1 \mathrm{e}^{\mathrm{i}\varphi_1}}{2\sqrt{\mu_{11}}}\operatorname{sech}\left(\theta_1+\ln\sqrt{\mu_{11}}\right), \tag{56}$$

2) superposition soliton 2 ($\theta_2 \approx 0$, $\theta_1 \to \infty$)

$$\begin{aligned}\phi_{j,2} \to{}& \frac{1}{2}\gamma_0 \mathrm{e}^{\mathrm{i}\eta_0} Z_1\left[(1+Z_2)\right. \\ &\left.-(1-Z_2)\tanh\left(\theta_2+\ln\sqrt{\mu_{22}}-x_{02}\right)\right] \\ &+(-1)^{3-j}\frac{\alpha_2 \mathrm{e}^{\mathrm{i}(\varphi_2+\delta_{02})}}{2\sqrt{\mu_{22}}} \\ &\times \operatorname{sech}\left(\theta_2+\ln\sqrt{\mu_{22}}-x_{02}\right).\end{aligned} \tag{57}$$

(ii) After collision (limit $t \to \infty$): 1) superposition soliton 1 ($\theta_1 \approx 0$, $\theta_2 \to \infty$)

$$\begin{aligned}\phi_{j,1} \to{}& \frac{1}{2}\gamma_0 \mathrm{e}^{\mathrm{i}\eta_0} Z_2\left[(1+Z_1)\right. \\ &\left.-(1-Z_1)\tanh\left(\theta_1+\ln\sqrt{\mu_{11}}-x_{01}\right)\right] \\ &+(-1)^{3-j}\frac{\alpha_1 \mathrm{e}^{\mathrm{i}(\varphi_1+\delta_{01})}}{2\sqrt{\mu_{11}}} \\ &\times \operatorname{sech}\left(\theta_1+\ln\sqrt{\mu_{11}}-x_{01}\right),\end{aligned} \tag{58}$$

2) superposition soliton 2 ($\theta_2 \approx 0$, $\theta_1 \to -\infty$)

$$\phi_{j,2} \to \frac{1}{2}\gamma_0 \mathrm{e}^{\mathrm{i}\eta_0}\left[(1+Z_2)-(1-Z_2)\tanh\left(\theta_2+\ln\sqrt{\mu_{22}}\right)\right] + (-1)^{3-j}\frac{\alpha_2 \mathrm{e}^{\mathrm{i}\varphi_2}}{2\sqrt{\mu_{22}}}\operatorname{sech}\left(\theta_2+\ln\sqrt{\mu_{22}}\right). \tag{59}$$

Here $j = 1,2$, $Z_j = (\mathrm{i}k_0 - k_j)/(\mathrm{i}k_0 + \bar{k}_j)$, $x_{0j} = [\ln(\mu_{11}\mu_{22}) - \kappa_2]/2$, and

$$\delta_{0j} = \arctan\frac{2k_{j\mathrm{R}}\left[c_j n_j l_j + q_j\left(k_{j\mathrm{R}}^2 c_j^2 - n_j^2\right)\right]}{\left(k_{j\mathrm{R}}^2 c_j^2 - n_j^2\right) l_j - 4k_{j\mathrm{R}}^2 q_j c_j n_j},$$

where

$$\begin{aligned} c_j &= d \mp a^2 + 2b_j a, \\ n_j &= -b_j d \pm b_j a^2 + 2ak_{1\mathrm{R}}^2, \\ q_j &= k_{(3-j)\mathrm{R}}^2 b_j + b_{3-j}^2 b_j + 2b_{3-j} g|\gamma_0|^2, \\ l_j &= k_{j\mathrm{R}}^2 k_{(3-j)\mathrm{R}}^2 - \left(b_j b_{3-j} + 2g|\gamma_0|^2\right)^2 \\ &\quad \pm \left(b_{3-j} k_{j\mathrm{R}}^2 - b_j^2 k_{(3-j)\mathrm{R}}^2\right), \end{aligned}$$

with $a = k_{1\mathrm{I}} - k_{2\mathrm{I}}$, $d = k_{1\mathrm{R}}^2 - k_{2\mathrm{R}}^2$, and $b_j = k_{j\mathrm{I}} - k_0$, and we take the down (up) operator when $j = 1$ ($j = 2$). By analyzing the asymptotic behavior in detail, we know that there is no change of the background, width, peak, and trough for each soliton solution in the collision, while we should notice $|Z_j|^2 = 1$. However, equations (56)–(59) show that there is a phase shift in the process of two-soliton collision. It shows that the above solution (53) corresponds to an elastic collision, which is illustrated in Fig. 3.

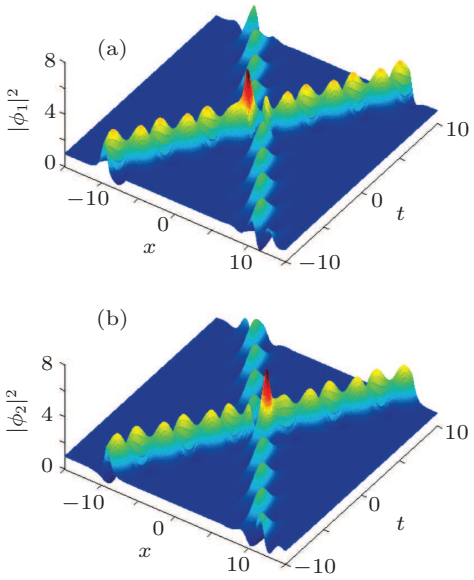


Fig. 3. (color online) The head-on collisions of two SPSs in (a) component 1 and (b) component 2 with parameters $k_1 = 2+\mathrm{i}$, $k_2 = 1-\mathrm{i}$, $\alpha_1 = 1$, $\alpha_2 = 1+\mathrm{i}$, $k_0 = 1$, $g = 1$, and $\gamma_0 = 1$.

3. Conclusion

In conclusion, the Hirota bilinear method is developed for two-component BECs, which are described by the coupled GP equations. The exact SPSs are obtained analytically, and the collision of two SPSs is studied. It is surprising to find that the single- and the two-BB soliton solutions can be obtained in special conditions. We also obtain the energies of the BB solitons, the DB solitons, and the SPSs. It shows that the latter two kinds of solitons represent the excitation states for this system. The phase separation can be obtained in the course of transition from the ground state to the excitation state. Such results are a novel scenario for the study of BECs and can be extended to more complex models.

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