Quantum information entropies of the eigenstates for the Pöschl–Teller-like potential*

Guo-Hua Sun[a)†], M. Avila Aoki[a)], and Shi-Hai Dong[b)]

[a)] Centro Universitario Valle de Chalco, Universidad Autónoma del Estado de México, Valle de Chalco Solidaridad, Estado de México, 56615, Mexico
[b)] Departamento de Física, Escuela Superior de Física y Matemátics, Instituto Politécnico Nacional, Edificio 9, Unidad Profesional Adolfo López Mateos, Mexico D. F. 07738, Mexico

(Received 22 September 2012; revised manuscript received 31 October 2012)

Shannon entropy for lower position and momentum eigenstates of Pöschl–Teller-like potential is evaluated. Based on the entropy densities demonstrated graphically, we note that the wave through of the position information entropy density $\rho(x)$ moves right when the potential parameter V_1 increases and its amplitude decreases. However, its wave through moves left with the increase in the potential parameter $|V_2|$. Concerning the momentum information entropy density $\rho(p)$, we observe that its amplitude increases with increasing potential parameter V_1, but its amplitude decreases with increasing $|V_2|$. The Bialynicki–Birula–Mycielski (BBM) inequality has also been tested for a number of states. Moreover, there exist eigenstates that exhibit squeezing in the momentum information entropy. Finally, we note that position information entropy increases with V_1, but decreases with $|V_2|$. However, the variation of momentum information entropy is contrary to that of the position information entropy.

Keywords: bound states, quantum information entropy, Pöschl–Teller-like potential

PACS: 03.65.–w, 03.65.Ge, 03.67.–a **DOI:** 10.1088/1674-1056/22/5/050302

1. Introduction

Recently, there has been a growing interest in dealing with information theoretic measures for quantum-mechanical systems. Particularly, entropic uncertainty relations, which serve as alternatives to the Heisenberg uncertainty relation, have been tested by some authors.[1,2] Among various measures of information entropy, Shannon entropy[2–6] plays a particularly important role in the measure of uncertainty. The entropic uncertainty relation, which is related to the position and momentum spaces, was obtained by Beckner, Bialynicki-Birula, and Mycielski (BBM),[7–9] and given by

$$S_x + S_p \geq D(1 + \ln \pi), \tag{1}$$

where D represents the spatial dimension. The position and momentum space information entropies S_x and S_p are defined, respectively, by

$$\begin{aligned} S_x &= -\int_{-\infty}^{\infty} |\psi(x)|^2 \ln |\psi(x)|^2 \, \mathrm{d}x, \\ S_p &= -\int_{-\infty}^{\infty} |\phi(p)|^2 \ln |\phi(p)|^2 \, \mathrm{d}p, \end{aligned} \tag{2}$$

where $\psi(x)$ is the normalized eigenfunction in spatial coordinate, and $\phi(p)$ is the corresponding Fourier transform.

Except for their intrinsic interest, the entropic uncertainty relations[1,2] have been widely used to study squeezing localization[10] and fractional revival,[11–13] and to reconstruct the charge and momentum densities of atomic and molecular systems by means of maximum-entropy procedures.[13,14] We should realize that it is not easy to calculate exactly the Shannon entropies (2) or the entropic uncertainty relations[1,2] due to the logarithmic factors $\ln |\psi|^2$ and $\ln |\phi|^2$ involved in the integral. Up to now, they have been obtained only for a few special cases such as the harmonic oscillator,[1,2,15] Pöschl–Teller potential,[12,13] Morse potential,[16–19] Coulomb potential for highly excited states,[20] some potential isospectral to the Pöschl–Teller potential,[21] and classical orthogonal polynomials.[22] In addition, other related studies involving the confined atoms, modified Yukawa and Hulthén potentials, the relativistic effect of the hydrogen atom, as well as the new estimation for the power-law distribution parameter based on the Shannon entropy have also been carried out.[23,24] In this work, we study the Shannon entropy for an interesting quantum system, i.e., the Pöschl–Teller-like potential,[25–27] whose solutions are expressed by the hypergeometric functions. In particular, we find that there exist eigenstates that exhibit squeezing in the momentum information entropy. The entropy squeezing, which occurs in momentum space, will be compensated by increasing the position entropy to satisfy the BBM inequality. To our knowledge, the novelty of squeezing in momentum has not been considered.

This paper is organized as follows. In Section 2, we present the exact solutions of this quantum system. The position and momentum information entropy densities $\rho(x)$ and $\rho(p)$ are presented. The position and momentum information entropies S_x and S_p are evaluated numerically in Section 3. Some concluding remarks are given in Section 4.

*Project supported by COFAA-IPN (Grant No. 20120876-SIP-IN).
†Corresponding author. E-mail: sunghdb@yahoo.com
© 2013 Chinese Physical Society and IOP Publishing Ltd

2. Pöschl–Teller-like potential

The stationary one-dimensional Schrödinger equation ($m=\hbar=1$) is given by[20]

$$-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x)+V(x)\psi(x)=E\psi(x), \tag{3}$$

with

$$V(x)=\frac{V_1+V_2\cosh(\alpha x)}{\sinh^2(\alpha x)},\quad (V_2<0), \tag{4}$$

where V_1, V_2, and α represent three potential parameters. It should be noted that previously discussed Pöschl–Teller potential takes the form of $-D/\cosh^2(\alpha x)$. The present potential (4) is absolutely different from it. Defining

$$z=\tanh^2\left(\frac{\alpha x}{2}\right),\quad \varepsilon=\frac{\sqrt{-2E}}{\alpha}, \tag{5}$$

we obtain the following differential equation:

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\psi(z)+\frac{3z-1}{2z(z-1)}\frac{\mathrm{d}}{\mathrm{d}z}\psi(z) +\left\{\frac{(z-1)[V_1+V_2+(V_2-V_1)z]}{2\alpha^2z^2(z-1)^2}-\frac{\varepsilon^2}{z(z-1)^2}\right\}\psi(z)=0. \tag{6}$$

It is shown from Eq. (5) that $z\to 0$ if $x\to\pm\infty$, and $z\to 1$ if $x\to 0$. Thus, we may obtain the wave function $\psi(z)$ of the form

$$\psi(z)=(1-z)^{\varepsilon}z^{(\mu+1/2)/2}F(z), \tag{7}$$

where

$$\mu=\frac{1}{2\alpha}\sqrt{8(V_1+V_2)+\alpha^2}. \tag{8}$$

Substitution of Eq. (7) into Eq. (8) leads to the hypergeometric differential equation

$$\frac{1}{2\alpha^2}\left[-2V_2-\alpha^2\left(\mu+\frac{1}{2}+\varepsilon\right)(1+2\varepsilon)\right]F(z) +[\mu+1-z\,(\mu+2\varepsilon+2)]\,F'(z) +(1-z)\,zF''(z)=0, \tag{9}$$

whose solutions are nothing but the hypergeometric functions $F(z)={}_2\mathrm{F}_1(a,b;c;z)$. Here, the parameters are given by

$$a=\frac{1}{2}+\frac{1}{2}\mu+\varepsilon-\frac{\nu}{2},\quad b=\frac{1}{2}+\frac{1}{2}\mu+\varepsilon+\frac{\nu}{2},\quad c=1+\mu, \tag{10}$$

with

$$\nu=\frac{1}{2\alpha}\sqrt{8(V_1-V_2)+\alpha^2}. \tag{11}$$

From the properties of the hypergeometric functions, this series $F(z)$ approaches infinity unless $a=\frac{1}{2}+\frac{\mu}{2}+\varepsilon-\frac{\nu}{2}$ is a negative integer. Therefore, the wave functions $\psi(z)$ will not be finite everywhere unless

$$a=\frac{1}{2}+\frac{\mu}{2}+\varepsilon-\frac{\nu}{2}=-n,\quad (n=0,1,\ldots), \tag{12}$$

and then we have

$$\varepsilon=\frac{1}{2}(\nu-\mu-2n-1). \tag{13}$$

Substituting this equation into Eq. (5) allows us to obtain the following eigenvalues:

$$E_n=-\frac{\alpha^2}{8}[\nu-\mu-(1+2n)]^2,\quad (n=0,1,\ldots,[(\nu-\mu-1)/2]), \tag{14}$$

where $[f]$ means the largest integer, but smaller than f. It is shown from the definitions of ν and μ given by Eqs. (11) and (8) that the bound states $E<0$ exist only if $(V_1+V_2)>0$ and $V_2<0$, with $x>0$, which coincides with the conclusion in Ref. [20].

On the other hand, the eigenfunctions can be written as

$$\psi_n(z)=\mathscr{C}z^{\delta}(1-z)^{\varepsilon}\times{}_2\mathrm{F}_1(-n,n+2(\delta+\varepsilon+1/4);2\delta+1/2;z), \tag{15}$$

where

$$\delta=\frac{1}{4}+\frac{\mu}{2}. \tag{16}$$

Now, let us find the normalization constant $\mathscr{C}$. The normalization condition

$$\int_0^{\infty}\psi_n(z)^2\,\mathrm{d}x=\int_0^1\frac{\psi_n(z)^2}{(1-z)\sqrt{z}\alpha}\,\mathrm{d}z=1 \tag{17}$$

requires

$$\frac{\mathscr{C}^2}{\alpha}\int_0^1(1-z)^{2\varepsilon-1}z^{2\delta-1/2}\times{}_2\mathrm{F}_1\left(-n,n+2(\delta+\varepsilon+1/4);2\delta+1/2;z\right)^2\,\mathrm{d}z=1, \tag{18}$$

from which we finally obtain the normalization constant as follows:

$$\mathscr{C}=\sqrt{\frac{2\alpha\,\varepsilon\Gamma(1/2+n+2\delta)\Gamma(1/2+n+2\,\delta+2\varepsilon)}{n!\Gamma(1/2+2\delta)^2\Gamma(1+n+2\varepsilon)}} =\sqrt{\frac{2\alpha\,\varepsilon\Gamma(n+\mu+1)\Gamma(n+\mu+2\varepsilon+1)}{n!\Gamma(\mu+1)^2\Gamma(n+2\varepsilon+1)}}. \tag{19}$$

In the calculation, we have used the following formula:[22]

$$\int_0^1(1-z)^{2\varepsilon-1}z^{2\delta-\frac{1}{2}}\times{}_2\mathrm{F}_1\left(-n,n+2(\delta+\varepsilon+1/4),2\delta+1/2,z\right)^2\,\mathrm{d}z =\frac{n!\Gamma(1/2+2\delta)^2\Gamma(1+n+2\varepsilon)}{2\varepsilon\Gamma(1/2+n+2\delta)\Gamma(1/2+n+2\delta+2\varepsilon)}. \tag{20}$$

3. Information entropy

The position and momentum space information entropies for the one-dimensional potential can be calculated by using Eqs. (2). Note that their analytical derivations for the Pöschl–Teller-like potential are quite difficult. Therefore, the present work is devoted to the numerical study of the information entropies by taking the eigenfunctions (15) into account. Instead of evolution of position and momentum space information entropies, we plot entropy densities $\rho(x) = |\psi(x)|^2 \ln|\psi(x)|^2$ and $\rho(p) = |\phi(p)|^2 \ln|\phi(p)|^2$ for both position and momentum spaces in Figs. 1–4, respectively. Due to the difficulty in the calculation, we only consider the lower excited states. For the ground state, we have

$$\psi_0(z) = c z^{1/4+\mu/2}(1-z)^{(-1+\nu-\mu)/2},$$
$$c = \sqrt{\frac{\alpha\Gamma(\nu)}{\Gamma(-1+\nu-\mu)\Gamma(1+\mu)}}, \tag{21}$$

from which we may derive the corresponding momentum representation as

$$\phi_0(p) = \frac{c}{\alpha}\sqrt{\frac{2}{\pi}}\Gamma\left(\mu+\frac{3}{2}\right) {}_2\tilde{\mathrm{F}}_1\left(\mu+\frac{3}{2},\frac{1}{2}(\mathrm{i}\,p_1+\mu-\nu+3);\right.$$
$$\left.\frac{1}{2}(\mathrm{i}\,p_1+\mu+\nu+2);-1\right)$$
$$\times\Gamma\left[\frac{1}{2}(\mathrm{i}\,p_1-\mu+\nu-1)\right], \tag{22}$$

by means of the Fourier transform

$$\phi_0(p) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \psi_0(x)\,\mathrm{e}^{-\mathrm{i}px}\,\mathrm{d}x, \tag{23}$$

where $p_1 = 2p/\alpha$ and ${}_2\tilde{\mathrm{F}}_1$ are the regularized hypergeometric functions.

For the first excited state, we have

$$\psi_1(z) = d\,z^{(\mu+1/2)/2}(1-z)^{(\nu-\mu-3)/2}[1+\mu-z(\nu-1)],$$
$$d = \sqrt{\frac{\alpha(\nu-\mu-3)\Gamma(\nu-1)}{\Gamma(\mu+2)\Gamma(\nu-\mu-1)}}. \tag{24}$$

Likewise, one is able to obtain the corresponding momentum representation, but the expression is too complicated and not written here for simplicity. Other excited states with $n = 2$ and 3 are not discussed here due to their complexity.

Let us first analyze the properties of the position information entropy densities. It is interesting to note that the wave through moves right when the potential parameter V_1 increases and its amplitude decreases, as shown in Figs. 1 and 2. On the contrary, the wave through moves left when the potential parameter $|V_2|$ increases and its amplitude increases. This is because the potential well depth decreases as the potential parameter V_1 increases. However, when $|V_2|$ increases, the potential well depth increases. With respect to the momentum information entropy density, it is shown in Fig. 3 that its amplitude increases with the increase in the potential parameter V_1. This can be explained well by analyzing the potential well depth. As we know, as the potential V_1 increases, the potential well depth decreases. Thus, the fact that particle can move easily results in increasing amplitude of the momentum information entropy density. The variation in Fig. 4 is contrary to that in Fig. 3.

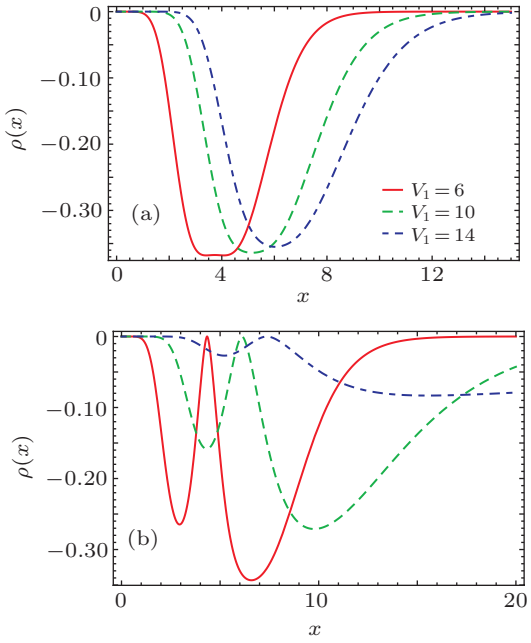


Fig. 1. (color online) Plots of the position space entropy density $\rho(x)$ of the Pöschl–Teller-like potential for (a) $n = 0$ and (b) $n = 1$, respectively. The potential parameters are $V_2 = -4$, $\alpha = 0.5$, and $V_1 = 6, 10, 14$.

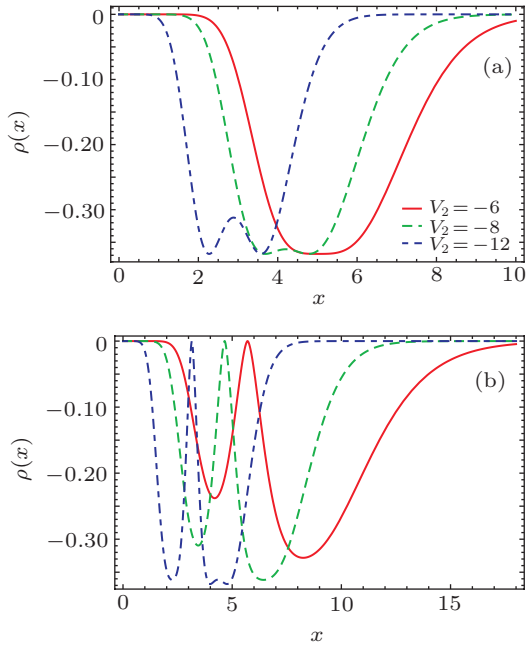


Fig. 2. (color online) Plots of the position space entropy density $\rho(x)$ for (a) $n = 0$ and (b) $n = 1$, respectively. The potential parameters are $V_1 = 15$, $\alpha = 0.5$, and $V_2 = -6, -8, -12$.

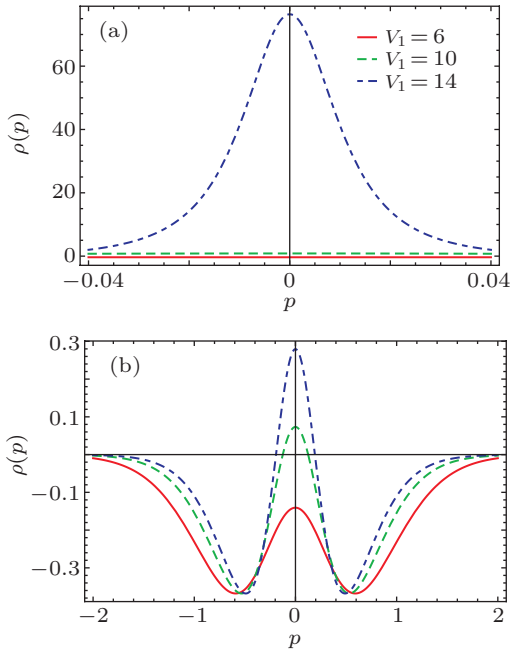


Fig. 3. (color online) Plots of the momentum space entropy density $\rho(p)$ for (a) $n = 0$ and (b) $n = 1$, respectively. The potential parameters are $V_2 = -4$, $\alpha = 0.5$, and $V_1 = 6, 10, 14$.

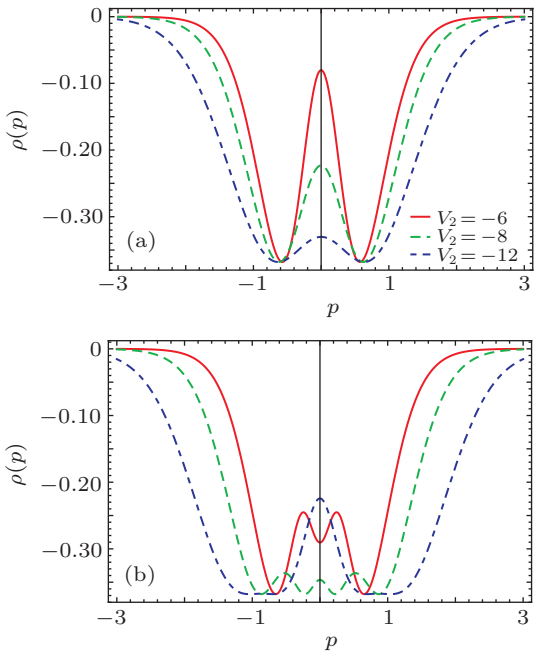


Fig. 4. (color online) Plots of the momentum space entropy density $\rho(p)$ for (a) $n = 0$ and (b) $n = 1$, respectively. The potential parameters are $V_1 = 15$, $\alpha = 0.5$, and $V_2 = -6, -8, -12$.

In Tables 1 and 2, we present the numerical results of information entropies S_x and S_p, and their sum for some given quantum numbers n and potential parameters V_1 and V_2. It is clearly seen from Tables 1 and 2 that BBM inequality is satisfied for the Pöschl–Teller-like potential. With one entropy increasing, the other entropy decreases, but only to the extent that their sum stays above the stipulated lower bound of the value $(1+\ln\pi)$. We also note that S_x increases with the potential parameter V_1, but decreases with $|V_2|$.

Table 1. Information entropies of the eigenstates for $V_2 = -2$, $\alpha = 0.5$, and various V_1.

n	V_1	S_x	S_p	$S_x + S_p$	$1+\ln\pi$
0	3	1.71604	0.53463	2.25067	2.1447
	2.5	1.60858	0.60440	2.21298	2.1447
	2.1	1.46269	0.76499	2.22768	2.1447
1	3.0	2.88432	0.73692	3.62124	2.1447
	2.5	2.43364	0.58855	3.02219	2.1447
	2.1	2.09346	0.87709	2.97055	2.1447

Table 2. Information entropies of the eigenstates for $V_1 = 15, \alpha = 0.5$, and various V_2.

n	V_2	S_x	S_p	$S_x + S_p$	$1+\ln\pi$
0	–6	1.53504	0.65511	2.19015	2.1447
	–8	1.32877	0.84954	2.17831	2.1447
	–12	1.03531	1.13445	2.16976	2.1447
1	–6	2.20059	0.77163	2.97222	2.1447
	–8	1.39266	1.43247	2.90606	2.1447
	–12	1.17588	1.62175	2.83281	2.1447

4. Concluding remarks

In this paper, we have presented the exact solutions of one-dimensional Schrödinger equation with a Pöschl–Teller-like potential. The complicated normalized wavefunctions are also found. Based on the solutions, we have studied the Shannon entropy of the position and momentum eigenstates for lower states. In terms of those entropy densities demonstrated graphically, we have found that the wave through of the position information entropy density $\rho(x)$ moves right when the potential parameter V_1 increases and its amplitude decreases. However, its wave through moves left when the potential parameter $|V_2|$ increases. Concerning the momentum information entropy density $\rho(p)$, we note that its amplitude increases with increasing V_1, but decreases as the potential parameter $|V_2|$ increases. Finally, the BBM inequality has also been examined for lower states. Before ending this work, we give a useful remark on BBM inequality. It should be pointed out that some authors[23] claimed that there exists an implicit relationship between the entropic uncertainty relation and the Heisenberg uncertainty relation $\Delta x \Delta p \geq \hbar/2$. In fact, there is not any essential relation between them from their individual definitions except for their so-called common terminology "uncertainty relation". For example, for fixed parameters $V_2 = -4$ and $\alpha = 0.5$, we have $(\Delta p)^2 = 0.02809$ for $V_1 = 10$, while $(\Delta p)^2 = 0.00934$ for $V_1 = 20$. However, $(\Delta x)^2 = 27.2006, 55.0718$ for $V_1 = 10, 20$, respectively. This means that the squeezing in p-quadratures of the field is visible by considering the fact that a state is squeezed if $(\Delta x)^2 < 0.5$ or $(\Delta p)^2 < 0.5$ in the Heisenberg uncertainty relation.

References

[1] Yáñez R J, van Assche W and Dehesa J S 1994 *Phys. Rev. A* **50** 3065
[2] Aptekarev A I, Dehesa J S and Yáñez R J 1994 *J. Math. Phys.* **35** 4423
[3] Shannon C E 1948 *Bell Syst. Tech. J.* **27** 623

[4] Shannon C E 1993 *Collected Papers* (New York: IEEE Press)
[5] Everett H 1973 *The Many World Interpretation of Quantum Mechanics* (Princeton, NJ: Princeton University Press)
[6] Hirschmann Jr I I 1957 *Am. J. Math.* **79** 152
[7] Beckner W 1975 *Ann. Math.* **102** 159
[8] Bialynicki-Birula I and Mycielski J 1975 *Commun. Math. Phys.* **44** 129
[9] Bialynicki-Birula I and Rudnicki L 2010 arXive:1001.4668 [quant-ph]
[10] Orlowski A 1997 *Phys. Rev. A* **56** 2545
[11] Atre R, Kumar A, Kumar C N and Panigrahi P 2004 *Phys. Rev. A* **69** 052107
[12] Romera E and de los Santos F 2007 *Phys. Rev. Lett.* **99** 263601
[13] Galindo A and Pascual P 1978 *Quantum Mechanics* (Berlin: Springer)
[14] Angulo J C, Antolin J, Zarzo A and Cuchi J C 1999 *Eur. Phys. J. D* **7** 479
[15] Majernik V and Opatrný T 1996 *J. Phys. A* **29** 2187
[16] Dehesa J S, Martínez-Finkelshtein A and Sorokin V N 2006 *Mol. Phys.* **104** 613
[17] Coffey M W 2007 *Can. J. Phys.* **85** 733
[18] Aydiner E, Orta C and Sever R 2008 *Int. J. Mod. Phys. B* **22** 231
[19] Popov D 2001 *Buletinul Stiintific al Universitatii "Politehnica" din Timisoara, Seria Matematica-Fizica* **46** 60
[20] Dehesa J S, van Assche W and Yáñez R J 1997 *Methods Appl. Math.* **4** 91
[21] Kumar A 2005 *Ind. J. Pure. Appl. Phys.* **43** 958
[22] Buyarov V S, López-Artés P, Martínez-Finkelshtein A and van Assche W 2000 *J. Phys. A: Math. Gen.* **33** 6549
[23] Katriel J and Sen K D 2010 *J. Comp. Appl. Math.* **233** 1399
[24] Fu B B and Gao Z Y 2006 *Chin. Phys. Lett.* **23** 520
[25] Bagrov V G and Gitman D M 1990 *Exact Solution of Relativistic Wave Equations* (Dordrecht: Kluwer)
[26] Dong S H and Gonzalez-Cisneros A 2008 *Ann. Phys.* **323** 1136
[27] Miranda M G, Sun G H and Dong S H 2010 *Int. J. Mod. Phys. E* **19** 123
[28] Ghasemi A, Hooshmandasl M R and Tavassoly M K 2011 *Phys. Scr.* **84** 035007