Exact solutions of (3+1)-dimensional nonlinear incompressible non-hydrostatic Boussinesq equations*

Liu Ping(刘 萍)[a)†] and Li Zi-Liang(李子良)[b)]

[a)] College of Electron and Information Engineering, University of Electronic Science and Technology of China Zhongshan Institute, Zhongshan 528402, China
[b)] Physical Oceanography Laboratory, College of Physical and Environment Oceanography, Ocean University of China, Qingdao 266100, China

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The symmetries and the exact solutions of the (3+1)-dimensional nonlinear incompressible non-hydrostatic Boussinesq (INHB) equations, which describe atmospheric gravity waves, are studied in this paper. The calculation on symmetry shows that the equations are invariant under the Galilean transformations, the scaling transformations, and the space–time translations. Three types of symmetry reduction equations and similar solutions for the (3+1)-dimensional INHB equations are proposed. Traveling and non-traveling wave solutions of the INHB equations are demonstrated. The evolutions of the wind velocities in latitudinal, longitudinal, and vertical directions with space–time are demonstrated. The periodicity and the atmosphere viscosity are displayed in the (3+1)-dimensional INHB system.

Keywords: (3+1)-dimensional nonlinear incompressible non-hydrostatic Boussinesq equations, atmospheric gravity waves, symmetries, exact solutions

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1. Introduction

It has been recognized in the previous work on atmospheric gravity waves (GWs) that GWs bring about a profound impact on Earth's weather and climate systems. Whether at the largest scale of the general circulation or the smaller scale of a surface turbulence, calculations of the atmospheric budgets routinely include contributions from wave transport and mixing. Waves generated by flow over mountains, convection, and jet-stream instability cause the vertical advection that can produce organized cloud and has a great effect on severe hazardous weather and climate, such as typhoon rainstorms, deep convection, and orographic precipitation. These kinds of waves can break, causing severe downslope windstorms and rotors over the surface with significant vertical mixing in the upper troposphere and lower stratosphere and in the mesosphere and lower thermosphere.[1–4]

The governing equations for GWs consist of the nonlinear horizontal momentum equations, the mass continuity equations, and the nonlinear thermodynamic equations. Generally speaking, the governing equations for gravity waves can be considered as (3+1)-dimensional nonlinear incompressible non-hydrostatic Boussinesq (INHB) equations[5]

$$u_t + u u_x + v u_y + w u_z + p_x/\rho = 0, \tag{1}$$

$$v_t + u v_x + v v_y + w v_z + p_y/\rho = 0, \tag{2}$$

$$w_t + u w_x + v w_y + w w_z + p_z/\rho - gT/T_0 = 0, \tag{3}$$

$$T_t + u\, T_x + v\, T_y + w\, T_z + (\gamma_{\rm d} - \gamma) w = 0, \tag{4}$$

$$u_x + v_y + w_z = 0, \tag{5}$$

where u, v, and w are the velocities in x, y, and z directions, x, y, and z denote the coordinates in latitudinal, longitudinal, and vertical directions, respectively, ρ denotes the density, T_0 is the background temperature, T is the temperature perturbation, p is the perturbation pressure, g denotes the gravitational constant, $\gamma_{\rm d}$ and γ are the dry adiabatic lapse rate and the moisture lapse rate, respectively, and t is the time.

The symmetry study is universally held as one of the most effective and systematic methods to solve partial differential equations (PDEs). Many methods researching on the symmetries of PDEs have been proposed.[6–11] In view of the non-integrability and high nonlinearity of the system, researchers usually study atmospheric and oceanic circulations by numerical models or ideally theoretical models. Though many analytical solutions of some atmospheric and oceanic models have been reported, the exact solutions and the symmetries for the INHB equations, to our knowledge, have not been systematically studied yet. In this paper, we will concentrate on the symmetries, similarity reductions, and exact solutions of the INHB Eqs. (1)–(5). The paper is organized as follows. Resorting to the classical Lie group theory, we try to find the symmetries of the INHB equations in the next section. Section 3 is devoted to similarity reductions and similar solutions of the INHB equations. In Section 4, the exact solutions for the INHB equations are proposed. The conclusion is drawn in Section 5.

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†Corresponding author. E-mail: liuping49@126.com

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2. Symmetry

The Lie point symmetry components $\{\eta_i(x,y,z,t), i = 1,2,\ldots,5\}$ of Eqs. (1)–(5) are the solutions of the linearized nonlinear equations of Eqs. (1)–(5), namely,

$$\eta_{1,t}+u\eta_{1,x}+v\eta_{1,y}+w\eta_{1,z}+\eta_1 u_x + \eta_2 u_y+\eta_3 u_z+\frac{1}{\rho}\eta_{4,x}=0, \tag{6}$$

$$\eta_{2,t}+u\eta_{2,x}+v\eta_{2,y}+w\eta_{2,z} + \eta_1 v_x+\eta_2 v_y+\eta_3 v_z+\frac{1}{\rho}\eta_{4,y}=0, \tag{7}$$

$$\eta_{3,t}+u\eta_{3,x}+v\eta_{3,y}+w\eta_{3,z}+\eta_1 w_x + \eta_2 w_y+\eta_3 w_z+\frac{1}{\rho}\eta_{4,z}-\frac{g\eta_5}{T_0}=0, \tag{8}$$

$$\eta_{5,t}+u\eta_{5,x}+v\eta_{5,y}+w\eta_{5,z}+\eta_1 T_x + \eta_2 T_y+\eta_3 T_z+(\gamma_d-\gamma)\eta_3=0, \tag{9}$$

$$\eta_{1,x}+\eta_{2,y}+\eta_{3,z}=0. \tag{10}$$

The five symmetry components $\{\eta_i(x,y,z,t), i = 1,2,\ldots,5\}$ can be supposed to have the forms

$$\eta_1 = \bar{x}u_x+\bar{y}u_y+\bar{z}u_z+\bar{t}u_t-\bar{u}, \tag{11}$$

$$\eta_2 = \bar{x}v_x+\bar{y}v_y+\bar{z}v_z+\bar{t}v_t-\bar{v}, \tag{12}$$

$$\eta_3 = \bar{x}w_x+\bar{y}w_y+\bar{z}w_z+\bar{t}w_t-\bar{w}, \tag{13}$$

$$\eta_4 = \bar{x}p_x+\bar{y}p_y+\bar{z}p_z+\bar{t}p_t-\bar{p}, \tag{14}$$

$$\eta_5 = \bar{x}T_x+\bar{y}T_y+\bar{z}T_z+\bar{t}u_t-\bar{T}, \tag{15}$$

where $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{t}$, $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{p}$, $\bar{T}$ are functions of the variables $\{x,y,z,t,u,v,w,p,T\}$. The expressions imply that equations (1)–(5) are formal invariant under the transformations

$$x\longrightarrow x+\varepsilon\bar{x},\ \ y\longrightarrow y+\varepsilon\bar{y},\ \ z\longrightarrow z+\varepsilon\bar{z},$$
$$t\longrightarrow t+\varepsilon\bar{t},\ \ u\longrightarrow u+\varepsilon\bar{u},\ \ v\longrightarrow v+\varepsilon\bar{v},$$
$$w\longrightarrow w+\varepsilon\bar{w},\ \ p\longrightarrow p+\varepsilon\bar{p},\ \ T\longrightarrow T+\varepsilon\bar{T},$$

with ε being an infinitesimal parameter.

The substitution of Eqs. (11)–(15) into Eq. (6) leads to

$$\bar{x}_v v_x v u_y+\bar{x}_w w_x v u_y+\frac{1}{\rho}\bar{y}_v u_y p_y + S_1(x,y,z,t,u,v,w,p,T)=0, \tag{16}$$

where $S_1(x,y,z,t,u,v,w,p,T)$ is a polynomial excluding the terms $v_x v u_y$ $w_x v u_y$, and $u_y p_y$. All the coefficients of various derivatives of $\{u,v,w,p,T\}$ should be zero. From Eq. (16), we have

$$\bar{x}_v=\bar{x}_w=\bar{y}_v=0, \tag{17}$$

which means

$$\bar{x}(x,y,z,t,u,v,w,p,T) = \bar{x}(x,y,z,t,u,p,T), \tag{18}$$

$$\bar{y}(x,y,z,t,u,v,w,p,T) = \bar{y}(x,y,z,t,u,w,p,T). \tag{19}$$

Substituting Eqs. (11)–(15), (18), and (19) into Eq. (7) and identifying all coefficients of derivatives of $\{u,v,w,p,T\}$, we conclude

$$\bar{x}_u u_y u w_z-\bar{x}_T T_y v u_y+\frac{1}{\rho}\bar{z}_u p_z u_y + S_2(x,y,z,t,u,v,w,p,T)=0, \tag{20}$$

where $S_2(x,y,z,t,u,v,w,p,T)$ is another polynomial excluding the terms $u_y u w_z$, $p_z u_y$, and $T_y v u_y$. From the coefficients of the three terms, it is not difficult to find

$$\bar{x}_u=\bar{x}_T=\bar{z}_u=0,$$

which leads to

$$\bar{x}(x,y,z,t,u,p,T)=\bar{x}(x,y,z,t,p), \tag{21}$$

$$\bar{z}(x,y,z,t,u,v,w,p,T)=\bar{z}(x,y,z,t,v,w,p,T). \tag{22}$$

Substituting Eqs. (11)–(15) and (18)–(22) into Eqs. (6)–(10), and simplifying some terms by the original equations, we obtain an over-determined set of equations for the unknown functions $\bar{x}(x,\ y,\ z,\ t,\ p)$, $\bar{y}(x,\ y,\ z,\ t,\ u,\ w,\ p,\ T)$, $\bar{z}(x,\ y,\ z,\ t,\ v,\ w,\ p,\ T)$, $\bar{t}(x,\ y,\ z,\ t,\ u,\ v,\ w,\ p,\ T)$, $\bar{u}(x,\ y,\ z,\ t,\ u,\ v,\ w,\ p,\ T)$, $\bar{v}(x,\ y,\ z,\ t,\ u,\ v,\ w,\ p,\ T)$, $\bar{w}(x,\ y,\ z,\ t,\ u,\ v,\ w,\ p,\ T)$, $\bar{p}(x,\ y,\ z,\ t,\ u,\ v,\ w,\ p,\ T)$, and $\bar{T}(x,\ y,\ z,\ t,\ u,\ v,\ w,\ p,\ T)$. Solving the determinant equations, we obtain

$$\begin{aligned}
&\bar{x}=C_1x+C_2y+F_1(t),\ \ \bar{y}=C_1y-C_2x+F_2(t),\\
&\bar{z}=C_1z+F_3(t),\\
&\bar{t}=C_3t+C_4,\ \ \bar{u}=F_{1,t}+(C_1-C_3)u+C_2v,\\
&\bar{v}=F_{2,t}+(C_1-C_3)v-C_2u,\ \ \bar{w}=F_{3,t}+(C_1-C_3)w,\\
&\bar{p}=C_5z-\rho(F_{1,tt}x+F_{2,tt}y+F_{3,tt}z)+2(C_1-C_3)p\\
&\quad+F_4-z\rho\Gamma g(C_3z+F_3)/T_0,\\
&\bar{T}=(C_1-2C_3)T-\Gamma(2zC_3+F_3)+\frac{C_5T_0}{\rho g},
\end{aligned} \tag{23}$$

where $\Gamma\equiv\gamma_d-\gamma$, C_1, C_2, C_3, C_4, and C_5 are arbitrary constants, F_1, F_2, and F_3 are arbitrary functions of t. Then the concrete forms of the symmetry components $\{\eta_i(x,\ y,\ z,\ t),\ i = 1,\ 2,\ldots,\ 5\}$ can be written as

$$\begin{aligned}
\eta_1 &= (C_3t+C_4)u_t+(C_1x+C_2y+F_1)u_x\\
&\quad+(C_1y-C_2x+F_2)u_y+(C_1z+F_3)u_z\\
&\quad-(C_1-C_3)u-C_2v-F_{1,t},\\
\eta_2 &= (C_3t+C_4)v_t+(C_1x+C_2y+F_1)v_x\\
&\quad+(C_1y-C_2x+F_2)v_y+(C_1z+F_3)v_z\\
&\quad-(C_1-C_3)v+C_2u-F_{2,t},\\
\eta_3 &= (C_3t+C_4)w_t+(C_1x+C_2y+F_1)w_x\\
&\quad+(C_1y-C_2x+F_2)w_y+(C_1z+F_3)w_z\\
&\quad-(C_1-C_3)w-F_{3,t},\\
\eta_4 &= (C_3t+C_4)p_t+(C_1x+C_2y+F_1)p_x
\end{aligned}$$

$$
\begin{aligned}
&+(C_1y-C_2x+F_2)p_y+(C_1z+F_3)p_z\\
&+z\rho\Gamma g(C_3z+F_3)/T_0-2(C_1-C_3)p-C_5z\\
&-F_4+\rho(F_{1,tt}x+F_{2,tt}y+F_{3,tt}z),\\
\eta_5 \;=\;&(C_3t+C_4)T_t+(C_1x+C_2y+F_1)T_x\\
&+(C_1y-C_2x+F_2)T_y+(C_1z+F_3)T_z-(C_1-2C_3)T\\
&+\Gamma(F_3+2C_3z)-C_5T_0/(\rho g). \qquad (24)
\end{aligned}
$$

A general element of this algebra is described as

$$
\begin{aligned}
&(C_1x+C_2y+F_1)\frac{\partial}{\partial x}+(C_1y-C_2x+F_2)\frac{\partial}{\partial y}+(C_1z+F_3)\frac{\partial}{\partial z}\\
&+(C_3t+C_4)\frac{\partial}{\partial t}+(F_{1,t}+C_1u-C_3u+C_2v)\frac{\partial}{\partial u}\\
&+(F_{2,t}+C_1v-C_3v-C_2u)\frac{\partial}{\partial v}\\
&+\left[C_5z-\rho(F_{1,tt}x+F_{2,tt}y+F_{3,tt}z)+2(C_1-C_3)p\right.\\
&\left.+F_4-\frac{\Gamma z\rho g(C_3z+F_3)}{T_0}\right]\frac{\partial}{\partial p}\\
&+(F_{3,t}+C_1w-C_3w)\frac{\partial}{\partial w}+\left(C_1T-2C_3T\right.\\
&\left.-2\Gamma zC_3-\Gamma F_3+\frac{C_5T_0}{\rho g}\right)\frac{\partial}{\partial T}. \qquad (25)
\end{aligned}
$$

All invariant group vectors can be written as

$$
\begin{aligned}
V \;=\;&C_1V+C_2V_2+C_3V_3+C_4V_4+C_5V_5\\
&+V_6(F_1)+V_7(F_2)+V_8(F_3)+V_9(F_4),
\end{aligned}
$$

where

$$
\begin{aligned}
V \;=\;&x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}+w\frac{\partial}{\partial w}\\
&+2p\frac{\partial}{\partial p}+T\frac{\partial}{\partial T},\\
V_2 \;=\;&y\frac{\partial}{\partial x}-x\frac{\partial}{\partial y}+v\frac{\partial}{\partial u}-u\frac{\partial}{\partial v},\\
V_3 \;=\;&t\frac{\partial}{\partial t}-u\frac{\partial}{\partial u}-v\frac{\partial}{\partial v}-w\frac{\partial}{\partial w}\\
&-\left(2p+\frac{\Gamma\rho gz^2}{T_0}\right)\frac{\partial}{\partial p}-2(T+\Gamma z)\frac{\partial}{\partial T},\\
V_4 \;=\;&C_4\frac{\partial}{\partial t},\;\; V_5=z\frac{\partial}{\partial p}+\frac{T_0}{\rho g}\frac{\partial}{\partial T},\\
V_6 \;=\;&F_1\frac{\partial}{\partial x}+F_{1,t}\frac{\partial}{\partial u}-\rho F_{1,tt}x\frac{\partial}{\partial p},\\
V_7 \;=\;&F_2\frac{\partial}{\partial y}+F_{2,t}\frac{\partial}{\partial v}-\rho F_{2,tt}y\frac{\partial}{\partial p},\\
V_8 \;=\;&F_3\frac{\partial}{\partial z}-\left(\rho F_{3,tt}z+\frac{\Gamma z\rho gF_3}{T_0}\right)\frac{\partial}{\partial p}\\
&+F_{3,t}\frac{\partial}{\partial w}-\Gamma F_3\frac{\partial}{\partial T},\\
V_9 \;=\;&F_4\frac{\partial}{\partial p}, \qquad (26)
\end{aligned}
$$

where V and V_3 are the scaling transformations, V_2 indicates the rotational symmetry in latitudinal and longitudinal directions, V_4 denotes the time-translation invariance, and the others are related to Galilean boosts.

3. Symmetry reductions

In this section, we will find symmetry reductions of the (3+1)-dimensional Boussinesq system. In order to find symmetry reductions, we have to solve the characteristic equations

$$
\frac{\mathrm{d}x}{\bar{x}}=\frac{\mathrm{d}y}{\bar{y}}=\frac{\mathrm{d}z}{\bar{z}}=\frac{\mathrm{d}t}{\bar{t}}=\frac{\mathrm{d}u}{\bar{u}}=\frac{\mathrm{d}v}{\bar{v}}=\frac{\mathrm{d}w}{\bar{w}}=\frac{\mathrm{d}p}{\bar{p}}=\frac{\mathrm{d}T}{\bar{T}}, \qquad (27)
$$

where $\bar{x}$, $\bar{y}$, $\bar{z}$, $\bar{t}$, $\bar{u}$, $\bar{v}$, $\bar{w}$, $\bar{p}$, and $\bar{T}$ are given by formula (23). Specifically, three types of similarity reduction solutions can be found.

3.1. First type of reduction

In the case of $C_3\neq 0$, we obtain the first type of group invariants

$$
\begin{aligned}
\xi \;=\;&\int\frac{\sin\left(C_2\ln(\frac{C_3t+C_4}{C_3})/C_3\right)[C_2F_2-C_1F_1+(C_3t+C_4)F_{1,t}]C_3^{\frac{C_1}{C_3}}}{C_2(C_3t+C_4)^{\frac{C_1}{C_3}+1}}\mathrm{d}t\\
&+\frac{C_2\,x\cos\left(C_2\ln(\frac{C_3t+C_4}{C_3})/C_3\right)-(C_2y+F_1)\sin\left(C_2\ln(\frac{C_3t+C_4}{C_3})/C_3\right)}{C_2(\frac{C_3t+C_4}{C_3})^{\frac{C_1}{C_3}}},\\
\eta \;=\;&\int\frac{\cos\left(C_2\ln(\frac{C_3t+C_4}{C_3})/C_3\right)[C_1F_1-C_2F_2-(C_3t+C_4)F_{1,t}]C_3^{\frac{C_1}{C_3}}}{C_2(C_3t+C_4)^{\frac{C_1}{C_3}+1}}\mathrm{d}t\\
&+\frac{C_2\,x\sin\left(C_2\ln(\frac{C_3t+C_4}{C_3})/C_3\right)+(C_2y+F_1)\cos\left(C_2\ln(\frac{C_3t+C_4}{C_3})/C_3\right)}{C_2(\frac{C_3t+C_4}{C_3})^{\frac{C_1}{C_3}}},\\
\zeta \;=\;&\int\frac{F_3}{(C_3t+C_4)^{\frac{C_1}{C_3}+1}}\mathrm{d}t+\frac{z}{(C_3t+C_4)^{\frac{C_1}{C_3}}}. \qquad (28)
\end{aligned}
$$

The similarity solution and the similarity reduction equations are extremely complicated and lengthy. On condition that $F_1=0$, $F_2=0$, $F_3=0$, and $F_4=0$, the group invariants ξ, η, and ζ transform to

$$\begin{aligned}
\xi &= \frac{x\cos\left(C_2\ln(\frac{C_3t+C_4}{C_3})/C_3\right)-y\sin\left(C_2\ln(\frac{C_3t+C_4}{C_3})/C_3\right)}{(\frac{C_3t+C_4}{C_3})^{\frac{C_1}{C_3}}},\\
\eta &= \frac{x\sin\left(C_2\ln(\frac{C_3t+C_4}{C_3})/C_3\right)+y\cos\left(C_2\ln(\frac{C_3t+C_4}{C_3})/C_3\right)}{(\frac{C_3t+C_4}{C_3})^{\frac{C_1}{C_3}}},\\
\zeta &= \frac{z}{(C_3t+C_4)^{\frac{C_1}{C_3}}},
\end{aligned} \tag{29}$$

and the similarity solutions are

$$\begin{aligned}
u &= (C_3t+C_4)^{\frac{C_1}{C_3}-1}\left[\sin\left(C_2\ln\right]\left(\frac{C_3t+C_4}{C_3}\right)/C_3\right)U\\
&\quad+\cos\left(C_2\ln\left(\frac{C_3t+C_4}{C_3}\right)/C_3\right)V\Big],\\
v &= (C_3t+C_4)^{\frac{C_1}{C_3}-1}\left[\cos\left(C_2\ln\left(\frac{C_3t+C_4}{C_3}\right)/C_3\right)U\right.\\
&\quad\left.-\sin\left(C_2\ln\left(\frac{C_3t+C_4}{C_3}\right)/C_3\right)V\right],\\
w &= (C_3t+C_4)^{\frac{C_1}{C_3}-1}W,\\
p &= (C_3t+C_4)^{\frac{2C_1}{C_3}-2}P-\frac{(C_3t+2C_4)\rho gt\Gamma C_3\zeta^2}{2T_0}\\
&\quad\times(C_3t+C_4)^{\frac{2C_1}{C_3}-2}-\frac{(C_3t+C_4)^{\frac{C_1}{C_3}}C_5\zeta}{C_1-2C_3},\\
T &= (C_3t+C_4)^{\frac{C_1}{C_3}-2}\widetilde{T}-(C_3t+2C_4)\Gamma C_3t\\
&\quad\times(C_3t+C_4)^{\frac{C_1}{C_3}-2}\zeta-\frac{C_5T_0}{(C_1-2C_3)\rho g},
\end{aligned} \tag{30}$$

where U, V, W, P, and $\widetilde{T}$ are functions of variables ξ, η, and ζ. The corresponding similarity reduction equations are

$$\begin{aligned}
&\frac{1}{\rho}C_3^{\frac{2C_1-C_3}{C_3}}P_\xi+(C_2\xi+C_3U-C_1\eta)V_\eta\\
&-(C_1\xi-C_3V+C_2\eta)V_\xi+(W-C_1\zeta)V_\zeta\\
&+C_2U+(C_1-C_3)V=0,
\end{aligned} \tag{31}$$

$$\begin{aligned}
&\frac{1}{\rho}C_3^{\frac{2C_1-C_3}{C_3}}P_\eta+(C_2\xi+C_3U-C_1\eta)U_\eta\\
&-(C_1\xi-C_3V+C_2\eta)U_\xi+(W-C_1\zeta)U_\zeta-C_2V\\
&+(C_1-C_3)U=0,
\end{aligned} \tag{32}$$

$$\begin{aligned}
&\frac{1}{\rho}P_\zeta+(C_2\xi+C_3U-C_1\eta)W_\eta-(C_1\xi-C_3V+C_2\eta)W_\xi\\
&-(C_1\zeta-W)W_\zeta+(C_1-C_3)W-g\frac{\widetilde{T}}{T_0}=0,
\end{aligned} \tag{33}$$

$$\begin{aligned}
&(C_2\xi+C_3U-C_1\eta)\widetilde{T}_\eta-(C_1\xi-C_3V+C_2\eta)\widetilde{T}_\xi\\
&+(W-C_1\zeta)\widetilde{T}_\zeta+\Gamma C_4^2W+(C_1-2C_3)\widetilde{T}\\
&-2\Gamma C_3C_4^2\zeta=0,
\end{aligned} \tag{34}$$

$$C_3V_\xi+C_3U_\eta+W_\zeta=0. \tag{35}$$

3.2. Second type of reduction

When $C_3=0$ and $C_4\neq0$, we have the second type of group invariants

$$\begin{aligned}
\xi &= \frac{1}{C_2C_4}\int\sin\left(\frac{C_2t}{C_4}\right)(C_2F_2-C_1F_1+C_4F_{1,t})\mathrm{e}^{-\frac{C_1t}{C_4}}\,\mathrm{d}t\\
&\quad-\left(y+\frac{F_1}{C_2}\right)\sin\left(\frac{C_2t}{C_4}\right)\mathrm{e}^{-\frac{C_1t}{C_4}}+x\cos\left(\frac{C_2t}{C_4}\right)\mathrm{e}^{-\frac{C_1t}{C_4}},\\
\eta &= \frac{1}{C_2C_4}\int\cos\left(\frac{C_2t}{C_4}\right)(C_1F_1-C_2F_2-C_4F_{1,t})\mathrm{e}^{-\frac{C_1t}{C_4}}\,\mathrm{d}t\\
&\quad+\left(y+\frac{F_1}{C_2}\right)\cos\left(\frac{C_2t}{C_4}\right)\mathrm{e}^{-\frac{C_1t}{C_4}}+x\sin\left(\frac{C_2t}{C_4}\right)\mathrm{e}^{-\frac{C_1t}{C_4}},\\
\zeta &= z\mathrm{e}^{-\frac{C_1t}{C_4}}-\frac{1}{C_4}\int F_3\,\mathrm{e}^{-\frac{C_1t}{C_4}}\,\mathrm{d}t.
\end{aligned} \tag{36}$$

When $F_1=0$, $F_2=0$, and $F_3=0$, the variables degenerate to

$$\begin{aligned}
\xi &= -y\sin\left(\frac{C_2t}{C_4}\right)\mathrm{e}^{-\frac{C_1t}{C_4}}+x\cos\left(\frac{C_2t}{C_4}\right)\mathrm{e}^{-\frac{C_1t}{C_4}},\\
\eta &= y\cos\left(\frac{C_2t}{C_4}\right)\mathrm{e}^{-\frac{C_1t}{C_4}}+x\sin\left(\frac{C_2t}{C_4}\right)\mathrm{e}^{-\frac{C_1t}{C_4}},\\
\zeta &= z\,e^{-\frac{C_1t}{C_4}}.
\end{aligned} \tag{37}$$

The corresponding similar solutions are

$$\begin{aligned}
u &= \mathrm{e}^{\frac{C_1t}{C_4}}\left[\sin\left(\frac{C_2t}{C_4}\right)U+\cos\left(\frac{C_2t}{C_4}\right)V\right],\\
v &= -\mathrm{e}^{\frac{C_1t}{C_4}}\left[\sin\left(\frac{C_2t}{C_4}\right)V-\cos\left(\frac{C_2t}{C_4}\right)U\right],\\
w &= \mathrm{e}^{\frac{C_1t}{C_4}}W,\\
p &= -\mathrm{e}^{\frac{C_1t}{C_4}}\frac{C_5\zeta}{C_1}+\mathrm{e}^{\frac{2C_1t}{C_4}}P+\mathrm{e}^{\frac{2C_1t}{C_4}}\frac{1}{C_4}\int F_4\,\mathrm{e}^{-\frac{2C_1t}{C_4}}\,\mathrm{d}t,\\
T &= \mathrm{e}^{\frac{C_1t}{C_4}}\widetilde{T}-\frac{C_5T_0}{C_1\rho g},
\end{aligned} \tag{38}$$

where the variables U, V, W, P, and $\widetilde{T}$ are determined by the similarity reduction equations

$$\begin{aligned}
&(C_2\xi-C_1\eta+C_4U)U_\eta-(C_1\xi+C_2\eta-C_4V)U_\xi\\
&-(C_1\zeta-C_4W)U_\zeta+(C_1U-C_2V)+\frac{C_4}{\rho}P_\eta=0,
\end{aligned} \tag{39}$$

$$\begin{aligned}
&(C_2\xi-C_1\eta+C_4U)V_\eta-(C_1\xi+C_2\eta-C_4V)V_\xi\\
&-(C_1\zeta-C_4W)V_\zeta+(C_1V-C_2U)+\frac{C_4}{\rho}P_\eta=0,
\end{aligned} \tag{40}$$

$$\begin{aligned}
&(C_2\xi-C_1\eta+C_4U)W_\eta-(C_1\xi+C_2\eta-C_4V)W_\xi\\
&-(C_1\zeta-C_4W)W_\zeta+\left(C_1W-\frac{gC_4\widetilde{T}}{T_0}\right)+\frac{C_4}{\rho}P_\zeta=0,
\end{aligned} \tag{41}$$

$$\begin{aligned}
&(C_2\xi-C_1\eta+C_4U)\widetilde{T}_\eta-(C_1\xi+C_2\eta-C_4V)\widetilde{T}_\xi\\
&-(C_1\zeta-C_4W)\widetilde{T}_\zeta+C_1\widetilde{T}-\Gamma C_4W=0,
\end{aligned} \tag{42}$$

$$V_\xi+U_\eta+W_\zeta=0. \tag{43}$$

3.3. Third type of reduction

For $C_3 = C_4 = 0$ and $F_1 = 0 = F_2 = F_3 = 0$, we have the similarity solutions

$$u = zV\sin\left(\frac{C_2\ln z}{C_1}\right) + zU\cos\left(\frac{C_2\ln z}{C_1}\right), \tag{44}$$

$$v = zV\cos\left(\frac{C_2\ln z}{C_1}\right) - zU\sin\left(\frac{C_2\ln z}{C_1}\right), \tag{45}$$

$$w = zW, \tag{46}$$

$$p = -\frac{2zC_5 + F_4}{2C_1} + z^2 P, \tag{47}$$

$$T = z\widetilde{T} - \frac{C_5 T_0}{\rho g C_1}, \tag{48}$$

where U, V, W, P, and $\widetilde{T}$ are the similarity reduction functions with respect to the similarity variables

$$\begin{aligned}
\xi &= \frac{y}{z}\cos\left(\frac{C_2\ln z}{C_1}\right) + \frac{x}{z}\sin\left(\frac{C_2\ln z}{C_1}\right),\\
\eta &= -\frac{y}{z}\sin\left(\frac{C_2\ln z}{C_1}\right) + \frac{x}{z}\cos\left(\frac{C_2\ln z}{C_1}\right),\\
\zeta &= t.
\end{aligned} \tag{49}$$

The corresponding similarity reduction equations are

$$\left(U - \frac{C_2}{C_1}\xi W - \eta W\right)U_\eta + \left(V + \frac{C_2}{C_1}\eta W - \xi W\right)U_\xi + \frac{P_\eta}{\rho} + U_\zeta + W\left(U + \frac{C_2}{C_1}V\right) = 0, \tag{50}$$

$$\left(U - \frac{C_2}{C_1}\xi W - \eta W\right)V_\eta + \left(V + \frac{C_2}{C_1}\eta W - \xi W\right)V_\xi + \frac{P_\xi}{\rho} + V_\zeta + W\left(V - \frac{C_2}{C_1}U\right) = 0, \tag{51}$$

$$\left(U - \frac{C_2}{C_1}\xi W - \eta W\right)W_\eta + \left(V + \frac{C_2}{C_1}\eta W - \xi W\right)W_\xi - \left(\frac{\eta}{\rho} + \frac{C_2\xi}{C_1\rho}\right)P_\eta + \left(\frac{C_2\eta}{C_1\rho} - \frac{\xi}{\rho}\right)P_\xi + W_\zeta + \frac{2}{\rho}P - \frac{g}{T_0}\widetilde{T} + W^2 = 0, \tag{52}$$

$$\left(U - \frac{C_2}{C_1}\xi W - \eta W\right)\widetilde{T}_\eta + \left(V + \frac{C_2}{C_1}\eta W - \xi W\right)\widetilde{T}_\xi + \widetilde{T}_\zeta + \left(\Gamma + \widetilde{T}\right)W = 0, \tag{53}$$

$$U_\eta + V_\xi - \left(\frac{C_2\xi}{C_1} + \eta\right)W_\eta + \left(\frac{C_2\eta}{C_1} - \xi\right)W_\xi + W = 0. \tag{54}$$

4. Exact solutions

We can obtain various exact solutions of the INHB equations by means of the symmetry method. Different reduction equations lead to different results. In this section, we will discuss non-traveling and traveling wave solutions with the help of the second type of reduction equations and the corresponding similarity solutions.

4.1. Non-traveling wave solutions

In conditions that $C_4 \neq 0$ and $C_3 = F_1 = F_2 = F_3 = 0$, the group invariants ξ, η, ζ and the similarity solutions are described by formulas (37) and (38), respectively. The corresponding reduction equations are Eqs. (39)–(43). For Eqs. (39)–(43), there exist the following solutions

$$\begin{aligned}
U &= k_1 - \frac{1}{2}\frac{k_3(k_{12} - 2k_2k_4)\xi}{k_5k_7} + k_2\eta + k_3\zeta,\\
V &= -\frac{k_5k_6}{k_4} - \frac{k_5k_2(C_1k_9 + C_4k_1k_{10})}{k_{10}C_4k_4k_3} + k_4\xi + \frac{1}{2}\frac{k_{12}\eta k_3}{k_5k_2} + \frac{k_5k_2\eta}{k_3} + k_5\zeta,\\
W &= k_6 + k_7\xi - \frac{k_4k_{12}\eta k_3}{{k_5}^2k_2} - \frac{k_2(k_2 + k_4)\eta}{k_3} - (k_2 + k_4)\zeta,\\
P &= k_8 + \frac{\rho k_5C_1k_{13}\xi}{k_{10}{C_4}^2k_4k_3} + \frac{\rho C_1(-k_1k_{10} + k_9k_2)\eta}{k_{10}C_4} - \rho k_{14}\zeta + k_{15}(\xi^2 + \eta^2) + \frac{1}{2}\rho k_{16}\zeta^2,\\
\widetilde{T} &= k_9 + \frac{k_{10}k_3k_4\xi}{k_5k_2} + k_{10}\eta + k_{11}\zeta,
\end{aligned} \tag{55}$$

where the parameters satisfy

$$\begin{aligned}
k_2 &= \frac{k_3k_{10}}{\Gamma + k_{11}},\quad k_{15} = \frac{1}{8}\frac{\rho k_{12}^2{k_3}^2}{{k_5}^2{k_2}^2},\\
k_{10} &= \frac{T_0({k_4}^2 + k_2k_4 + k_5k_7)({k_2}^2{k_5}^2 + {k_3}^2{k_4}^2 + {k_3}^2k_7k_5)}{g{k_5}^2k_3k_2},\\
k_{12} &= k_2k_4 + {k_4}^2 + k_7k_5,\\
k_{13} &= k_2(k_4C_4 + C_1)k_9 + C_4k_{10}(k_3k_6 + k_2k_1),\\
k_{14} &= -\frac{k_{12}({k_4}^2k_9 + k_1k_{10}k_4 + k_9k_7k_5)k_3}{{k_5}^2k_2k_{10}} - \frac{k_7(k_1k_{10} + k_4k_9)k_{12}}{k_{10}k_4k_3} - \frac{k_6k_{12}}{k_4} + \frac{C_1k_6}{C_4} - \frac{k_5k_7k_2C_1k_9}{C_4k_{10}k_4k_3},\\
k_{16} &= \frac{k_4k_{12}{k_3}^2}{{k_5}^2k_2} + \frac{k_{11}({k_{10}}^2 + k_7k_5)k_{12}k_3}{{k_5}^2k_7k_{10}} - k_{12} + \frac{k_{11}k_2k_{12}}{k_{10}k_3}.
\end{aligned}$$

The substitution of $U, V, W, P, \widetilde{T}$ and ξ, η, ζ into Eq. (38) leads to

$$u = k_4x - \left\{\frac{k_5(C_1k_9k_2 + k_{10}k_3C_4k_6 + C_4k_1k_{10}k_2)\exp\left(\frac{C_1t}{C_4}\right)}{k_{10}C_4k_3k_4} - \left[\frac{({k_5}^2k_2 + k_4{k_3}^2)x}{k_5k_3} + (k_2 - k_4)y\right]\sin\left(\frac{C_2t}{C_4}\right) - k_5z\right\}$$

$$\times\cos\left(\frac{C_2t}{C_4}\right)+\sin\left(\frac{C_2t}{C_4}\right)k_1\exp\left(\frac{C_1t}{C_4}\right)+\left[(k_2-k_4)x-\frac{({k_5}^2k_2+k_4{k_3}^2)y}{k_5k_3}\right]\sin\left(\frac{C_2t}{C_4}\right)^2+k_3z\sin\left(\frac{C_2t}{C_4}\right)$$
$$+\frac{1}{2}\frac{({k_3}^2{k_4}^2+{k_3}^2k_5k_7+{k_3}^2k_2k_4+2{k_5}^2{k_2}^2)y}{k_5k_2k_3},$$

$$v=\left[(k_7-k_4)x-\frac{({k_5}^2k_2+k_4{k_3}^2)y}{k_5k_3}\right]\sin\left(\frac{C_2t}{C_4}\right)\cos\left(\frac{C_2t}{C_4}\right)+\frac{k_5(C_1k_9k_7+k_{10}k_3C_4k_6+C_4k_1k_{10}k_2)\sin\left(\frac{C_2t}{C_4}\right)\exp\left(\frac{C_1t}{C_4}\right)}{k_{10}C_4k_3k_4}$$
$$-\left[\frac{({k_5}^2k_2+k_4{k_3}^2)x}{k_5k_3}+(k_2-k_4)y\right]\sin\left(\frac{C_2t}{C_4}\right)^2-k_5z\sin\left(\frac{C_2t}{C_4}\right)+\frac{1}{2}\frac{k_3(k_2k_4-{k_4}^2-k_5k_7)x}{k_5k_2}+k_2y$$
$$+\left[k_1\exp\left(\frac{C_1t}{C_4}\right)+k_3z\right]\cos\left(\frac{C_2t}{C_4}\right),$$

$$w=\left[xk_7-\frac{(k_4k_{12}{k_3}^2+k_4{k_5}^2{k_2}^2+{k_5}^2{k_2}^3)y}{{k_5}^2k_2k_3}\right]\cos\left(\frac{C_2t}{C_4}\right)+k_6\exp\left(\frac{C_1t}{C_4}\right)-\left[\frac{(k_4k_{12}{k_3}^2+k_4{k_5}^2{k_2}^2+{k_5}^2{k_2}^3)x}{{k_5}^2k_2k_3}+yk_7\right]$$
$$\times\sin\left(\frac{C_2t}{C_4}\right)-(k_2+k_4)z,$$

$$p=\left[\frac{\rho k_5C_1k_{13}x}{k_{10}{C_4}^2k_4k_3}+\frac{\rho C_1(-k_1k_{10}+k_9k_2)y}{C_4k_{10}}\right]\exp\left(\frac{C_1t}{C_4}\right)\cos\left(\frac{C_2t}{C_4}\right)+\left\{\left[\frac{\rho C_1(-k_1k_{10}+k_9k_2)x}{C_4k_{10}}-\frac{\rho k_5C_1k_{13}y}{k_{10}{C_4}^2k_4k_3}\right]\right.$$
$$\left.\times\sin\left(\frac{C_2t}{C_4}\right)-\rho k_{14}z\right\}\exp\left(\frac{C_1t}{C_4}\right)+k_{15}x^2+\left[\frac{1}{C_4}\int\frac{\mathrm{F}_4}{\exp\left(\frac{2C_1t}{C_4}\right)}\mathrm{d}t+k_8\right]\exp\left(\frac{2C_1t}{C_4}\right)+k_{15}y^2+\frac{1}{2}\rho k_{16}z^2-\frac{C_5z}{C_1},$$

$$T=\left[\frac{k_{10}k_4k_3x}{k_5k_2}+k_{10}y\right]\cos\left(\frac{C_2t}{C_4}\right)+\left[k_{10}x-\frac{k_4k_{10}k_3y}{k_5k_2}\right]\sin\left(\frac{C_2t}{C_4}\right)+k_9\exp\left(\frac{C_1t}{C_4}\right)+k_{11}z-\frac{C_5T_0}{g\rho C_1}. \tag{56}$$

The above formulas show that the solutions for u,v,w,p, and T are all non-traveling wave solutions.

4.2. Traveling wave solutions

When $C_1=C_2=C_3=0$ and $F_1=C_{11}$, $F_2=C_{12}$, $F_3=C_{13}$, then $\bar{x}=C_{11}$, $\bar{y}=C_{12}$, $\bar{z}=C_{13}$, and $\bar{t}=C_4$, which means that independent variables are translational invariant. The corresponding one-parameter group ϕ is

$$\mu(x,y,z,t)=\mu(x+\varepsilon C_{11},y+\varepsilon C_{12},z+\varepsilon C_{13},t+\varepsilon C_4),\quad(\mu=u,v,w,p,T). \tag{57}$$

Then we can obtain the traveling wave transformation

$$u=U(\psi),\ v=V(\psi),\ w=W(\psi),\ p=P(\psi),$$
$$T=\widetilde{T}(\psi),\ \psi=k_{17}x+k_{18}y+k_{19}z+k_{20}t.$$

Equations (1)–(5) read

$$(k_{17}U+k_{18}V+k_{19}W+k_{20})U_\psi+\frac{k_{17}}{\rho}P_\psi=0, \tag{58}$$
$$(k_{17}U+k_{18}V+k_{19}W+k_{20})V_\psi+\frac{k_{18}}{\rho}P_\psi=0, \tag{59}$$
$$(k_{17}U+k_{18}V+k_{19}W+k_{20})W_\psi+\frac{k_{19}}{\rho}P_\psi-\frac{g\widetilde{T}}{T_0}=0, \tag{60}$$
$$(k_{17}U+k_{18}V+k_{19}W+k_{20})\widetilde{T}_\psi+\Gamma W=0, \tag{61}$$
$$k_{17}U_\psi+k_{18}V_\psi+k_{19}W_\psi=0. \tag{62}$$

Integrating ψ to Eq. (62), we obtain

$$k_{17}U+k_{18}V+k_{19}W+C_{15}=0, \tag{63}$$

where C_{15} is an arbitrary integral constant, then

$$U=-\frac{k_{18}V+k_{19}W+C_{15}}{k_{17}}. \tag{64}$$

The substitution of Eq. (64) into Eq. (58) leads to

$$\rho k_{18}(C_{15}-k_{20})V+\rho k_{19}(C_{15}-k_{20})W+k_{17}^2P+C_{16}=0, \tag{65}$$

which means

$$V=-\frac{k_{19}W}{k_{18}}-\frac{k_{17}^2P+C_{16}}{\rho k_{18}(C_{15}-k_{20})}. \tag{66}$$

From Eqs. (59), (64), and (66), we have

$$W=-\frac{(k_{18}^2+k_{17}^2)P+C_{17}k_{18}}{k_{19}\rho(C_{15}-k_{20})}. \tag{67}$$

Substituting Eqs. (64), (66), and (67) into Eq. (60), we obtain

$$\widetilde{T}=\frac{T_0(k_{17}^2+k_{18}^2+k_{19}^2)P_\psi}{g\rho k_{19}}. \tag{68}$$

Therefore, equation (61) can be written as

$$\Gamma g(k_{17}^2+k_{18}^2)P+T_0(k_{17}^2+k_{18}^2+k_{19}^2)(C_{15}-k_{20}^2)P_\psi-\Gamma C_{17}gk_{18}=0. \tag{69}$$

Then we obtain

$$P(\psi)=\sin(\theta)C_{18}+\cos(\theta)\frac{C_{17}C_{19}k_{18}}{k_{17}^2+k_{18}^2}, \tag{70}$$

where

$$\theta = \sqrt{\frac{\Gamma g(k_{17}^2+k_{18}^2)}{T_0(k_{17}^2+k_{18}^2+k_{19}^2)}}\frac{\psi}{C_{15}-k_{20}} = \sqrt{\frac{\Gamma g(k_{17}^2+k_{18}^2)}{T_0(k_{17}^2+k_{18}^2+k_{19}^2)}}\frac{k_{17}x+k_{18}y+k_{19}z+k_{20}t}{C_{15}-k_{20}}. \quad (71)$$

Furthermore, we find

$$\begin{aligned} U &= \frac{k_{17}(C_{19}\cos(\theta)+C_{18}\sin(\theta))}{\rho(C_{15}-k_{20})} - \frac{C_{15}}{k_{17}} \\ &\quad + \frac{C_{16}}{(C_{15}-k_{20})k_{17}\rho} - \frac{k_{17}k_{18}C_{17}}{(C_{15}-k_{20})(k_{17}^2+k_{18}^2)\rho}, \\ V &= \frac{k_{18}(C_{19}\cos(\theta)+C_{18}\sin(\theta))}{\rho(C_{15}-k_{20})} - \frac{C_{16}}{(C_{15}-k_{20})k_{18}\rho} \\ &\quad + \frac{k_{17}^2C_{17}}{(C_{15}-k_{20})(k_{17}^2+k_{18}^2)\rho}, \\ W &= -\frac{k_{17}^2+k_{18}^2}{k_{19}\rho(C_{15}-k_{20})}[C_{19}\cos(\theta)+C_{18}\sin(\theta)], \\ \widetilde{T} &= \frac{\sqrt{T_0\Gamma(k_{17}^2+k_{18}^2)(k_{17}^2+k_{18}^2+k_{19}^2)}}{k_{19}\rho(C_{15}-k_{20})\sqrt{g}} \\ &\quad \times [\cos(\theta)C_{18}-C_{19}\sin(\theta)]. \end{aligned} \quad (72)$$

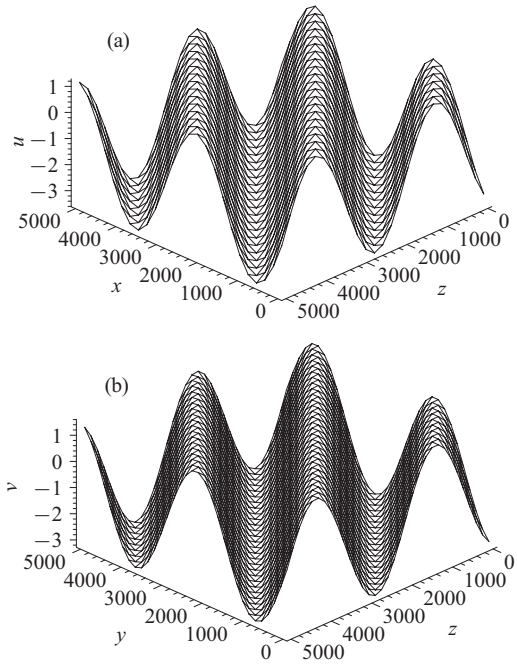


Fig. 1. Evolutions of wind velocities in (a) latitudinal and (b) longitudinal directions. The parameters are $k_{17}=k_{18}=k_{19}=k_{20}=1, C_{15}=C_{16}=C_{17}=C_{18}=C_{19}=2, \rho=1.2, T_0=290, \Gamma=0.0002, g=9.81$.

Figure 1 describes the evolutions of the wind velocities in latitudinal and longitudinal directions. Not surprisingly, both u and v have trigonometric forms. Figure 2 poses the evolution of the vertical velocity with height at different time. The solid line shows the evolution with $x=1000$ m, $y=1000$ m, $t=0$, and the dot line is related to the evolution with $x=1000$ m, $y=1000$ m, $t=500$ s. The evolution tells us that the waves spread down. Figure 3 shows the variations of perturbation pressure and temperature perturbation in space and time.

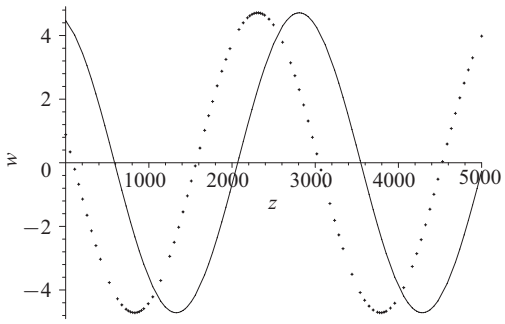


Fig. 2. Evolution of vertical velocity with height at $t=0$ (line) and $t=500$ s (dot). The other parameters are the same as those in Fig. 1.

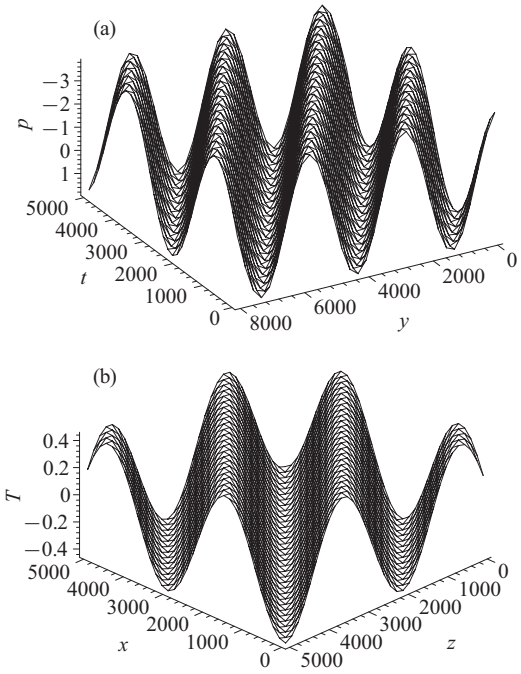


Fig. 3. Variations of (a) perturbation pressure and (b) temperature perturbation in space and time.

5. Summary and discussion

The (3+1)-dimensional INHB Eqs. (1)–(5) are of great importance due to their applications in atmospheric dynamics. We focus our attention on its symmetries, similarity reductions, and exact solutions in this paper. Calculations show that the INHB system is invariant under some Galilean transformations, scaling transformations, and space–time translations, which is consistent with the real observation on atmosphere gravity waves.

The symmetry reduction equations are classified to three types according to different symmetry parameter constraints. The solutions of the reduction equations are figured out. Simultaneously, we obtain the non-traveling and the period wave solutions for the original system by means of the symmetry

method, and some evolution curves of physical quantities with time are demonstrated.

The periodicity is one basic nature of wave phenomena. Noticing the formulas of u, v and group invariants ξ, η, and ζ, we can conclude that the periodicity is fully displayed in the (3+1)-dimensional INHB system. There exists the expression $\exp(C_1 t/C_4)$ in formula (38). When $C_1/C_4 < 0$, the velocities in all directions, the temperature perturbation, and the perturbation pressure will decay with time, which is consistent with the real observation on atmosphere gravity waves.

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References

[1] Li Z L 2008 *J. Phys. A: Math. Theor.* **41** 145206
[2] Li Z L 2009 *Chin. Phys. B* **18** 4074
[3] Qing J R, Chen S M, Li D W, Liang B and Liu B W 2012 *Chin. Phys. B* **21** 089401
[4] Liu P, Li Z L and Lou S Y 2010 *Applied Mathematics and Mechanics* **31** 1383
[5] McWilliams J C 2006 *Fundamentals of Geophysical Fluid Dynamics* (London: Cambridge University Press)
[6] Olver P 1986 *Applications of Lie Group to Differential Equations* (New York: Spring-Verlag)
[7] Bluman G W and Kumei S 1989 *Symmetries and Differential Equations* (New York: Spring-Verlag)
[8] Wang X X, Sun X T, Zhang M L, Han Y L and Jia L Q 2012 *Chin. Phys. B* **21** 050201
[9] Wang X Z, Fu H and Fu J L 2012 *Chin. Phys. B* **21** 040201
[10] Liu P and Fu P K 2011 *Chin. Phys. B* **20** 090203
[11] Liu P and Lou S Y 2009 *Commun. Theor. Phys.* **51** 27