Painlevé integrability of a generalized fifth-order KdV equation with variable coefficients: Exact solutions and their interactions*

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By means of singularity structure analysis, the integrability of a generalized fifth-order KdV equation is investigated. It is proven that this equation passes the Painlevé test for integrability only for three distinct cases. Moreover, the multi-soliton solutions are presented for this equation under three sets of integrable conditions. Finally, by selecting appropriate parameters, we analyze the evolution of two solitons, which is especially interesting as it may describe the overtaking and the head-on collisions of solitary waves of different shapes and different types.

Keywords: generalized fifth-order KdV equation, Painlevé integrability, soliton solution, symbolic computation

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1. Introduction

The nonlinear evolution equations (NLEEs), particularly the integrable ones, are of great interest both from mathematical and physical points of view. In the past decades, a vast variety of powerful methods to solve integrable models, such as the inverse scattering transformation,[1] the Bäcklund transformations, the Darboux transformations,[2] the Painlevé analysis,[3] and the Hirota bilinear approach,[4] have been established by many famous mathematical physicists.

Recently there is considerable interest in the study of variable-coefficient NLEEs.[5–8] The need for studying them is due to the fact that the equations with variable coefficients and nonisospectral parameters may provide more realistic models. Many physical and mechanical situations are governed by the variable-coefficient Korteweg–de Vries (KdV) equation,[9–13] e.g., the nonlinear excitations of a gas of impenetrable bosons with longitudinal confinement, the nonlinear waves in various types of rods. A variable-coefficient NLS equation can describe the propagation of pulses in an inhomogeneous optical fiber.[14–19]

In this paper, we will consider a generalized fifth-order KdV equation with variable coefficients[20]

$$u_t + a(t)uu_{xxx} + b(t)u_xu_{xx} + c(t)u^2u_x + d(t)uu_x + e(t)u_{xxx} + l(t)u_{xxxxx} + m(t)u + n(t)u_x = 0, \quad (1)$$

where u is a real function of x and t, and coefficients $a(t)$, $b(t)$, $c(t)$, $d(t)$, $e(t)$, $l(t)$, $m(t)$, and $n(t)$ are analytic functions of t. Equation (1) describes the interaction between a water wave and a floating ice cover or gravity-capillary waves in fluid dynamics. References [20] and [21] have investigated Eq. (1) under the parametric constraints

$$a(t) = b(t) = \frac{15l(t)}{\rho}\exp\left(\int m(t)\,\mathrm{d}t\right),$$
$$c(t) = \frac{45l(t)}{\rho^2}\exp\left(\int 2m(t)\,\mathrm{d}t\right),\ d(t) = e(t) = 0, \quad (2)$$

and presented Bäcklund transformation, a Lax pair, multi-soliton solutions, as well as double-period solutions. When $l(t) = m(t) = n(t) = 0$, references [22]– [24] have also given some analytic solutions. To the author's knowledge, the integrable properties of Eq. (1) with general forms have not been discussed before.

This paper is organized as follows. In Section 2, we will derive the integrable conditions for Eq. (1) by performing the Painlevé test. In Section 3, under three sets of Painlevé-integrable conditions, the multisoliton solutions for Eq. (1) are constructed by applying the Hirota method. In Section 4, we will show the interaction of multiple solitons. Finally, the conclusions will be given in Section 5.

2. Painlevé test

In this section, we discuss the constraints on coefficients $\{a(t), b(t), c(t), d(t), e(t), l(t), m(t), n(t)\}$ under which equation (1) is integrable. Various techniques have been developed to determine whether a nonlinear model belongs to the integrable system. Among those, the Painlevé analysis is a widely applied and quite successful technique to investigate the integrability by analyzing the singularity structure of the solutions.[25–32]

According to the Weiss–Tabor–Carnevale (WTC) method,[3] the Painlevé test of Eq. (1) consists three steps. The first step is determining the leading order exponent and

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the leading order coefficient. Inserting $u(x,t) = u_0\phi^\alpha$ into Eq. (1) and balancing the most dominant terms, we obtain two branches as follows:

$$\alpha = -2,\ u_0 = -3(b(t)+2a(t)+g(t))\phi_x^2/c(t), \quad (3)$$

$$\alpha = -2,\ u_0 = -3(b(t)+2a(t)-g(t))\phi_x^2/c(t), \quad (4)$$

where $g(t) = \sqrt{b(t)^2+4a(t)b(t)+4a(t)^2-40c(t)l(t)}$, and $c(t)\neq 0$. In order to simplify the calculations, Kruskal's ansatz $\phi(x,t) = x+\psi(t)$ will be adopted in the following two steps.

The next step is to find the resonant points. For the branch given by Eq. (3), we substitute

$$u(x,t) = u_0\phi^{-2} + u_j\phi^{j-2} \quad (5)$$

into Eq. (1) and only retain its most singular terms. Its resonance equation reads

$$\begin{aligned}&\phi^{j-7}u_j(j+1)(j-6)[l(t)c(t)j^3 - 15l(t)c(t)j^2\\ &-(86l(t)c(t)+3a(t)g(t)-6a(t)^2-3a(t)b(t))j\\ &-6b(t)g(t)-12a(t)g(t)+6g(t)^2] = 0. \quad (6)\end{aligned}$$

In Eq. (6), two resonances $j = -1, 6$ can be determined uniquely, the other undetermined resonances are denoted as j_1, j_2, j_3. From Eq. (6), we can obtain the relation between j_1, j_2, j_3 and parameters $a(t)$, $b(t)$, $c(t)$, $g(t)$ as follows:

$$\begin{aligned}&j_1+j_2+j_3 = 15,\\ &j_1j_2j_3c(t)l(t) = 6g(t)(b(t)-2a(t)-g(t)),\\ &(j_1j_2+j_2j_3+j_1j_3)c(t)l(t)\\ &= 3a(t)g(t)-6a(t)^2-3b(t)a(t)+86. \quad (7)\end{aligned}$$

For Eq. (1) to possess the Painlevé property, undetermined resonances j_1, j_2, j_3 should be positive integers. Taking into account all possible values of j_1, j_2, and j_3, we have to study nine different cases, which are listed in Table 1. For the second branch (4), the relation between the resonant points and the coefficients is also given by Eq. (7), so its resonance analysis is omitted here.

Table 1. Three undetermined resonances and parametric choices for the principle branch.

Case	Parameter choices	j_1, j_2, j_3	Case	Parameter choices	j_1, j_2, j_3
1	$b(t)=\frac{a(t)}{4}, l(t)=\frac{a(t)^2}{8c(t)}$	1, 2, 12	2	$b(t)=\frac{17a(t)}{26}, l(t)=\frac{29a(t)^2}{169c(t)}$	1, 3, 11
3	$b(t)=\frac{9a(t)}{8}, l(t)=\frac{15a(t)^2}{64c(t)}$	1, 4, 10	4	$b(t)=\frac{29a(t)}{18}, l(t)=\frac{25a(t)^2}{81c(t)}$	1, 5, 9
5	$b(t)=a(t), l(t)=\frac{a(t)^2}{5c(t)}$	2, 3, 10	6	$b(t)=\frac{3a(t)}{2}, l(t)=\frac{a(t)^2}{4c(t)}$	2, 4, 9
7	$b(t)=2a(t), l(t)=\frac{3a(t)^2}{10c(t)}$	2, 5, 8	8	$b(t)=2a(t), l(t)=\frac{2a(t)^2}{9c(t)}$	3, 4, 8
9	$b(t)=\frac{5a(t)}{2}, l(t)=\frac{a(t)^2}{5c(t)}$	3, 5, 7			

Setting the coefficients equal to particular values in determining the resonant points may affect the results of leading orders and leading coefficients at the former steps. Therefore, the Painlevé test of Eq. (1) under all parametric constraints listed in Table 1 should be performed again.

For case 1 in Table 1, equation (1) admits two different branches: (i) $\alpha = -2$, $u_0 = -a(t)/[6c(t)]$, the resonances occur at -1, 1, 2, 6, 12; (ii) $\alpha = -2$, $u_0 = -15a(t)/[2c(t)]$, the resonance equation is $(j+1)(j-3)(j-6)(j^2-12j-10) = 0$, thus the resonances occur at -1, 3, 6, $6+\sqrt{46}$, $6-\sqrt{46}$. It is obvious that there are non-integer resonances for the secondary branch, thus equation (1) is not integrable in this case. Similarly, for cases 2–4 and 6 in Table 1, equation (1) fails the Painlevé test because there are non-integer resonance points in the second branches.

For case 8 in Table 1, equation (1) also admits two different branches: (i) $\alpha = -2$, $u_0 = -4a(t)/c(t)$, the resonances occur at -1, 3, 4, 6, 8; (ii) $\alpha = -2$, $u_0 = -20a(t)/c(t)$, the resonances occur at -1, -5, 6, 8, 12. Subsequently, for the principal branch (i), the coefficients of the Laurent series read

$$u_1 = 0, \quad u_2 = \frac{a(t)d(t)-3c(t)e(t)}{a(t)c(t)},$$

where u_3 is an arbitrary function. And then the resonant condition at level $j = 4$ becomes

$$\begin{aligned}&a(t)^2c(t)\psi'(t)+2a(t)^2d(t)^2-9a(t)c(t)d(t)e(t)\\ &+9c(t)^2e(t)^2+a(t)^2c(t)n(t) = 0. \quad (8)\end{aligned}$$

Since ψ is an arbitrary function, and $a(t)$, $c(t)$ cannot be zero, the resonant condition (8) cannot hold identically. Thus we can conclude that equation (1) fails the Painlevé test in case 8.

For case 5, equation (1) also admits two different branches: (i) $\alpha = -2$, $u_0 = -6a(t)/c(t)$, the resonances occur at -1, 2, 3, 6, 10; (ii) $\alpha = -2$, $u_0 = -12a(t)/c(t)$, the resonances occur at -1, -2, 5, 6, 12. The final step is to verify the compatibility conditions. For branch (i), we may suppose

$$u(x,t) = \sum_{j=0}^{10} u_j\phi^{j-2}, \quad (9)$$

where $\phi(x,t)=x+\psi(t)$, and $u_j=u_j(t)$. Inserting Eq. (9) into Eq. (1) and setting the coefficients of ϕ^j ($j=-6,\ldots,3$) to zero lead to

$$
\begin{aligned}
&\phi^{-6}: \ 36u_1a(t)^2/c(t)=0,\\
&\phi^{-5}: \ 8a(t)\left[18c(t)e(t)+2u_1^2c(t)^2-9a(t)d(t)\right]/c(t)^2=0,\\
&\quad\cdots\\
&\phi^{3}: \ 128u_1u_9a(t)+120u_8e(t)+4u_2u_6d(t)\\
&\quad+8u_1u_2u_7c(t)+4u_3^2u_4c(t)+8u_4u_6a(t)\\
&\quad+u_5m(t)+4u_6\psi'(t)-24u_8d(t)a(t)/c(t)\\
&\quad+4u_1^2u_8c(t)+4u_3u_5d(t)+2u_4^2d(t)\\
&\quad+8u_1u_4u_5c(t)+u_5'+4u_2^2u_6c(t)\\
&\quad+8u_1u_3u_6c(t)+4u_2u_4^2c(t)+8u_2u_3u_5c(t)\\
&\quad+4u_1u_7d(t)+32u_3u_7a(t)+72u_2u_8a(t)=0.
\end{aligned}
$$

From the coefficient of ϕ^{-6}, we obtain $u_1=0$. By inserting the value of u_1 to the coefficient of ϕ^{-5}, the resonant condition at level $j=2$ becomes

$$72a(t)[2e(t)c(t)-d(t)a(t)]=0, \tag{10}$$

where $a(t)$ is a nonzero function of t. From Eq. (10), we have

$$e(t)=\frac{d(t)a(t)}{2c(t)}. \tag{11}$$

Under condition (11), the coefficients u_1, u_4, u_5, u_7, u_8, u_9 in Eq. (9) are determined. Subsequently, the resonant conditions at levels $j=2,3,6$ all hold identically, which indicates that u_2, u_3, u_6 are arbitrary functions of t.

However, the compatibility condition at level $j=10$ is reduced to

$$
\begin{aligned}
&u_3a(t)c(t)[a(t)c(t)d(t)m(t)-a(t)c(t)d'(t)\\
&\quad-a(t)d(t)c'(t)+2c(t)d(t)a'(t)]\\
&\quad+4u_2u_3a(t)c(t)^2[m(t)a(t)c(t)+a'(t)c(t)\\
&\quad-a(t)c'(t)]+[a(t)^2c'(t)^2-a(t)^2c(t)c''(t)\\
&\quad+a(t)^2c(t)^2m'(t)+m(t)^2a(t)^2c(t)^2\\
&\quad-a(t)^2c(t)m(t)c'(t)+a'(t)c(t)a(t)c'(t)\\
&\quad+a(t)c(t)^2a''(t)-2a'(t)^2c(t)^2]=0.
\end{aligned}
\tag{12}
$$

Considering the arbitrariness of ψ, u_2, and u_3, we easily check that if parameters $a(t)$, $c(t)$, $d(t)$, $m(t)$ satisfy the following constraints:

$$c(t)=\mu_1a(t)\exp\left(\int m(t)\,\mathrm{d}t\right),\quad d(t)=2\mu_2a(t), \tag{13}$$

with μ_1 and μ_2 being arbitrary real constants, then condition (12) holds identically.

When we choose $b(t)=a(t)$, $l(t)=a(t)^2/[5c(t)]$, together with Eqs. (11) and (13), the Painlevé test for Eq. (1) shows that all compatibility conditions at non-negative resonances are satisfied identically, thus we obtain a new set of integrable conditions

$$
\begin{aligned}
&b(t)=a(t),\quad c(t)=\mu_1a(t)\exp\left(\int m(t)\,\mathrm{d}t\right),\\
&d(t)=2\mu_2a(t),\quad e(t)=\frac{\mu_2a(t)}{\mu_1\exp\left(\int m(t)\,\mathrm{d}t\right)},\\
&l(t)=\frac{a(t)}{5\mu_1\exp\left(\int m(t)\,\mathrm{d}t\right)}.
\end{aligned}
\tag{14}
$$

Under the above conditions, equation (1) can be rewritten as

$$
\begin{aligned}
&u_t+a(t)uu_{xxx}+a(t)u_xu_{xx}+\mu_1a(t)\mathrm{e}^{\int m(t)\,\mathrm{d}t}u^2u_x\\
&\quad+2\mu_2a(t)uu_x+\frac{\mu_2a(t)}{\mu_1\mathrm{e}^{\int m(t)\,\mathrm{d}t}}u_{xxx}\\
&\quad+\frac{a(t)}{5\mu_1\mathrm{e}^{\int m(t)\,\mathrm{d}t}}u_{xxxxx}+m(t)u+n(t)u_x=0,
\end{aligned}
\tag{15}
$$

where $\mu_1\neq0$, μ_2 are arbitrary constants, and $a(t)\neq0$, $m(t)$, $n(t)$ are arbitrary functions of t.

After finishing similar analyses for cases 7 and 9, we obtain two other Painlevé-integrable models. One reads

$$
\begin{aligned}
&u_t+a(t)uu_{xxx}+2a(t)u_xu_{xx}+\mu_3a(t)\mathrm{e}^{\int m(t)\,\mathrm{d}t}u^2u_x\\
&\quad+2\mu_3e(t)\mathrm{e}^{\int m(t)\,\mathrm{d}t}uu_x+e(t)u_{xxx}\\
&\quad+\frac{3a(t)}{10\mu_3\mathrm{e}^{\int m(t)\,\mathrm{d}t}}u_{xxxxx}+m(t)u+n(t)u_x=0,
\end{aligned}
\tag{16}
$$

where $\mu_3\neq0$, $a(t)\neq0$, and $e(t)$, $m(t)$, $n(t)$ are arbitrary functions of t. Another one is

$$
\begin{aligned}
&u_t+a(t)uu_{xxx}+\frac{5a(t)}{2}u_xu_{xx}+\mu_5a(t)\mathrm{e}^{\int m(t)\,\mathrm{d}t}u^2u_x\\
&\quad+2\mu_4a(t)uu_x+\frac{\mu_4a(t)}{\mu_5\mathrm{e}^{\int m(t)\,\mathrm{d}t}}u_{xxx}\\
&\quad+\frac{a(t)}{5\mu_5\mathrm{e}^{\int m(t)\,\mathrm{d}t}}u_{xxxxx}+m(t)u+n(t)u_x=0,
\end{aligned}
\tag{17}
$$

where $\mu_5\neq0$, μ_4 are arbitrary constants, and $a(t)\neq0$, $m(t)$, $n(t)$ are arbitrary functions of t.

To sum up, we derive three Painlevé-integrable conditions of Eq. (1). To the author's knowledge, these three Painlevé-integrable models (15)–(17) are reported here first. When $\mu_1=\rho/3$, $\mu_2=0$, and $a(t)=15l(t)\exp(\int m(t)\mathrm{d}t)$, equation (15) is reduced to the integrable model considered in Ref. [20].

3. Bilinear form and multisoliton solutions

In the present section, we solve three integrable models (15)–(17) by the Hirota method,[4] which is a powerful tool for obtaining a wide class of exact solutions of NLEEs.[33–41] The

crucial step of this method is to find a suitable dependent variable transformation, which transforms a given equation into the bilinear form. In the following, we make use of the truncated Painlevé expansions to derive the bilinear forms,[42] and the multisoliton solutions are constructed with the aid of symbolic computation.

3.1. The *N*-soliton solutions for Eq. (15)

Performing the Painlevé analysis, we can obtain the truncated Painlevé expansion

$$u=\frac{6}{\mu_1 \mathrm{e}^{\int m(t)\,\mathrm{d}t}}(\ln f)_{xx}+u_2, \tag{18}$$

where u_2 is the solution of Eq. (15). For simplicity, we take the trivial solution $u_2=0$. Substituting transformation (18) into Eq. (15) and integrating once with respect to x yield

$$\begin{aligned}&5\mu_1 \mathrm{e}^{\int m(t)\,\mathrm{d}t}(\ln f)_{xt}+5\mu_1 n(t)\mathrm{e}^{\int m(t)\,\mathrm{d}t}(\ln f)_{xx}\\&\quad+5\mu_2 a(t)\left[(\ln f)_{4x}+6(\ln f)_{xx}^2\right]\\&\quad+a(t)\left[(\ln f)_{6x}+30(\ln f)_{xx}(\ln f)_{4x}+60(\ln f)_{xx}^3\right]=0,\end{aligned} \tag{19}$$

which can be transformed into the bilinear form

$$\begin{aligned}&\left[5\mu_1 \mathrm{e}^{\int m(t)\,\mathrm{d}t}D_xD_t+5\mu_1 n(t)\mathrm{e}^{\int m(t)\,\mathrm{d}t}D_x^2\right.\\&\quad\left.+5\mu_2 a(t)D_x^4+a(t)D_x^6\right]f\cdot f=0,\end{aligned} \tag{20}$$

where D is the well-known Hirota bilinear operator[4] defined by

$$\begin{aligned}&D_x^m D_t^n a(x,t)\cdot b(x,t)\\&=\frac{\partial^m}{\partial s^m}\frac{\partial^n}{\partial y^n}a(x+s,t+y)b(x-s,t-y)|_{s=0,y=0},\\&\quad m,n=1,2,\ldots\end{aligned}$$

According to the Hirota method, equation (20) can be solved by introducing the following power series expansions for f:

$$f=1+\varepsilon f_1+\varepsilon^2 f_2+\varepsilon^3 f_3+\cdots. \tag{21}$$

Substituting Eq. (21) into Eq. (20) and equating the coefficients of ε yield

$$\begin{aligned}&\left[5\mu_1 \mathrm{e}^{\int m(t)\,\mathrm{d}t}D_xD_t+5\mu_1 n(t)\mathrm{e}^{\int m(t)\,\mathrm{d}t}D_x^2\right.\\&\quad\left.+5\mu_2 a(t)D_x^4+a(t)D_x^6\right]f_1\cdot 1=0,\end{aligned} \tag{22}$$

$$\begin{aligned}&\left[5\mu_1 \mathrm{e}^{\int m(t)\,\mathrm{d}t}D_xD_t+5\mu_1 n(t)\mathrm{e}^{\int m(t)\,\mathrm{d}t}D_x^2\right.\\&\quad\left.+5\mu_2 a(t)D_x^4+a(t)D_x^6\right](f_2\cdot 1+f_1\cdot f_1+1\cdot f_2)=0,\end{aligned} \tag{23}$$

$$\begin{aligned}&\left[5\mu_1 \mathrm{e}^{\int m(t)\,\mathrm{d}t}D_xD_t+5\mu_1 n(t)\mathrm{e}^{\int m(t)\,\mathrm{d}t}D_x^2\right.\\&\quad\left.+5\mu_2 a(t)D_x^4+a(t)D_x^6\right](f_3\cdot 1+f_2\cdot f_1)=0,\\&\quad\cdots\end{aligned} \tag{24}$$

to be solved.

To obtain the one-soliton solution of Eq. (15), we assume

$$f_1=\mathrm{e}^{\eta_1},\quad \eta_1=k_1x-\omega_1(t)+\xi_1^0, \tag{25}$$

where k_1 and ξ_1^0 are arbitrary constants, and $\omega_1(t)$ is a differentiable function of t to be determined. From Eqs. (22)–(24) we have

$$\begin{aligned}\omega_1(t)&=\int\left(k_1n(t)+\frac{\mu_2k_1^3a(t)}{\mu_1\mathrm{e}^{\int m(t)\,\mathrm{d}t}}+\frac{k_1^5a(t)}{5\mu_1\mathrm{e}^{\int m(t)\,\mathrm{d}t}}\right)\mathrm{d}t,\\&\quad f_i=0,\quad i\geq 2.\end{aligned} \tag{26}$$

Without loss of generality, we take $\varepsilon=1$, thus the one-soliton solution of Eq. (15) is

$$\begin{aligned}u&=\frac{6}{\mu_1\mathrm{e}^{\int m(t)\,\mathrm{d}t}}\left[\ln(1+\mathrm{e}^{\eta_1})\right]_{xx}\\&=\frac{3k_1^2}{2\mu_1\mathrm{e}^{\int m(t)\,\mathrm{d}t}}\operatorname{sech}^2\left(\frac{k_1x-\omega_1(t)+\xi_1^0}{2}\right),\end{aligned} \tag{27}$$

where $\mu_1\neq 0,\mu_2,k_1,\xi_1^0$ are arbitrary real constants, $a(t)\neq 0,m(t),n(t)$ are arbitrary functions of t, and $\omega_1(t)$ is given by Eq. (26).

Similar to the one-soliton solution, we can proceed to construct two-soliton and three-soliton solutions. In general, the N-soliton ($N\geq 2$) solutions of Eq. (15) can be expressed as

$$\begin{aligned}u&=\frac{6}{\mu_1\mathrm{e}^{\int m(t)\,\mathrm{d}t}}\\&\quad\times\left[\ln\left(\sum_{\nu=0,1}\exp\left(\sum_{j=1}^N\nu_j\eta_j+\sum_{1\leq i<j}^N\nu_i\nu_jA_{ij}\right)\right)\right]_{xx},\\\eta_j&=k_jx-\omega_j(t)+\xi_j^0,\omega_j(t)\\&=\int\left(k_jn(t)+\frac{\mu_2k_j^3a(t)}{\mu_1\mathrm{e}^{\int m(t)\,\mathrm{d}t}}+\frac{k_j^5a(t)}{5\mu_1\mathrm{e}^{\int m(t)\,\mathrm{d}t}}\right)\mathrm{d}t,\\&\quad j=1,\ldots,N,\\A_{ij}&=\ln(a_{ij}),\\a_{ij}&=\frac{(k_i-k_j)^2(3\mu_2+k_j^2-k_ik_j+k_i^2)}{(k_i+k_j)^2(3\mu_2+k_j^2+k_ik_j+k_i^2)},\end{aligned} \tag{28}$$

where $\mu_1\neq 0,\mu_2,k_j,\xi_j^0$ ($j=1,\ldots,N$) are arbitrary real constants, $a(t)\neq 0,m(t),n(t)$ are arbitrary functions of t, $\sum_{\nu=0,1}$ is a summation over all possible combinations of $\nu_1=0,1$, $\nu_2=0,1,\ldots,\nu_N=0,1$, and $\sum_{1\leq i<j}^N$ means a summation over all possible pairs (i,j) chosen from set $(1,2,\ldots,N)$ with the condition $1\leq i<j$.

3.2. The *N*-soliton solutions for Eq. (16)

For integrable model (16), substituting its truncated Painlevé expansion

$$u=\frac{6}{\mu_3\mathrm{e}^{\int m(t)\,\mathrm{d}t}}(\ln f)_{xx} \tag{29}$$

into it yields

$$\frac{a(t)}{2}\left[(\ln f)_{6x}+30(\ln f)_{xx}(\ln f)_{4x}+60(\ln f)_{xx}^3\right]_x$$

$$+\frac{5a(t)}{2}\frac{\partial}{\partial x}\left[\partial_x^2+6(\ln f)_{xx}\right]\left[(\ln f)_{4x}+6(\ln f)_{xx}^2\right]$$
$$+10\mu_3 e^{\int m(t)\,dt}\left[(\ln f)_{xt}+n(t)(\ln f)_{xx}\right.$$
$$\left.+e(t)(\ln f)_{4x}+6e(t)(\ln f)_{xx}^2\right]_x=0. \quad (30)$$

If we introduce an auxiliary variable z and impose a subsidiary constraint condition

$$(\ln f)_{4x}+6(\ln f)_{xx}^2=(\ln f)_{xz}, \quad (31)$$

then integrating Eq. (30) once with respect to x yields the bilinear forms

$$a(t)(D_x^6+5D_x^3D_z)f\cdot f+20\mu_3 e^{\int m(t)\,dt}$$
$$\times[D_xD_t+n(t)D_x^2+e(t)D_x^4]f\cdot f=0,$$
$$(D_x^4-D_xD_z)f\cdot f=0. \quad (32)$$

Expand function f as a series

$$f=1+\varepsilon f_1+\varepsilon^2 f_2+\varepsilon^3 f_3+\cdots. \quad (33)$$

Substituting it into Eq. (32) and equating various powers of ε, we obtain the following recursion relations for f_i $(i=1,2,\ldots)$

$$\left[a(t)(D_x^6+5D_x^3D_z)+20\mu_3 e^{\int m(t)\,dt}\right.$$
$$\left.\times(D_xD_t+n(t)D_x^2+e(t)D_x^4)\right]f_1\cdot 1=0, \quad (34)$$
$$(D_x^4-D_xD_z)f_1\cdot 1=0, \quad (35)$$
$$\left[a(t)(D_x^6+5D_x^3D_z)+20\mu_3 e^{\int m(t)\,dt}\right.$$
$$\left.\times(D_xD_t+n(t)D_x^2+e(t)D_x^4)\right]$$
$$\times(f_2\cdot 1+f_1\cdot f_1+1\cdot f_2)=0, \quad (36)$$
$$(D_x^4-D_xD_z)(f_2\cdot 1+f_1\cdot f_1+1\cdot f_2)=0,$$
$$\cdots \quad (37)$$

to be solved.

To obtain the one-soliton solution, we use the ansatz

$$f_1=e^{\eta_1},\eta_1=k_1x-\omega_1(t)+\lambda_1 z+\xi_1^0, \quad (38)$$

where k_1,λ_1,ξ_1^0 are real constants, and $\omega_1(t)$ is an arbitrary function. From Eqs. (34)–(38), we have

$$\lambda_1=k_1^3,$$
$$\omega_1(t)=\int\left(k_1n(t)+k_1^3e(t)+\frac{3k_1^5a(t)}{10\mu_3 e^{\int m(t)\,dt}}\right)dt,$$
$$f_i=0,\quad i\geq 2. \quad (39)$$

So the one-soliton solution of Eq. (16) reads

$$u=\frac{6}{\mu_3 e^{\int m(t)\,dt}}[\ln(1+e^{\eta_1})]_{xx}$$
$$=\frac{3k_1^2}{2\mu_3 e^{\int m(t)\,dt}}\operatorname{sech}^2\left(\frac{k_1x-\omega_1(t)+\xi_1^0}{2}\right), \quad (40)$$

where $\mu_3\neq 0,k_1,\xi_1^0$ are arbitrary real constants, $a(t)\neq 0,e(t),m(t)$, and $n(t)$ are arbitrary functions of t, and $\omega_1(t)$ is given by Eq. (39).

Generally, the N-soliton ($N\geq 2$) solutions of Eq. (16) can be denoted as

$$u=\frac{6}{\mu_3 e^{\int m(t)\,dt}}$$
$$\times\left[\ln\left(\sum_{\nu=0,1}\exp\left(\sum_{j=1}^N\nu_j\eta_j+\sum_{1\leq i<j}^N\nu_i\nu_jA_{ij}\right)\right)\right]_{xx},$$
$$\eta_j=k_jx-\omega_j(t)+\xi_j^0,$$
$$\omega_j(t)=\int\left(k_jn(t)+k_j^3e(t)+\frac{3k_j^5a(t)}{10\mu_3 e^{\int m(t)\,dt}}\right)dt,$$
$$j=1,\ldots,N,$$
$$A_{ij}=\ln(a_{ij}),\quad a_{ij}=\left(\frac{k_i-k_j}{k_i+k_j}\right)^2, \quad (41)$$

where $\mu_3\neq 0,k_j,\xi_j^0$ $(j=1,\ldots N)$ are arbitrary real constants, $a(t)\neq 0,e(t),m(t)$, and $n(t)$ are arbitrary functions of t, $\sum_{\nu=0,1}$ is a summation over all possible combinations of $\nu_1=0,1$, $\nu_2=0,1,\cdots,\nu_N=0,1$, and $\sum_{1\leq i<j}^N$ means a summation over all possible pairs (i,j) chosen from set $(1,2,\ldots,N)$ with the condition $1\leq i<j$.

3.3. One-soliton and two-soliton solutions of Eq. (17)

Introducing the dependent variable transformation

$$u=\frac{3}{\mu_5 e^{\int m(t)\,dt}}(\ln f)_{xx}, \quad (42)$$

we can transform Eq. (17) into

$$15[(\ln f)_{xt}+n(t)(\ln f)_{xx}]+\frac{3a(t)}{\mu_5 e^{\int m(t)\,dt}}$$
$$\times[(\ln f)_{6x}+15(\ln f)_{xx}(\ln f)_{4x}+15(\ln f)_{xx}^3$$
$$+\frac{45}{4}(\ln f)_{3x}^2+15\mu_4(\ln f)_{xx}^2+5\mu_4(\ln f)_{4x}]=0. \quad (43)$$

If we take $\mu_4=0$ and introduce an auxiliary function g

$$\frac{g}{f}=\frac{3}{2\mu_5 e^{\int m(t)\,dt}}\frac{D_x^4f\cdot f}{f^2}, \quad (44)$$

then combining Eqs. (43) and (44), we can obtain the bilinear form of Eq. (17) as follows:

$$\left[80D_xD_t+80n(t)D_x^2+\frac{a(t)}{\mu_5 e^{\int m(t)\,dt}}D_x^6\right]f\cdot f$$
$$+10a(t)D_x^2f\cdot g=0,$$
$$3D_x^4f\cdot f-2\mu_5 e^{\int m(t)\,dt}g\cdot f=0. \quad (45)$$

We expand f and g into the power series of a small parameter ε as

$$f=1+\varepsilon f_1+\varepsilon^2 f_2+\varepsilon^3 f_3+\cdots,$$
$$g=\varepsilon g_1+\varepsilon^2 g_2+\varepsilon^3 g_3+\cdots. \quad (46)$$

Substituting Eq. (46) into Eq. (45) and equating the coefficients of ε yield

$$3D_x^4 f_1\cdot 1-\mu_5 e^{\int m(t)\,dt}g_1=0, \tag{47}$$

$$\left[80D_xD_t+80n(t)D_x^2+\frac{a(t)}{\mu_5 e^{\int m(t)\,dt}}D_x^6\right]f_1\cdot 1 + 5a(t)D_x^2 g_1\cdot 1=0, \tag{48}$$

$$3D_x^4[f_2\cdot 1+1\cdot f_2+f_1\cdot f_1] - 2\mu_5 e^{\int m(t)\,dt}(g_2+g_1f_1)=0, \tag{49}$$

$$\left[80D_xD_t+80n(t)D_x^2+\frac{a(t)}{\mu_5 e^{\int m(t)\,dt}}D_x^6\right] \times[f_2\cdot 1+1\cdot f_2+f_1\cdot f_1] + 10a(t)D_x^2(g_2\cdot 1+f_1\cdot g_1)=0, \tag{50}$$

$$3D_x^4[f_3\cdot 1+f_2\cdot f_1]-\mu_5 e^{\int m(t)\,dt} \times(g_3+g_2f_1+g_1f_2)=0, \tag{51}$$

$$\left[80D_xD_t+80n(t)D_x^2+\frac{a(t)}{\mu_5 e^{\int m(t)\,dt}}D_x^6\right][f_3\cdot 1+f_2\cdot f_1] + 5a(t)D_x^2(g_3\cdot 1+f_2\cdot g_1+f_1\cdot g_2)=0, \tag{52}$$

$$3D_x^4[f_4\cdot 1+f_3\cdot f_1+f_2\cdot f_2+f_1\cdot f_3+1\cdot f_4] - 2\mu_5 e^{\int m(t)\,dt}(g_4+g_3f_1+g_2f_2+g_1f_3)=0, \tag{53}$$

$$\left[80D_xD_t+80n(t)D_x^2+\frac{a(t)}{\mu_5 e^{\int m(t)\,dt}}D_x^6\right] \times[f_4\cdot 1+f_3\cdot f_1+f_2\cdot f_2+f_1\cdot f_3+1\cdot f_4] + 10a(t)D_x^2(g_4\cdot 1+g_3\cdot f_1+g_2\cdot f_2+g_1\cdot f_3)=0, \tag{54}$$

$$\cdots$$

to be solved.

To find the one-soliton solution, we take

$$f_1=e^{\eta_1},\quad \eta_1=k_1x-\omega_1(t)+\xi_1^0. \tag{55}$$

From Eqs. (47)–(54), we can work out

$$\omega_1(t)=\int\left(k_1n(t)+\frac{k_1^5a(t)}{5\mu_5 e^{\int m(t)\,dt}}\right)dt,$$

$$g_1=\frac{3k_1^4}{\mu_5 e^{\int m(t)\,dt}}e^{\eta_1},\quad f_2=\frac{e^{2\eta_1}}{16}, \tag{56}$$

$$f_{i+1}=g_i=0,\quad i\geq 2. \tag{57}$$

Therefore, the one-soliton solution to Eq. (17) is

$$u=\frac{48k_1^2e^{\eta_1}(16+4e^{\eta_1}+e^{2\eta_1})}{\mu_5 e^{\int m(t)\,dt}(16+16e^{\eta_1}+e^{2\eta_1})^2},$$

$$\eta_1=k_1x-\omega_1(t)+\xi_1^0, \tag{58}$$

where $\mu_4=0$, $\mu_5\neq 0$, k_1, ξ_1^0 are arbitrary real constants, $a(t)\neq 0$, $m(t)$, and $n(t)$ are arbitrary functions of t, and $\omega_1(t)$ is given by Eq. (56).

For the two-soliton solution, we may suppose

$$f_1=e^{\eta_1}+e^{\eta_2},\quad \eta_j=k_jx-\omega_j(t)+\xi_j^0,\ (j=1,2). \tag{59}$$

From Eqs. (47)–(52), after detailed computation with the aid of symbolic computation, we have

$$\omega_j(t)=\int\left(k_jn(t)+\frac{k_j^5a(t)}{5\mu_5\,e^{\int m(t)\,dt}}\right)dt,\quad (j=1,2),$$

$$g_1=\frac{3k_1^4e^{\eta_1}+3k_2^4e^{\eta_2}}{\mu_5 e^{\int m(t)\,dt}},\quad f_2=\frac{1}{16}e^{2\eta_1}+\frac{1}{16}e^{2\eta_2}+\lambda_{12}e^{\eta_1+\eta_2},$$

$$g_2=\delta_{12}e^{\eta_1+\eta_2},\quad f_3=\rho_{12}(e^{2\eta_1+\eta_2}+e^{\eta_1+2\eta_2}),$$

$$g_3=\sigma_1e^{2\eta_1+\eta_2}+\sigma_2e^{\eta_1+2\eta_2},$$

$$f_4=\rho_{12}^2e^{2\eta_1+2\eta_2},\quad f_{i+1}=g_i=0,\quad i\geq 4, \tag{60}$$

where

$$\lambda_{12}=\frac{2k_1^4+2k_2^4-k_1^2k_2^2}{2(k_1+k_2)^2(k_1^2+k_1k_2+k_2^2)},$$

$$\delta_{12}=\frac{3(k_1-k_2)^2(2k_1^4+2k_2^4+3k_1^2k_2^2)}{2\mu_5(k_1^2+k_1k_2+k_2^2)e^{\int m(t)\,dt}},$$

$$\rho_{12}=\frac{(k_1-k_2)^2(k_1^2+k_2^2-k_1k_2)}{16(k_1+k_2)^2(k_1^2+k_2^2+k_1k_2)},$$

$$\sigma_1=\frac{3k_2^4\rho_{12}}{\mu_5 e^{\int m(t)\,dt}},\quad \sigma_2=\frac{3k_1^4\rho_{12}}{\mu_5 e^{\int m(t)\,dt}}. \tag{61}$$

Thus the two-soliton solution of Eq. (17) reads

$$u=\frac{3}{\mu_5 e^{\int m(t)\,dt}}(\ln f)_{xx}, \tag{62}$$

where f is given by

$$\begin{aligned} f &= 1+e^{\eta_1}+e^{\eta_2}+\frac{e^{2\eta_1}}{16}+\frac{e^{2\eta_2}}{16}+\lambda_{12}e^{\eta_1+\eta_2} \\ &\quad +\rho_{12}(e^{2\eta_1+\eta_2}+e^{\eta_1+2\eta_2})+\rho_{12}^2e^{2\eta_1+2\eta_2}, \\ \eta_j &= k_jx-\int\left(k_jn(t)+\frac{k_j^5a(t)}{5\mu_5 e^{\int m(t)\,dt}}\right)dt+\xi_j^0, \\ &\quad j=1,2, \end{aligned} \tag{63}$$

here $\mu_4=0$, $\mu_5\neq 0$, k_1,k_2,ξ_1^0,ξ_2^0 are arbitrary real constants, $a(t)\neq 0$, $m(t)$, and $n(t)$ are arbitrary functions of t, and λ_{12}, ρ_{12} are given by Eq. (61).

In comparison with the former two integrable models, equation (17) is more difficult to solve. The involved computations in deriving the three-soliton solution are very lengthy. As far as we know, the multi-soliton solutions obtained in this section are reported here first.

4. Interactions of multiple solitons

For the first integrable model (15), the N-soliton solution (30) under condition (2) is in agreement with the results reported in Ref. [20], in which the solitonic propagation and collision under the effects of variable coefficients have been discussed. For simplicity, we only analyze the bidirectional two-soliton interactions with Eqs. (16) and (17) through several plots.

By selecting appropriate coefficients in Eq. (16), the head-on collision of two solitary waves are studied and shown in Fig. 1. It is clearly seen that the soliton with the small amplitude has changed its direction twice after the collision, in other words, there exists an overtaking collision during the head-on collision, which is different from the head-on collision of many nonlinear models with constant coefficients.

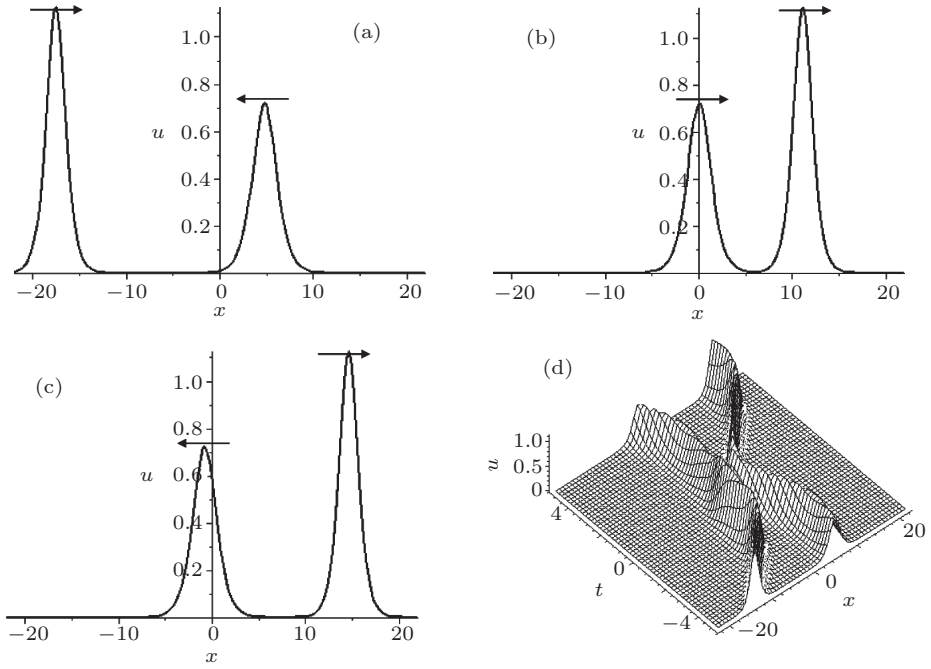


Fig. 1. The head-on collision of one right-going soliton and one left-going soliton (Eq. (41)) with $\mu_3 = 3$, $a(t) = 10$, $e(t) = \sin(t)$, $m(t) = 0$, $n(t) = -2.5$, $k_1 = 1.2$, $k_2 = 1.5$, $\xi_1^0 = \xi_2^0 = 0$. Panels (a)–(c) show the solitons at (a) $t = -6$, (b) $t = 2.5$, and (c) $t = 4$, respectively. Panel (d) shows the evolution plot of the solitons.

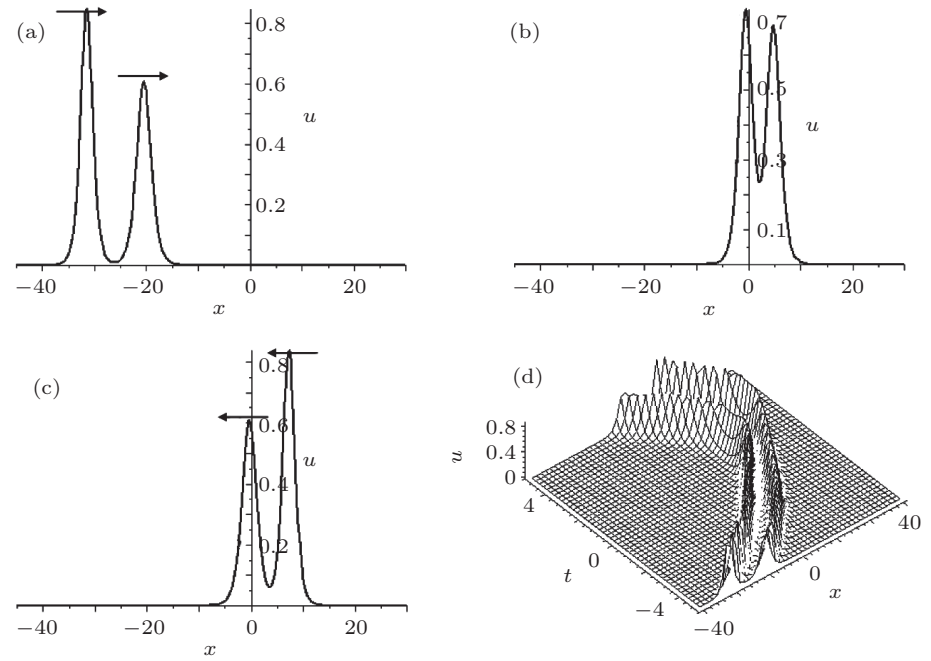


Fig. 2. Overtaking collision of two solitons (Eq. (41)) with $\mu_3 = 3$, $a(t) = 10$, $e(t) = \cos(t)$, $m(t) = 0$, $n(t) = -1.5t$, $k_1 = 1.1$, $k_2 = 1.3$, $\xi_1^0 = \xi_2^0 = 0$. Panels (a)–(c) show the solitons at (a) $t = -5$, (b) $t = 0$, and (c) $t = 2.5$, respectively. Panel (d) shows the evolution plot of the solitons.

Figure 2 depicts the overtaking collision of two solitary waves along the same direction of propagation. In Figs. 2(a) and 2(b), the large-amplitude soliton overtakes the small-amplitude one, but both are right going. However, after the collision of the two solitons, they both change the directions of propagation, which are shown in Figs. 2(c) and 2(d). That is to say, the overtaking collisions of two left-going solitary waves and two right-going solitary waves coexist in the evolution of two solitons.

Similarly, equation (17) also admits the bidirectional wave interactions. For instance, if

$$a(t) = \exp(\cos(t)), \quad m(t) = -\sin(t), \quad n(t) = \cos(t) - 2, \tag{64}$$

and the other parameters take appropriate values, we can ob-

tain several types of collisions for Eq. (62), as exhibited in Fig. 3, from which we know that coefficient $m(t)$ plays an important role in modulating the amplitude of the solution. It is seen that the soliton amplitude periodically oscillates along the t axis. The head-on collision of two solitary waves along opposite directions is depicted in Fig. 3(a), and the overtaking collision of two left-going solitary waves is shown in Fig. 3(b).

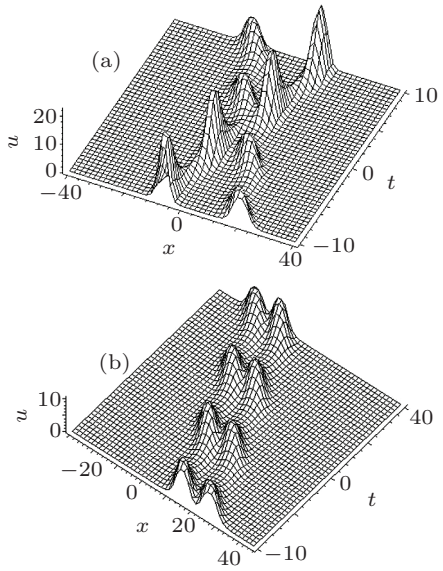


Fig. 3. Bidirectional wave interactions of two solitons (Eq. (62)) with $\mu_5 = 0.2$, $\xi_1^0 = \xi_2^0 = 0$, (a) $k_1 = 0.9$, $k_2 = 1.3$; (b) $k_1 = 0.8$, $k_2 = 0.9$.

5. Conclusions

In this paper, with the aid of symbolic computation, we have investigated the generalized fifth-order KdV equation with multiple variable coefficients, which describes the interaction between a water wave and a floating ice cover or the gravity-capillary waves in fluid dynamics. We first verify whether the given equation is Painlevé-integrable and under which parameter constraints the equation is integrable. As a result, we obtain all possible Painlevé integrable conditions, and derive three new integrable equations systematically.

Furthermore, for the three obtained integrable equations, we have made use of the truncated Painlevé expansion to derive their bilinear forms and then construct the multi-soliton solutions. To the author's knowledge, these multi-soliton solutions are reported here for the first time.

Due to the arbitrariness of functions $a(t)$, $e(t)$, $m(t)$, and $n(t)$ in solutions (30), (41), and (62), there are many possible excitations. By selecting appropriate parameters, we analyze the evolution of two solitons for three Painlevé-integrable models (15)–(17), which indicates that these three integrable models possess interesting bidirectional wave interactions. For example, there exists an overtaking collision during the head-on collision, the overtaking collisions of two left-going solitary waves and of two right-going solitary waves coexist in the evolution of two solitons. These special bidirectional wave interactions are different from those of NLEEs with constant coefficients.

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