Comparative study of travelling-wave and numerical solutions for the coupled short pulse (CSP) equation

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The Lie symmetry analysis is performed for the coupled short plus (CSP) equation. We derive the infinitesimals that admit the classical symmetry group. Five types arise depending on the nature of the Lie symmetry generator. In all types, we find reductions in terms of system of ordinary differential equations, and exact solutions of the CSP equation are derived, which are compared with numerical solutions using the classical fourth-order Runge–Kutta scheme.

Keywords: coupled short pulse (CSP) equation, Lie symmetric analysis, Runge–Kutta scheme, exact solutions

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1. Introduction

The short pulse (SP) equation, which has the form of

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \tag{1}$$

was derived by Schäfer and Wayne[1] as a model equation describing the propagation of ultra-short light pulses in silica optical fibres. Here, $u = u(x,t)$ represents the magnitude of the electric field, while the subscripts t and x denote partial differentiations. The SP equation represents an alternative of nonlinear Schrödinger (NLS) equation to approximate the evaluation of very short optical pulses in the nonlinear media.

The coupled short pulse (CSP) equation[2] reads

$$\begin{aligned} u_{xt} &= u + \frac{1}{6}(u^3)_{xx} + \frac{1}{2}w^2 u_{xx}, \\ w_{xt} &= w + \frac{1}{6}(w^3)_{xx} + \frac{1}{2}u^2 w_{xx}. \end{aligned} \tag{2}$$

Such an equation is proposed for the propagation of ultra-short pulses in optical fibers. Based on two sets of bilinear equations of a two-dimensional Toda lattice linked by a Bäcklund transformation, and an appropriate hodograph transformation, the CSP equation is derived. In practice, system (2) can be written as

$$\begin{aligned} u_{xt} &= u + u(u^2)_x + \frac{1}{2}(u^2 + w^2)u_{xx}, \\ w_{xt} &= w + w(w^2)_x + \frac{1}{2}(u^2 + w^2)w_{xx}. \end{aligned} \tag{3}$$

If $u = 0$ or $w = 0$, the system is reduced to the SP equation (1). For more details on both the mathematical theory and the physical significance of systems (2) and (3), we refer the readers to Ref. [2]. In this paper, we study the general form of the CSP equation

$$\begin{aligned} u_{xt} &= \alpha_1 u + \beta_1 u(u^2)_x + \delta_1(u^2 + w^2)u_{xx}, \\ w_{xt} &= \alpha_2 w + \beta_2 w(w^2)_x + \delta_2(u^2 + w^2)w_{xx}, \end{aligned} \tag{4}$$

where α_i, β_i, and δ_i $(i = 1,2)$ are arbitrary constant coefficients for exact and numerical solutions by using Lie symmetry and 4th-order Runge–Kutta methods. Recently, Yildirim, Biswas, and his associates presented soliton solutions of some nonlinear partial differential equations.[17–19]

The rest of this paper is organized as follows. In Section 2, the classical Lie method is utilized to obtain the optimal system of sub-algebras for system (4). In Section 3, we derive some explicit exact solutions of the nonlinear system (4). In Section 4, we carry out comparative study of travelling-wave and numerical solutions for the CSP equation. In Section 5, some conclusions are drawn.

2. Lie symmetric analysis for the CSP equation

Firstly, let us consider a one-parameter Lie group of infinitesimal transformation

$$\begin{aligned} x^* &= x + \varepsilon\xi(x,t,u,w) + O(\varepsilon^2), \\ t^* &= t + \varepsilon\tau(x,t,u,w) + O(\varepsilon^2), \\ u^* &= u + \varepsilon\phi(x,t,u,w) + O(\varepsilon^2), \\ w^* &= w + \varepsilon\psi(x,t,u,w) + O(\varepsilon^2), \end{aligned} \tag{5}$$

where ε is a small parameter.

The nonlinear system (4) is said to admit a symmetry generated by the vector field

$$\begin{aligned} V = {} & \xi(x,t,u,w)\frac{\partial}{\partial x} + \tau(x,t,u,w)\frac{\partial}{\partial t} + \phi(x,t,u,w)\frac{\partial}{\partial u} \\ & + \psi(x,t,u,w)\frac{\partial}{\partial w}, \end{aligned} \tag{6}$$

if it is left invariant by the transformation $(x,t,u,w) \to (x^*,t^*,u^*,w^*)$.

Applying the second prolongation $\mathrm{pr}^{(2)}V$[3–5] to Eq. (4) results in an overdetermined system of linear partial differential equations. After some straightforward and lengthy calculations, the resultant overdetermined system gives the follow-

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ing infinitesimal elements:

$$\xi = ax + c, \quad \tau = -at + b, \quad \phi = au, \quad \psi = aw, \tag{7}$$

where a, b, and c are arbitrary constants. The Lie algebra associated with system (4) consists of the following three vector fields:

$$V_1 = x\frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u} + w\frac{\partial}{\partial w}, \quad V_2 = \frac{\partial}{\partial t}, \quad V_3 = \frac{\partial}{\partial x}. \tag{8}$$

The commutation relations between these vector fields are given in Table 1.

To compute the adjoint representation (see Table 2), we use the Lie series in the form of

$$\mathrm{Ad}(\exp(\varepsilon V_i))V_j = V_j - \varepsilon[V_i, V_j] + \frac{\varepsilon^2}{2}[V_i,[V_i,V_j]] - \cdots,$$

where ε is a parameter, and $[V_i, V_j]$ is the commutator of the Lie algebra $(i, j = 1, 2, 3)$.

Table 1. Commutation relations between these vector fields.

$[V_i, V_j]$	V_1	V_2	V_3
V_1	V_2	0	$-V_3$
V_2	$-V_2$	0	0
V_3	V_3	0	0

Table 2. Adjoint representation.

Ad	V_1	V_2	V_3
V_1	V_1	$V_2\,\mathrm{e}^{-\varepsilon}$	$V_3\,\mathrm{e}^{-\varepsilon}$
V_2	$V_1 + \varepsilon V_2$	V_2	V_3
V_3	$V_1 - \varepsilon V_3$	V_2	V_3

The classification of one-dimensional subalgebras of the whole symmetry algebra (8) is done by an inductive approach.[3] If $V = \sum_{i=1}^{3} a_i V_i$ is a none-zero vector field, we will simplify as many coefficients a_i as possible through judicious application of various adjoint transformations on this vector field. The types of one-dimensional subalgebras of system (4) are expressed as follows: (i) V_1, (ii) $V_2 + V_3$, (iii) $V_2 - V_3$, (iv) V_2, and (v) V_3.

3. Reduced systems and exact solutions

In the following, some exact solutions for the reduced systems are studied for each type of subalgebra. We first solve the associated Lagrange equations for subalgebra to obtain the group invariant solutions, and then obtain the reduced system of ODEs,[6–16] which are further studied for the exact and numerical solutions

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}u}{\phi} = \frac{\mathrm{d}w}{\psi}. \tag{9}$$

3.1. Subalgebra V_1

The symmetry in this type gives the invariant of the symmetry and group invariant solution as

$$z = xt, \quad u(x,t) = \frac{F(z)}{t}, \quad w(x,t) = \frac{G(z)}{t}. \tag{10}$$

Using this similarity variable and the forms of the similarity solution, the system of PDEs (4) reduces to the following system of ODEs:

$$F''z - \alpha_1 F - \beta_1 F(F')^2 - \delta_1(F^2+G^2)F'' = 0,$$
$$G''z - \alpha_2 G - \beta_2 G(G')^2 - \delta_2(F^2+G^2)G'' = 0. \tag{11}$$

The solutions of system (4) are discussed under the following cases.

Case (i) If

$$\alpha_1 = C_4^2\delta_1 C_3^2 + \delta_1 C_3^2 C_1^2 - zC_3^2,$$
$$\alpha_2 = -C_3^2\beta_2 C_4^2 - zC_3^2,$$
$$\beta_1 = \frac{-\delta_1(C_1^2+C_4^2)}{C_1^2}, \quad \delta_2 = -\frac{\beta_2 C_4^2}{C_1^2+C_4^2},$$

where β_2 and δ_1 are arbitrary constants, then the solution of system (11) is given by

$$F(z) = C_1\sin(C_2 + C_3 z), \quad G(z) = C_4\sin(C_2 + C_3 z). \tag{12}$$

The corresponding solution of system (4) can be expressed as

$$u(x,t) = C_1\sin(C_2 + C_3 xt), \quad w(x,t) = C_4\sin(C_2 + C_3 xt), \tag{13}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

Case (ii) When constants are under the following restrictions:

$$\alpha_1 = -C_4^2\beta_1 C_3^2 - \beta_1 C_3^2 C_1^2 - zC_3^2, \quad \alpha_2 = 0, \quad \delta_1 = -\beta_1,$$

where β_1, β_2, and δ_2 are arbitrary constants, then the solution of system (11) is given by

$$F(z) = C_1\sin(C_2 + C_3 z), \quad G(z) = C_4. \tag{14}$$

The solution of the main system (4) corresponding to the solution (14) is obtained as

$$u(x,t) = C_1\sin(C_2 + C_3 xt), \quad w(x,t) = C_4, \tag{15}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

3.2. Subalgebra $V_2 + V_3$

We derive the following expression of z, u, and w by solving the Lagrange equation in the form of

$$z = x + t, \quad u(x,t) = F(z), \quad w(x,t) = G(z). \tag{16}$$

Using the similarity variable and the forms of the similarity solution (16), the system of PDEs (4) is reduced to the following system of ODEs:

$$F'' + \alpha_1 F + \beta_1 F(F')^2 + \delta_1(F^2+G^2)F'' = 0,$$
$$G'' + \alpha_2 G + \beta_2 G(G')^2 + \delta_2(F^2+G^2)G'' = 0. \tag{17}$$

We derive the solutions of system (4) in the following cases.

Case (i) If

$$\alpha_1 = C_4^2\delta_1 C_3^2 + \delta_1 C_3^2 C_1^2 + C_3^2, \quad \alpha_2 = -C_3^2\beta_2 C_4^2 + C_3^2,$$
$$\beta_1 = \frac{-\delta(C_1^2+C_4^2)}{C_1^2}, \quad \delta_2 = -\frac{\beta_2 C_4^2}{C_1^2+C_4^2},$$

where β_2 and δ_1 are arbitrary constants, then the solution of

system (17) is given by

$$F(z) = C_1 \sin(C_2 + C_3 z)m, \quad G(z) = C_4 \sin(C_2 + C_3 z). \quad (18)$$

The corresponding solutions of system (4) can be expressed as follows:

$$u(x,t) = C_1 \sin(C_2 + C_3 xt), \; w(x,t) = C_4 \sin(C_2 + C_3 xt), \quad (19)$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

Case (ii) When constants satisfy the following restrictions:

$$\alpha_1 = 0, \quad \alpha_2 = -C_1^2\beta_2 C_4^2 - \beta_2 C_4^2 C_2^2 + C_4^2, \quad \delta_2 = -\beta_2,$$

then the solution of system (17) is obtained as

$$F(z) = C_1, \quad G(z) = C_2 \sin(C_3 + C_4 z). \quad (20)$$

Thus, using the back transformation, we obtain the solution of system (4) as

$$u(x,t) = C_1, \quad w(x,t) = C_2 \sin(C_3 + C_4 xt), \quad (21)$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

3.3. Subalgebra $V_2 - V_3$

Suitable invariant of the symmetry and group invariant solution of this type for reduction are expressed as follows:

$$z = x - t, \quad u(x,t) = F(z), \quad w(x,t) = G(z). \quad (22)$$

Using the similarity variable and the forms of the similarity solution (22), the system of PDEs (4) is reduced to the following system of ODEs:

$$\begin{aligned} &F'' - \alpha_1 F - \beta_1 F(F')^2 - \delta_1(F^2 + G^2)F'' = 0, \\ &G'' - \alpha_2 G - \beta_2 G(G')^2 - \delta_2(F^2 + G^2)G'' = 0. \end{aligned} \quad (23)$$

We derived the solution of system (4) as follows.

If $\alpha_2 = -C_3^2\beta_2 C_4^2 - C_3^2$, and $\delta_2 = -\beta_2$, then the solution of system (23) corresponding to arbitrary constants α_1, β_1, β_2, and δ_1 is

$$F(z) = 0, \quad G(z) = C_1 \sin(C_2 + C_3 z). \quad (24)$$

The corresponding solution of system (4) can be expressed as

$$u(x,t) = 0, \quad w(x,t) = C_1 \sin(C_2 + C_3(x - t)), \quad (25)$$

where C_1, C_2, C_3, and C_4 are arbitrary constants.

3.4. Subalgebras V_2 and V_3

For these two types of subalgebra, we only obtain constant solutions of system (4).

4. Comparative study of travelling-wave and numerical solutions for the CSP equation

In this section, we derive numerical solutions of the main system (4). For this purpose, the reduced system of ODEs obtained in Section 3 is solved numerically. We also compare the exact and numerical solutions of system (4).

4.1. Classical fourth-order Runge–Kutta (RK4) method

The classical fourth-order Runge–Kutta (RK4) method applied to the n-th order differential equation

$$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)}), \quad (26)$$

with initial conditions

$$y(t_0) = y_0, \; y'(t_0) = y_0, \; y''(t_0) = y_0, \ldots, y^{(n-1)}(t_0) = y_0,$$

evaluates the function $f(t, y, y', y'', \ldots, y^{(n-1)})$ four times per step, and can be derived by transforming the problem to a system of first-order differential equations. To solve the reduced ODEs, the RK4 method is given by

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \quad t_{n+1} = t_n + h, \quad (27)$$

where y_{n+1} is the RK4 approximation of $y(t_{n+1})$, and

$$\begin{aligned} &k_1 = hf(t_n, y_n), \quad k_2 = hf(t_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1), \\ &k_3 = hf(t_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2), \quad k_4 = hf(t_n + h, y_n + k_3). \end{aligned} \quad (28)$$

4.2. Numerical experiments

In this subsection, numerical solutions of the system (4) are obtained by using the 4th order Runge–Kutta method. The exact solutions of these reduced ODEs are obtained in Section 3 by using the Lie classical method. The Dev C++, Matlab, and Maple software are used in the paper. For each reduced ODEs, absolute errors E_a, relative error E_r, and percentage error E_p are computed according to the following formulas:

$$E_a = |u - \bar{u}|, \quad E_r = |\frac{u - \bar{u}}{u}|, \quad E_p = |\frac{u - \bar{u}}{u}| \times 100,$$

where u and $\bar{u}$ denote the exact and numerical solution of the problem.

4.2.1. Subalgebra V_1

For the purpose of numerical solutions of the main system (4), we solve numerically the reduced system of ODEs (11) under the following restrictions with the help of the RK4 method.

Example 1 For numerical solutions, we start with initial values at $z = 5$ and the initial approximation by solution (12). If constants are under the following restrictions:

$$\begin{aligned} &\alpha_1 = C_4^2\delta_1 C_3^2 + \delta_1 C_3^2 C_1^2 - zC_3^2, \quad \alpha_2 = -C_3^2\beta_2 C_4^2 - zC_3^2, \\ &\beta_1 = \frac{-\delta(C_1^2 + C_4^2)}{C_1^2}, \quad \delta_2 = -\frac{\beta_2 C_4^2}{C_1^2 + C_4^2}, \quad \beta_2 = \delta_1 = 0, \\ &C_1 = 1.5, \quad C_2 = 0, \quad C_3 = \frac{1}{5}, \quad C_4 = -1, \end{aligned}$$

then the reduced ODE system (11) for CSP equation transforms into the following system of a first-order differential equation:

$$\frac{dy_1}{dz} = y_2, \; \frac{dy_2}{dz} = -0.04y_1, \; \frac{dy_3}{dz} = y_4, \; \frac{dy_4}{dz} = -0.04y_3, \quad (29)$$

with initial conditions of $y_1 = 1.5\sin(1)$, $y_2 = 0.3\cos(1)$, $y_3 = -\sin(1)$, and $y_4 = -0.2\cos(1)$.

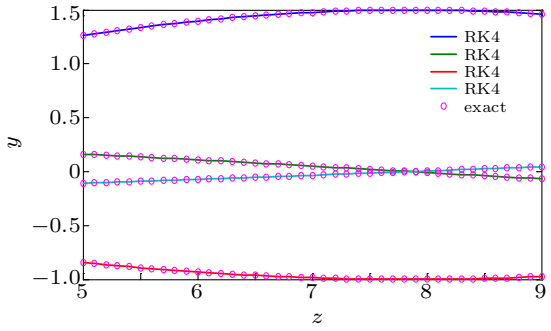


Fig. 1. (color online) Comparison between exact and numerical solutions for the reduced ODEs system (11) for CSP equation when we start with $z = 5$ and the initial approximation (12) at $h = 0.1$.

Example 2 In this example, we consider the constants as follows:

$$\alpha_1 = -C_4^2\beta_1 C_3^2 - \beta_1 C_3^2 C_1^2 - zC_3^2, \quad \delta_1 = -\beta_1, \quad \beta_1 = 0,$$
$$\beta_2 = 1, \quad \delta_2 = \frac{1}{2}, \quad C_1 = C_2 = C_4 = 1, \quad C_3 = -\frac{1}{5},$$

then the reduced ODE system (11) for the CSP equation transforms into the following system of a first-order differential equation:

$$\frac{dy_1}{dz} = y_2, \quad \frac{dy_2}{dz} = -0.04y_1, \quad \frac{dy_3}{dz} = y_4,$$
$$\frac{dy_4}{dz} = \frac{y_3(y_4)^2}{z - \frac{1}{2}(y_1^2 + y_3^2)}, \tag{30}$$

with initial conditions of $y_1 = \sin(0)$, $y_2 = -0.2\cos(0)$, $y_3 = 1$, and $y_4 = 0$.

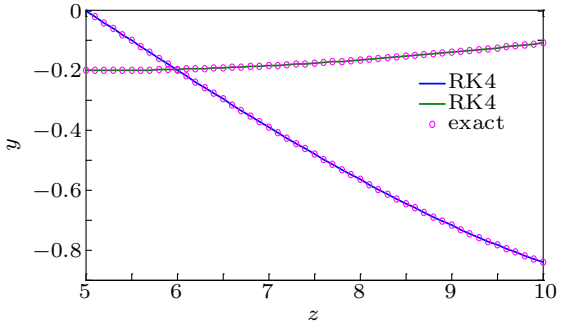


Fig. 2. (color online) Comparison between exact and numerical solutions for the reduced ODE system (11) for the CSP equation when we start with initial values at $z = 5$ and the initial approximation (14) at $h = 0.1$.

Table 3. Absolute error (E_a), relative error (E_r), and percentage error (E_p) for example 1 of subalgebra V_1.

	z	E_a	E_r	E_p
y_1	6	$2.4513724383 \times 10^{-13}$	$1.7534117579 \times 10^{-13}$	$1.7534117579 \times 10^{-11}$
	7	$1.4983125851 \times 10^{-11}$	$1.0136235531 \times 10^{-11}$	$1.0136235531 \times 10^{-9}$
	8	$1.7643597693 \times 10^{-10}$	$1.1767416053 \times 10^{-10}$	$1.1767416053 \times 10^{-8}$
	9	$1.0003580186 \times 10^{-9}$	$6.8481487721 \times 10^{-10}$	$6.8481487721 \times 10^{-8}$
	10	$3.7432299482 \times 10^{-9}$	$2.7444118486 \times 10^{-9}$	$2.7444118486 \times 10^{-7}$
y_2	6	$2.0997092953 \times 10^{-14}$	$1.9315251013 \times 10^{-13}$	$1.9315251013 \times 10^{-11}$
	7	$7.3100553388 \times 10^{-13}$	$1.4336212701 \times 10^{-13}$	$1.4336212701 \times 10^{-9}$
	8	$2.5900011302 \times 10^{-12}$	$2.9566706645 \times 10^{-10}$	$2.9566706645 \times 10^{-8}$
	9	$1.8625004316 \times 10^{-11}$	$2.7325165204 \times 10^{-10}$	$2.7325165204 \times 10^{-8}$
	10	$1.9830399822 \times 10^{-10}$	$1.5884136784 \times 10^{-9}$	$1.5884136784 \times 10^{-7}$
y_3	6	$1.6298074001 \times 10^{-13}$	$1.7486470521 \times 10^{-13}$	$1.7486470521 \times 10^{-11}$
	7	$9.9880104187 \times 10^{-12}$	$1.0135484454 \times 10^{-11}$	$1.0135484454 \times 10^{-9}$
	8	$1.1762302243 \times 10^{-10}$	$1.1767319792 \times 10^{-10}$	$1.1767319792 \times 10^{-8}$
	9	$6.6690508671 \times 10^{-10}$	$6.8481461120 \times 10^{-10}$	$6.8481461120 \times 10^{-8}$
	10	$2.4954870392 \times 10^{-9}$	$2.7444122963 \times 10^{-9}$	$2.7444122963 \times 10^{-7}$
y_4	6	$1.4002687898 \times 10^{-14}$	$1.9321634110 \times 10^{-13}$	$1.9321634110 \times 10^{-11}$
	7	$4.8699932975 \times 10^{-13}$	$1.4326278639 \times 10^{-11}$	$1.4326278639 \times 10^{-9}$
	8	$1.7269996197 \times 10^{-12}$	$2.9572395087 \times 10^{-10}$	$2.9572395087 \times 10^{-8}$
	9	$1.2416997985 \times 10^{-11}$	$2.7325887998 \times 10^{-10}$	$2.7325887998 \times 10^{-10}$
	10	$1.3220199934 \times 10^{-10}$	$1.5884056748 \times 10^{-9}$	$1.5884056748 \times 10^{-7}$

Table 4. Absolute error (E_a), relative error (E_r), and percentage error (E_p) for example 2 of subalgebra V_1.

	z	E_a	E_r	E_p
y_1	6	$2.6006974351 \times 10^{-14}$	$1.3090583356 \times 10^{-13}$	$1.3090583356 \times 10^{-11}$
	7	$3.3590352721 \times 10^{-12}$	$8.6257756946 \times 10^{-12}$	$8.6257756946 \times 10^{-10}$
	8	$5.5817017674 \times 10^{-11}$	$9.8853735424 \times 10^{-11}$	$9.8853735424 \times 10^{-9}$
	9	$4.0669900780 \times 10^{-10}$	$5.6694159701 \times 10^{-10}$	$5.6694159701 \times 10^{-8}$
	10	$1.8716309613 \times 10^{-9}$	$2.2242370742 \times 10^{-9}$	$2.2242370742 \times 10^{-7}$
y_2	6	$3.2002178684 \times 10^{-14}$	$1.6326533017 \times 10^{-13}$	$1.6326533017 \times 10^{-11}$
	7	$1.9509949211 \times 10^{-12}$	$1.0591019128 \times 10^{-11}$	$1.0591019128 \times 10^{-9}$
	8	$2.0566992553 \times 10^{-11}$	$1.2459775261 \times 10^{-10}$	$1.2459775261 \times 10^{-8}$
	9	$1.0410100559 \times 10^{-10}$	$7.4709346269 \times 10^{-10}$	$7.4709346269 \times 10^{-8}$
	10	$3.4276399774 \times 10^{-10}$	$3.1719649723 \times 10^{-9}$	$3.1719649723 \times 10^{-7}$

4.2.2. Subalgebra V_2+V_3

Using the similarity variable and the forms of the similarity solution, the system of PDEs (4) is reduced to the systemof ODEs (17). To obtain the numerical solutions of system (4), we solve the reduced system of ODEs (17) with the help of the RK4 method.

Example 1 We start with initial values at $z=5$ and the initial approximation (18). When

$$\alpha_1 = C_4^2\delta_1 C_3^2 + \delta_1 C_3^2 C_1^2 + C_3^2, \quad \delta_2 = -\frac{\beta_2 C_4^2}{C_1^2 + C_4^2},$$

$$\beta_2 = \delta_1 = 0, \quad C_1 = 5, \quad C_2 = 0,$$

$$C_3 = 0.1, \quad C_4 = -6,$$

then system (17) for the CSP equation becomes

$$\frac{dy_1}{dz} = y_2, \quad \frac{dy_2}{dz} = -0.01y_1,$$

$$\frac{dy_3}{dz} = y_4, \quad \frac{dy_4}{dz} = -0.01y_3, \tag{31}$$

with initial conditions of $y_1 = 5\sin(0.5)$, $y_3 = -6\sin(0.5)$, $y_2 = 0.5\cos(0.5)$, and $y_4 = -0.6\cos(0.5)$.

Table 5. Absolute error (E_a), relative error (E_r), and percentage error (E_p) for Example 1 of subalgebra V_2+V_3.

	z	E_a	E_r	E_p
y_1	6	$9.7699626167\times10^{-15}$	$3.4605836709\times10^{-15}$	$3.4605836709\times10^{-13}$
	7	$5.0004445029\times10^{-13}$	$1.5524083246\times10^{-13}$	$1.5524083246\times10^{-11}$
	8	$6.2598815020\times10^{-12}$	$1.7452647524\times10^{-12}$	$1.7452647524\times10^{-10}$
	9	$3.8620218134\times10^{-11}$	$9.8605620871\times10^{-12}$	$9.8605620871\times10^{-10}$
	10	$1.5941026276\times10^{-10}$	$3.7888475216\times10^{-11}$	$3.7888475216\times10^{-9}$
y_2	6	$9.9920072216\times10^{-16}$	$2.4213197737\times10^{-15}$	$2.4213197737\times10^{-13}$
	7	$6.3005156647\times10^{-14}$	$1.6475335093\times10^{-13}$	$1.6475335093\times10^{-11}$
	8	$6.7501559897\times10^{-13}$	$1.9377324487\times10^{-12}$	$1.9377324487\times10^{-10}$
	9	$3.5179636981\times10^{-12}$	$1.1318878002\times10^{-11}$	$1.1318878002\times10^{-9}$
	10	$1.2271017535\times10^{-11}$	$4.5422784253\times10^{-11}$	$4.5422784253\times10^{-9}$
y_3	6	$9.7699626167\times10^{-15}$	$2.8838197257\times10^{-15}$	$2.8838197257\times10^{-13}$
	7	$5.9996452250\times10^{-13}$	$1.5521785425\times10^{-13}$	$1.5521785425\times10^{-11}$
	8	$7.5193185011\times10^{-12}$	$1.7469981311\times10^{-12}$	$1.7469981311\times10^{-10}$
	9	$4.6339820869\times10^{-11}$	$9.8596172087\times10^{-12}$	$9.8596172087\times10^{-10}$
	10	$1.9130030892\times10^{-10}$	$3.7890058477\times10^{-11}$	$3.7890058477\times10^{-9}$
y_4	6	$1.9984014443\times10^{-15}$	$4.0355329561\times10^{-15}$	$4.0355329561\times10^{-13}$
	7	$7.5994766035\times10^{-14}$	$1.6560010090\times10^{-13}$	$1.6560010090\times10^{-11}$
	8	$8.0896400689\times10^{-13}$	$1.9352093595\times10^{-12}$	$1.9352093595\times10^{-10}$
	9	$4.2230108299\times10^{-12}$	$1.1322777533\times10^{-11}$	$1.1322777533\times10^{-9}$
	10	$1.4724998997\times10^{-11}$	$4.5422099313\times10^{-11}$	$4.5422099313\times10^{-9}$

Example 2 In this example, we start with initial values at and the initial approximation (20). When

$$\alpha_1 = 0, \ \alpha_2 = -C_1^2\beta_2 C_4^2 - \beta_2 C_4^2 C_2^2 + C_4^2, \ \delta_2 = -\beta_2,$$

$$\beta_2 = 0, \ \beta_1 = 1, \ \delta_1 = \frac{1}{2}, \ C_1 = C_2 = C_3 = 1, \ C_4 = 0.1,$$

then the system of ODEs (17) for the CSP equation can be expressed as

$$\frac{dy_1}{dz} = y_2, \ \frac{dy_2}{dz} = \frac{-y_1(y_2)^2}{z+(y_1^2+y_3^2)/2}, \ \frac{dy_3}{dz} = y_4, \ \frac{dy_4}{dz} = -0.01y_3, \tag{32}$$

with initial conditions of $y_1 = 1$, $y_2 = 0$, $y_3 = \sin(1.5)$, and $y_4 = 0.1\cos(1.5)$.

4.2.3. Subalgebra V_2-V_3

Using the similarity variable and the forms of the similarity solution, the system of PDEs (4) reduces to the system (23) of ODEs. To obtain the numerical solutions of the CSP equation, we solve system (23) under the restrictions given below.

Example Herein, we start with initial values at $z=5$ and the initial approximation (24) to construct the numerical solutions of system (4). When $\alpha_2 = -C_3^2\beta_2 C_4^2 - C_3^2$, $\delta_2 = -\beta_2$, $\beta_2 = 0$, $\alpha_1 = \beta_1 = C_1 = C_2 = C_3 = 1$, and $\delta_1 = 1/2$, then reduced first-order system of ODEs corresponding to the systems (23) for the CSP equation is expressed as follows:

$$\frac{dy_1}{dz} = y_2, \ \frac{dy_2}{dz} = \frac{y_1 + y_1(y_2)^2}{z-(y_1^2+y_3^2)/2}, \ \frac{dy_3}{dz} = y_4, \ \frac{dy_4}{dz} = -y_3, \tag{33}$$

with initial conditions of $y_1 = 0$, $y_2 = 0$, $y_3 = \sin(6)$, and $y_4 = \cos(6)$.

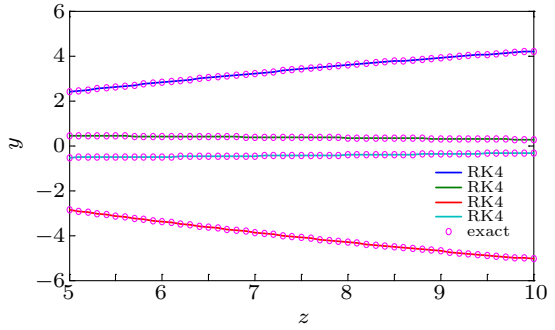


Fig. 3. (color online) Comparison between exact and numerical solutions for the reduced ODEs system (17) for the CSP equation when we start with initial values at $z=5$ and the initial approximation (18) at $h=0.1$.

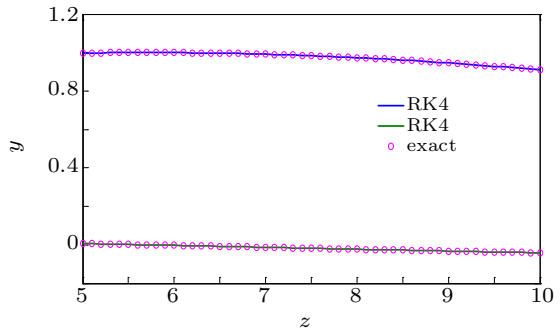


Fig. 4. (color online) Comparison between exact and numerical solutions for the reduced ODEs system (17) for the CSP equation when we start with initial values at $z = 5$ and the initial approximation (20) at $h = 0.1$.

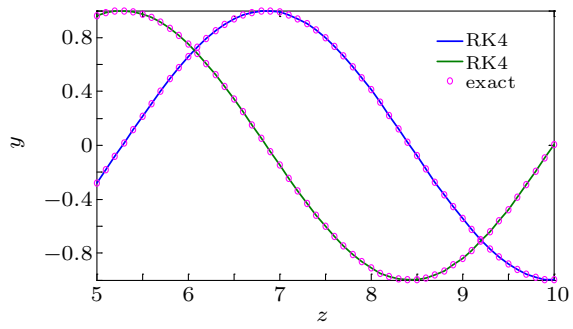


Fig. 5. (color online) Comparison between exact and numerical solutions for the reduced ODEs system (23) for the CSP equation when we start with initial values at $z = 5$ and the initial approximation (24) at $h = 0.1$.

Table 6. Absolute error (E_a), relative error (E_r), and percentage error (E_p) for Example 2 of subalgebra $V_2 + V_3$.

	z	E_a	E_r	E_p
y_3	6	$2.9976021664 \times 10^{-15}$	$2.9988808801 \times 10^{-15}$	$2.9988808801 \times 10^{-13}$
	7	$1.5998313784 \times 10^{-13}$	$1.6132783594 \times 10^{-13}$	$1.6132783594 \times 10^{-11}$
	8	$1.8119949984 \times 10^{-12}$	$1.8606555492 \times 10^{-12}$	$1.8606555492 \times 10^{-10}$
	9	$1.0095035918 \times 10^{-11}$	$1.0667901281 \times 10^{-11}$	$1.0667901281 \times 10^{-9}$
	10	$3.7877034841 \times 10^{-11}$	$4.1655275517 \times 10^{-11}$	$4.1655275517 \times 10^{-9}$
y_4	6	$1.9949319973 \times 10^{-17}$	$6.8320706646 \times 10^{-15}$	$6.8320706646 \times 10^{-13}$
	7	$1.6011497683 \times 10^{-15}$	$1.2426994083 \times 10^{-13}$	$1.2426994083 \times 10^{-11}$
	8	$3.2602393007 \times 10^{-14}$	$1.4349512512 \times 10^{-12}$	$1.4349512512 \times 10^{-11}$
	9	$2.6960378374 \times 10^{-13}$	$8.3393901745 \times 10^{-12}$	$8.3393901745 \times 10^{-10}$
	10	$1.3569007029 \times 10^{-12}$	$3.2606296233 \times 10^{-11}$	$3.2606296233 \times 10^{-9}$

Table 7. Absolute error (E_a), relative error (E_r), and percentage error (E_p) for the example of subalgebra $V_2 - V_3$.

	z	E_a	E_r	E_p
y_3	6	$1.3090309947 \times 10^{-9}$	$1.9924774679 \times 10^{-9}$	$1.9924774679 \times 10^{-7}$
	7	$1.6135250902 \times 10^{-7}$	$1.6308805184 \times 10^{-7}$	$1.6308805184 \times 10^{-5}$
	8	$1.5353387229 \times 10^{-6}$	$3.7254789046 \times 10^{-6}$	$3.7254789046 \times 10^{-4}$
	9	$1.0498759309 \times 10^{-6}$	$1.9298440997 \times 10^{-6}$	$1.9298440997 \times 10^{-4}$
	10	$2.8677316509 \times 10^{-5}$	$2.8677597361 \times 10^{-5}$	$2.8677597361 \times 10^{-3}$
y_4	6	$2.1546260331 \times 10^{-9}$	$2.8579647039 \times 10^{-9}$	$2.8579647039 \times 10^{-7}$
	7	$3.1689822987 \times 10^{-8}$	$2.1779942009 \times 10^{-7}$	$2.1779942009 \times 10^{-5}$
	8	$1.1455247350 \times 10^{-6}$	$1.2572568193 \times 10^{-6}$	$1.2572568193 \times 10^{-4}$
	9	$1.0913914822 \times 10^{-5}$	$1.3007132818 \times 10^{-5}$	$1.3007132818 \times 10^{-3}$
	10	$3.1303523470 \times 10^{-5}$	$7.0731268953 \times 10^{-3}$	$7.0731268953 \times 10^{-1}$

5. Discussion and concluding remarks

The Lie symmetry method is utilized to investigate the symmetries and invariant solutions of the CSP equations. By determining the transformation group under which a given system is invariant, we obtained information about the invariants and symmetries of that equation. This information, in turn, was used to determine similarity variables that reduce the number of independent variables. The vector fields of the optimal system lead to reduction of the nonlinear system of PDEs to ODEs. The infinitesimal generators in the optimal system are used for reductions and exact solutions. By using these vector fields, we have found group-invariant solutions. Here, we found new exact solutions which are compared with numerical solutions using the classical fourth-order Runge–Kutta scheme. The numerical solutions are presented in Tables 3–7 and Figs. 1–5. Form these results, we conclude that the numerical solutions are in good agreement with the analytical solution. The results reported here provide further evidence of the usefulness of the fourth-order Runge–Kutta method for a solving nonlinear system of the partial differential equation. It is worth mentioning here that the authenticity of all the solutions has been checked with the aid of Maple software.

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