Vortex solitons in the semi-infinite gap of optically induced periodic lattices*

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Vortex solitons with a ring vortex core residing in a single lattice site in the semi-infinite gap of square optical lattices are reported. These solitons are no longer bound states of the Bloch-wave unit (Bloch-wave distribution in one lattice site) at the band edge of the periodic lattice, and consequently they do not bifurcate from the corresponding band edge. For saturable nonlinearity, one family of such solitons is found, and its existing curve forms a closed loop, which is very surprising. For Kerr nonlinearity, two families of such vortex solitons are found.

Keywords: vortex soliton, Bloch band gap, nonlinear periodic lattice

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1. Introduction

Nonlinear wave propagation in periodic media exhibits many fundamental features that arise from the existence of Bloch bands and forbidden gaps in the linear spectrum of the media. One of the main feature is the formation of self-traped localized states, i.e., gap solitons or lattice solitons.[1–3] It has been shown that lattice solitons can exist in different open gaps of the periodic lattices. They either bifurcate from the edges of the corresponding Bloch bands,[4,5] or wind back before reaching the band edges,[6–8] depending on how they are constructed by the Bloch-wave unit (Bloch-wave distribution in one lattice site) at the band edges. In Ref. [4], all soliton families bifurcating from each Bloch band edge are classified as fundamental solitons, reduced-symmetry solitons, dipole-array solitons, and many others. These solitons share common properties: far from the band edge they bifurcate from, they become very localized and their main bodies consist of one Bloch-wave unit. When approaching the band edges, more and more Bloch mode tails appear resulting in the solitons being non-localized. Therefore, for these solitons, according to the Bloch modes at each band edge,[4] the soliton type existing in each gap is determined. For example, solitons bifurcating from the first band of a periodic square lattice are nodeless fundamental solitons in the semi-infinite gap under focusing nonlinearity, and fundamental solitons with neighboring out-of-phase tails in the first gap under defocusing nonlinearity (here we should mention that in-phase dipole solitons in the semi-infinite gap under focusing nonlinearity and out-of-phase dipole solitons in the first gap under defocusing nonlinearity also bifurcate from the edges of the first band);[1,9–11] solitons bifurcating from the second band are dipole solitons in the first and the second gap under self-focusing and self-defocusing nonlinearity, respectively.[4,12] Due to the x–y symmetry of the lattice, as the dipole solitons, there are vortex solitons in the first and the second gap which bifurcate from the second band.[4,13,14] Another family of soliton bifurcating from the band edge is the soliton train,[15,16] with its distribution in one direction being periodic (the same distribution as the edge Bloch mode) and in the other direction being localized (the main body is one Bloch wave unit). In the linear case, defect modes in periodic lattice with a single-sited defect have similar bifurcation properties.[17–19]

Except for the solitons bifurcating from the band edges, numerous types of solitons exist in the open gaps of the periodic lattice. These solitons disappear before reaching the band edges, and thus they no longer bifurcate from the edges of Bloch bands. They can be identified as bound states of the Bloch wave units in the lattice site. Therefore, different numbers of Bloch wave units in different lattice sites, with different phase difference, when bounded together, can form different types of solitons. Examples include out-of-phase dipole solitons in the semi-infinite gap and in-phase dipole solitons in the first gap,[20–22] quadrupole solitons,[22] soliton complexes,[23,24] vortex solitons,[6,10,25–30] truncated-Bloch-wave solitons,[8,31–33] and so on.

In this paper, we found another type of solitons in the semi-infinite gap of square periodic lattices. They are charge-1 vortex solitons, and are not bound states of Bloch-wave units at the lowest edge of the first band. The center of these solitons has a ring profile occupying a single lattice site, which is instead the typical Bloch-wave unit profile at the edge of the second band. This single-site ring center can be viewed as the nonlinear mode of the individual lattice site, because it does

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not come from the band-gap effect. Consequently, such vortex solitons do not bifurcate from the edge of the first band, i.e., their existent region cannot reach the first band.

2. Vortex solitons supported by saturable nonlinearity

For saturable nonlinearity, we consider one ordinarily polarized two-dimensional (2D) square-lattice beam and one extraordinarily polarized probe beam with very low intensity propagating simultaneously in a photorefractive crystal, and the two beams are mutually incoherent. The dynamics of the probe beam can be modeled by the dimensionless nonlinear Schröinger equation:[34]

$$\mathrm{i}\frac{\partial \psi}{\partial z}+\nabla_{\perp}^{2}\psi-\frac{E_0}{1+V(x,y)+|\psi|^2}\psi=0, \tag{1}$$

where ψ is the slowly-varying amplitude of the probe beam, $\nabla_{\perp}^{2}=\partial_x^2+\partial_y^2$, (x,y) are the the transverse distances (in units of D/π with D being the lattice spacing), z is the propagation distance (in units of $2k_0n_eD^2/\pi^2$), E_0 is the applied dc field [in units of $\pi^2/(k_0^2n_e^4D^2r_{33})$],

$$V(x,y)=V_0\cos(x)^2\cos(y)^2 \tag{2}$$

is the square lattice with V_0 being the modulation depth of the lattice (in units of the background irradiance). $k_0=2\pi/\lambda_0$ is the wave number in vacuum, n_e is the refractive index of the extraordinarily polarized beam, and r_{33} is the the electro–optic coefficient of the crystal. Typical physical parameters realized in experiment for this model can be found in Refs. [16] and [21].

Localized solutions of Eq. (1) are sought in the form of $\psi(x,y;z)=U(x,y)\exp(-\mathrm{i}\mu z)$ where μ is the propagation constant. We find votex solitons $U(x,y)$ by solving equation

$$\nabla_{\perp}^{2}U+\mu U-\frac{E_0}{1+V(x,y)+|U|^2}U=0 \tag{3}$$

numerically.[35] Without loss of generality, in this paper we take $E_0=15$ and $V_0=6$.

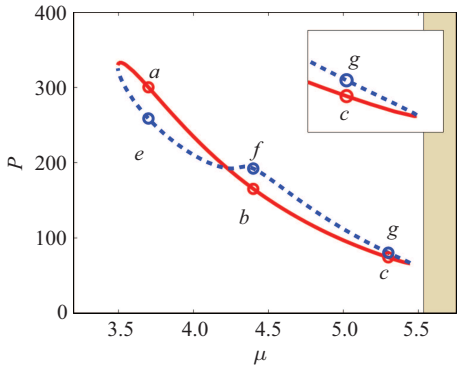


Fig. 1. (color online) Power diagram of vortex solitons in the semi-infinite gap. Shaded: the first Bloch band with the left band edge $\mu_{\mathrm{edge}}=5.536$. Soliton profiles at the position of circle markers are displayed in Fig. 2.

A new type of vortex solitons in the semi-infinite gap of the linear periodic potential $-E_0/(1+V)$ in Eq. (3) is found. They are on site solitons and have simple 2π-phase winding structures, thus they are charge-1 vortex solitons. The power versus propagation constant curve of these solitons with the power defined as $P=\iint|U|^2\mathrm{d}x\mathrm{d}y$ is shown in Fig. 1. Typical soliton profiles along the power curve are displayed in Fig. 2. Different types have been found in the semi-infinite gap, which are constructed by Bloch wave units of the lowest band edge,[6,10] vortex solitons found in this paper have a ring vortex core residing in a single lattice site, which is not the typical Bloch wave unit for forming solitons in this gap, but instead it is the typical Bloch wave unit of the second band edge. Due to the fact that this ring center does not come from the band gap effect, it can only be viewed as the nonlinear mode of the single lattice site.

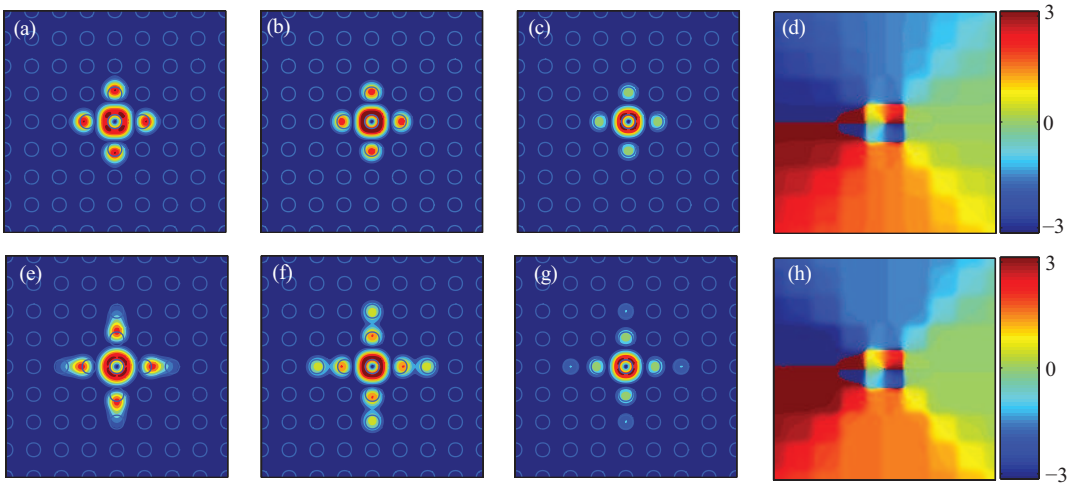


Fig. 2. (color online) (a)–(c) and (e)–(g): Intensity profiles ($|U|^2$) of the vortex solitons at the circle marker positions shown in Fig. 1; (d) and (h): phase of solitons shown in panels (b) and (f). The background circles represent the lattice sites.

Since the solitons' structure is not as the distribution of Bloch waves at the lowest band edge, it can be naturally inferred that their power curve cannot approach the first band, and it will turn back into the semi-infinite gap before reaching the band edge. As can be seen in Fig. 1, for example, the power curve of the red solid branch turns back into the semi-infinite gap at $\mu_{\rm r} = 5.445$ (the lowest band edge is at $\mu_{\rm edge} = 5.536$) and becomes a blue dashed branch. However, instead of going deeper into the gap independently with their power becoming higher and higher, these two branches merge together at $\mu_{\rm l} = 3.496$, forming a closed loop. Therefore, the vortex solitons can only exist in a certain region $[\mu_{\rm l}\ \mu_{\rm r}]$ in the semi-infinite gap. This phenomenon has not been found before in the field of solitons supported by periodic lattices to the best of our knowledge, but a power loop phenomenon of solitons supported by a non-periodic lattice has been found recently.[36] Now let us examine the solitons' profile evolution along their power curve. Figure 2(b) shows the intensity distribution of the red solid branch soliton shown in Fig. 1 at the middle of its existence region with $\mu = 4.4$. The soliton is composed of five parts which occupy the nearest five lattice sites. The central part is a ring vortex core located in a single lattice site, and the other four humps are Gaussian profile (Bloch wave units of the lowest band edge) located in each lattice site adjacent to the center. Figure 2(d) displays its phase structure, and it is obvious that winding around the soliton center, the phase changes 2π; the figure of the soliton's phase structure also shows that the vortex core and the outer humps are out of phase, i.e., soliton humps at different radii are out of phase. With the increase of the propagation constant μ, the soliton's power and the intensity decrease. When μ approaches the right cutoff value $\mu_{\rm r}$, four humps along the axes adjacent to the previous Gaussian humps appear and after the power curve winds back into the semi-infinite gap, the vortex soliton becomes composed of nine parts as shown in Fig. 2(g). Along this branch (blue dashed in Fig. 1), when μ decreases, the new appearing humps manifest themselves gradually [see Fig. 2(f)]; figure 2(h) is the phase of solitons shown in Fig. 2(f), and it is similar to that of the red branch soliton. But when the propagation constant μ decreases further to the left cutoff value $\mu_{\rm l}$, the most outer four Gaussian humps begin to shrink to the humps adjacent to them, and figure 2(e) demonstrates such profile. When μ reaches $\mu_{\rm l}$, the four lobe humps of the vortex soliton become totally single Gaussian in profile, the power curve winds back again toward the first band of the lattice, and the soliton translates to the red solid branch. Typical solitons shown in Fig. 2(a)–2(c) and 2(e)–2(g) show the reshaping process of such vortex solitons along their power curve. Even under the same model, the reshaping process is very different from that of the solitons composed of Bloch wave units where the soliton reshaping process is also very complicated.[6] These vortex solitons are very localized in the corresponding lattice sites even near the cutoff value of their right existence region.

3. Vortex solitons supported by Kerr nonlinearity

For Kerr nonlinearity, the model equation considered in this paper is:

$$\mathrm{i}\frac{\partial \psi}{\partial z} + \nabla_{\perp}^2 \psi + V(x,y)\psi + |\psi|^2\psi = 0, \tag{4}$$

where the periodic lattice potential

$$V(x,y) = -V_0[\sin(x)^2 + \sin(y)^2]. \tag{5}$$

Here, the spatial transverse and longitudinal coordinates (x,y) and z are scaled to the input beam width a and diffraction length $L_{\rm diff} = 2k_0 n_0 a^2$, respectively, where n_0 is the background refractive index. The depth of the lattice V_0 is scaled to the parameter $1/(2k_0^2 n_0 a^2)$. Therefore, from a physical perspective, for example, if $\lambda_0 = 1$ μm, $n_0 = 3$, and $a = 10$ μm, then $x = 1$ corresponds to 10 μm, $z = 1$ means the propagation distance of 3.77 mm, and $V_0 = 1$ corresponds to a maximum variation of the refractive index $n^{\max} \approx 4.22 \times 10^{-5}$. The lattice depth $V_0 = 6$ is selected in this paper; for such V_0 depth, the lowest band edge of the periodic lattice Eq. (5) is $\mu_{\rm dege} = 4.126$. Soliton solutions in the semi-infinite gap with $\mu < \mu_{\rm edge}$ of the lattice are derived by solving the resulting equation after inserting $\psi(x,y;z) = U(x,y)\exp(-\mathrm{i}\mu z)$ into Eq. (4).

Two families of vortex solitons with its center being a single-site vortex core are found, and the power curve of one family is shown in Fig. 3. Apparently, this soliton family is composed of two branches, which is similar to those found in Ref. [6]. Apart from the band edge in the semi-infinite gap, the lower branch soliton of this family is a pure single-charge ring vortex residing in the central lattice site; an example of such soliton's intensity is displayed in Fig. 4(d). In the previous work, for example, work in Refs. [4], [10]–[16] the solitons found in the band gap are all related to the Bloch

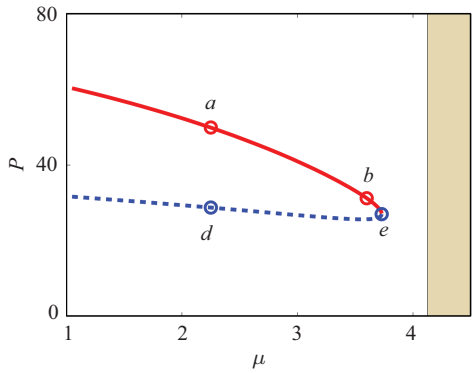


Fig. 3. (color online) Power curve of the first family Kerr vortex solitons in the semi-infinite gap. Shaded: the first Bloch band with the left band edge $\mu_{\rm edge} = 4.126$. Soliton profiles at the position of circle markers are displayed in Fig. 4.

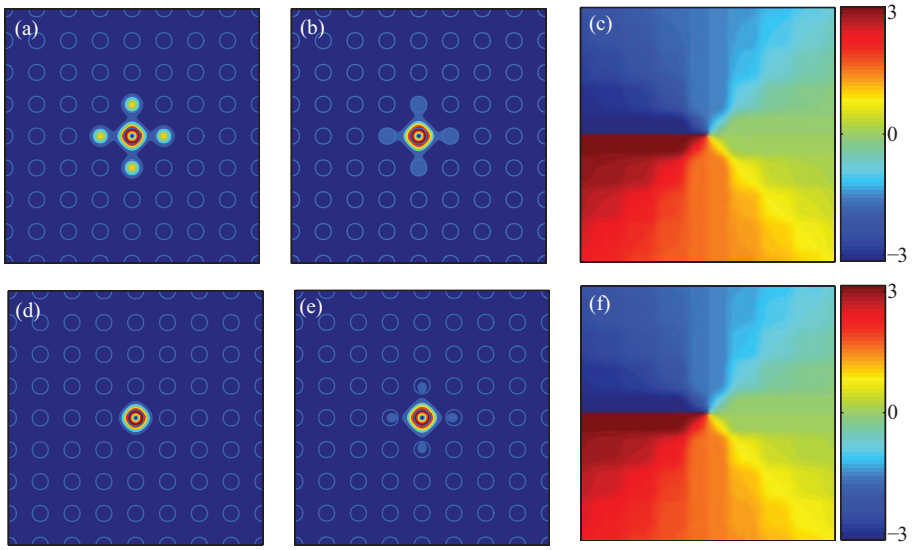


Fig. 4. (color online) Panels (a), (b), (d), and (e) show the intensity of the Kerr vortex solitons at the circle marker positions shown in Fig. 3; panels (c) and (f) show the phases of panels (b) and (e). The background circles represent the lattice sites.

waves at the corresponding band edges of the band they bifurcate from; they either bifurcate from the band edges with their formation resulted from localized Bloch waves in the forbidden gap,[4,10,13,14] or are formatted by Bloch wave units in a different way from the distribution of Bloch waves at the band edge.[6] Therefore, in the semi-infinite gap of the periodic lattice, formation of vortex solitons occupying a single lattice site is very unique because such vortex mode has nothing to do with the Bloch waves at the lowest edge of the first band, and it can only be considered as the nonlinear mode of the central lattice site. With increase of the propagation constant μ, the power curve of the lower branch slowly decreases (blue dashed line in Fig. 3), the solitons reshape themselves very little, and they are still localized in the central lattice site almost throughout their existence region. When approach the right cutoff value $\mu_r = 3.729$, the power curve of the lower branch winds back to the semi-infinite gap, and just near the cutoff value μ_r, the solitons of the lower branch develop four small Gaussian humps along the axis direction in the nearest four lattice sites. Figure 4(e) shows the soliton intensity at $\mu = \mu_r$, and it can be seen from this figure that the newly appearing four humps are very small comparing to the vortex core and the solitons are still very localized even at the right cutoff value of their existence region near the band. After the power curve winds back into the semi-infinite gap, the lower branch changes to the upper branch of this family, and the nearest Gaussian humps to the central vortex core become to manifest themselves. Figures 4(b) and 4(a) show the soliton intensity near μ_r with $\mu = 3.6$ and deep in the gap with $\mu = 2.25$. Unlike such type solitons supported by saturable nonlinearity (Fig. 1), for Kerr nonlinearity, power curve of the upper branch will not wind back again. Deeper into the gap, the two branches of this family will not merge, and they exist in the region $\mu < \mu_r$ independently.

Figures 4(c) and 4(f) show the phase structure of the upper and lower branch solitons at $\mu = 3.6$ and $\mu = 3.729$, respectively. Solitons of the whole family have the same phase structure. As can be seen from these two figures, they are the normal phase structure of a single charge vortex soliton, thus, for the upper branch, the vortex core in the central lattice site and the four Gaussian humps near to it are in phase.

The vortex solitons' power curve and spatial intensity profile of the other family are shown in Figs. 5 and 6. As in the first family, the soliton of this family also includes two branches. The two branches merge at $\mu_r = 4.069$, and they exist independently in the left region of μ_r, i.e., the region of $\mu < \mu_r$. But no solitons of this family are a simple ring vortex. Beyond the existence cutoff value μ_r in the semi-infinite gap, solitons of the low branch (blue dashed line in Fig. 5) are mainly composed of five parts, i.e., ring vortex core at the central lattice site and four Gaussian humps around it at the four nearest lattice sites. The soliton intensity at $\mu = 2.25$ of this branch is shown in Fig. 6(d). With μ increasing toward the band, the Gaussian humps around the vortex core decrease as compared to the central vortex, and an example of the soliton intensity near μ_r at $\mu = 4$ is displayed in Fig. 6(e). When approaching μ_r, the power curve of the lower branch winds back into the semi-infinite gap, becoming the upper branch of this family, and at the same time four small Gaussian humps appear adjacent to the previous four Gaussian humps. The soliton intensity of the upper branch at $\mu = 2.25$ and 4 are shown in Figs. 6(a) and 6(b). The reshaping behavior of this family is the same as the first family: for the lower branch, increasing μ, the soliton's power and the outer Gaussian humps (comparing to the central hump) decrease, at some value of $\mu = \mu_r$, the lower branch winds back becoming the upper branch with new Gaussian humps appearing at the nearest four lattice sites of the soliton, and with μ decreasing the newly appearing humps

increase. By comparing Fig. 4(a) and Fig. 6(d), Fig. 4(b) and Fig. 6(e), it looks like that the upper branch soliton of the first family and the lower branch soliton of the second family have the same spatial distribution. But in fact, they are very different. Figures 6(c) and 6(f) are the phase of the upper and lower branch solitons of Figs. 6(b) and 6(e) shown in these figures. It is very clear that for the solitons of this branch, their phase is similar to that of high order vortex solitons in bulk medium, i.e., humps at different radii are out of phase; for example, see the phase of the lower branch solitons (Fig. 6(f)), the center vortex and the four Gaussian humps around it are out of phase. However, for upper branch solitons of the first family, humps at different radii are all in phase (see Figs. 4(c) and 4(f)).

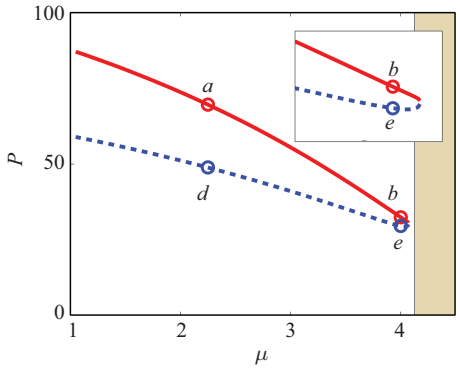


Fig. 5. (color online) Power curve of the second family Kerr vortex solitons in the semi-infinite gap. Shaded: the first Bloch band with the left band edge $\mu_{\rm edge} = 4.126$. Soliton profiles at the position of circle markers are displayed in Fig. 6.

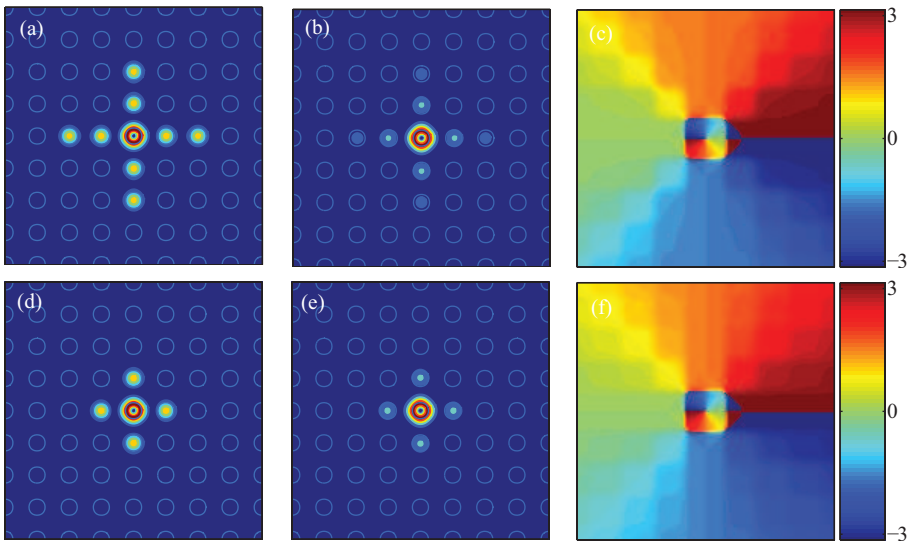


Fig. 6. (color online) Intensity of the Kerr vortex solitons at the circle marker positions shown in Fig. 5. The background circles represent the lattice sites.

4. Summary and discussion

In summary, new types of vortex solitons in the semi-infinite gap of a periodic lattice are found for both saturable and Kerr nonlinearity. These types of vortex solitons are no longer bound states of Bloch wave units at the lowest band edge of the first band, and the unique character of such vortex solitons is that their central part is a ring vortex core residing in a single lattice site. Such a vortex part in a single lattice site is the typical units of solitons bifurcating from the second band in the first gap and in the second gap;[4] therefore, the ring core of such new type vortex solitons can only be explained as the nonlinear mode of a single lattice site. For saturable nonlinearity, such vortex solitons can only exist in a finite region, and their power curve forms a closed loop. For Kerr nonlinearity, two families of such vortex solitons are found, both of them have two branches: the upper and lower branches, and the two branches merge at certain value of $\mu_{\rm r}$. The first family solitons have a similar phase structure as those of first order vortex solitons in bulk medium, i.e., humps at different radii are in phase, while the second family solitons have a similar phase structure as those of higher order vortex solitons in bulk medium, i.e., humps at different radii are out of phase.

A single lattice site can support not only the vortex mode, but also other type modes such as dipole mode, quadrupole mode, etc; so whether these modes are able to be bounded with each other or with the Bloch wave units at the band edge to form new type lattice solitons, is still an open question. The results in this paper provides a new way to construct solitons in a nonlinear periodic lattice system, and hopefully new types of other solitons constructed not only by Bloch wave units will be found, and the stability properties of such solitons studied.

References

[1] Efremidis N K, Hudock J, Christodoulides D N, Fleischer J W, Cohen O and Segev M 2003 *Phys. Rev. Lett.* **91** 213906
[2] Efremidis N K and Christodoulides D N 2003 *Phys. Rev. A* **67** 063608
[3] Louis Pear J Y, Ostrovskaya E A, Savage C M and Kivshar Y S 2003 *Phys. Rev. A* **67** 013602

[4] Shi Z Q and Yang J K 2007 *Phys. Rev. E* **75** 056602
[5] Pelinovsky D E, Sukhorukov A A and Kivshar Y S 2004 *Phys. Rev. E* **70** 036618
[6] Wang J D and Yang J K 2008 *Phys. Rev. A* **77** 033834
[7] Akylas T R, Hwang G and Yang J K 2012 *Proc. R. Soc. A* **468** 116
[8] Wang J D, Yang J K, Alexander T J and Kivshar Y S 2009 *Phys. Rev. A* **79** 043610
[9] Fleischer J W, Segev M, Efremidis N K and Christodoulides D N 2003 *Nature* **422** 147
[10] Yang J K and Musslimani Z H 2003 *Opt. Lett.* **28** 2094
[11] Ostrovskaya E and Kivshar Y S 2004 *Opt. Express* **12** 19
[12] Fischer R, Träger D, Neshev D N, Sukhorukov A A, Krolikowski W, Denz C and Kivshar Y S 2006 *Phys. Rev. Lett.* **96** 023905
[13] Bartal G, Manela O, Cohen O, Fleischer J W and Segev M 2005 *Phys. Rev. Lett.* **95** 053904
[14] Manela O, Cohen O, Bartal G, Fleischer J W and Segev M 2004 *Opt. Lett.* **29** 2049
[15] Chen Z G, Martin H, Eugenieva E D, Xu J J and Bezryadina A 2004 *Phys. Rev. Lett.* **92** 143902
[16] Lou C B, Wang X S, Xu J J, Chen Z G and Yang J K 2007 *Phys. Rev. Lett.* **98** 213903
[17] Fedele F, Yang J K and Chen Z G 2005 *Stud. Appl. Math.* **115** 279
[18] Wang J D, Yang J K and Chen Z G 2007 *Phys. Rev. A* **76** 013828
[19] Yang X Y, Zheng J B and Dong L W 2011 *Chin. Phys. B* **20** 034208
[20] Neshev D, Ostrovskaya E, Kivshar Y S and Chen Z G 2003 *Opt. Lett.* **28** 710
[21] Yang J K, Makasyuk I, Bezryadina A and Chen Z G 2004 *Opt. Lett.* **29** 1662
[22] Yang J K, Makasyuk I, Bezryadina A and Chen Z G 2004 *Stud. Appl. Math.* **113** 389
[23] Kartashov Y V, Egorov A A, Torner L and Christodoulides D N 2004 *Opt. Lett.* **29** 1918
[24] Yang J K, Zhang P, Yoshihara M, Hu Y and Chen Z G 2011 *Opt. Lett.* **36** 772
[25] Neshev D N, Alexander T J, Ostrovskaya E A, Kivshar Y S, Martin H, Makasyuk I and Chen Z G 2004 *Phys. Rev. Lett.* **92** 123903
[26] Fleischer J W, Bartal G, Cohen O, Manela O, Segev M, Hudock J and Christodoulides D N 2004 *Phys. Rev. Lett.* **92** 123904
[27] Yang J K 2004 *New J. Phys.* **6** 47
[28] Ostrovskaya E A, Alexander T J and Kivshar Y S 2006 *Phys. Rev. A* **74** 023605
[29] Alexander T J, Sukhorukov A A and Kivshar Y S 2004 *Phys. Rev. Lett.* **93** 063901
[30] Richter T and Kaiser F 2007 *Phys. Rev. A* **76** 033818
[31] Alexander T J, Ostrovskaya E A and Kivshar Y S 2006 *Phys. Rev. Lett.* **96** 040401
[32] Ostrovskaya A A and Kivshar Y S 2004 *Phys. Rev. Lett.* **93** 160405
[33] Alexander T J and Kivshar Y S 2006 *Appl. Phys. B* **82** 203
[34] Efremidis N K, Sears S, Christodoulides D N, Fleischer J W and Segev M 2002 *Phys. Rev. E* **66** 046602
[35] Yang J K and Lakoba T I 2007 *Stud. Appl. Math.* **118** 153
[36] Yang J K 2012 *Stud. Appl. Math.* **129** 133