Hysteresis behavior and nonequilibrium phase transition in a one-dimensional evolutionary game model*

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We investigate a simple evolutionary game model in one dimension. It is found that the system exhibits a discontinuous phase transition from a defection state to a cooperation state when the b payoff of a defector exploiting a cooperator is small. Furthermore, if b is large enough, then the system exhibits two continuous phase transitions between two absorbing states and a coexistence state of cooperation and defection, respectively. The tri-critical point is roughly estimated. Moreover, it is found that the critical behavior of the continuous phase transition with an absorbing state is in the directed percolation universality class.

Keywords: evolutionary game model, nonequilibrium phase transition with absorbing state, cooperation phenomenon, hysteresis behavior

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1. Introduction

Cooperation plays an important role in the evolution of biological and social systems. To understand the emergence of cooperation in these systems remains a great challenge for scientists of many different fields of natural and social sciences.[1–4] As an effective framework to study cooperation between unrelated individuals, some evolutionary game models have attracted much attention, including the well-known prisoner's dilemma (PD) game and the snowdrift game (SG).[4–9] In these models, two players simultaneously decide to cooperate or defect. They both receive R if they cooperate mutually, and they only receive P if both of them defect. A defector exploiting a cooperator obtains an amount T, and the exploited cooperator receives S. For the PD game model, $T > R > P > S$ and $T + S < 2R$. In a single round, it is best to defect regardless of the opponent's decision. This unfavorable scenario for cooperation stimulated the adoption of other games such as the SG game model, in which the order of P and S is exchanged and cooperation is more favorable in well-mixed populations.[2]

The evolution of cooperation has been investigated recently in more realistic situations, departing from the well-mixed population situation. The effects of the spatial structure are considered, where the individuals are constrained to play only with their immediate neighbors.[7,8] The heterogeneity of the contacts in a realistic social system is also considered, and the evolutionary games are studied extensively based on the contacts of complex networks.[5,10–19] It is disclosed that cooperation becomes a dominating trait in the PD and SG game models in scale-free networks. Most recently, it has become very interesting that many complex dynamical behaviors abound in these evolutionary game systems.[20] It is found that the cooperation evolution is dependent on the initial configuration of the players' strategies,[20] and a hysteresis behavior exists in the snowdrift game model in complex networks.[21] On the other hand, a discontinuous phase transition has been investigated extensively in the one-dimensional PD game model with a strong noise effect,[22] where it is an interesting problem of non-equilibrium statistic physics whether there is a discontinuous phase transition in one dimension without external fields, additional symmetric conditions, special boundary condition, and strong noise within short interactions.[23–31]

In this paper, we study a spatial game model to investigate the effects of the possible payoff T and P of a defector on the emergence of cooperation in one dimension as the possible payoff of a cooperator is fixed. It is found that there are complex phase transitions with an absorbing state when T and P vary.

2. Model

In our model, players locate on a one-dimensional lattice of size L with periodic boundary conditions. For a random initial strategy state, every player chooses one strategy randomly from cooperation and defection, which are described by $s_i = 1$ and $s_i = -1$, respectively. For simplicity, we fix the possible payoff of a cooperator $R = 1$ and $S = -1$ and assume $T = b$ and $P = -y$.

The evolution of cooperation is investigated through a

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sequential simulation. For a Monte Carlo attempt, a player X and one of its neighbors Y are firstly selected randomly, then they play games with all of their neighbors, and their payoffs are accumulated as E_x and E_y, respectively. Secondly, player X updates its strategy according to the following rules: it does not change its strategy if $E_x > E_y$, otherwise, X adopts Y's strategy with a probability $p = (E_y - E_x)/\{k[\max(1,b) - \min(-1,-y)]\}$,[10,11] where k is the number of interacting neighbors. When the system enters into a stationary state or the simulation is ended up until t Monte Carlo steps (a Monte Carlo step (MCS) means L Monte Carlo attempts), then the fraction ρ of the cooperators is calculated and averaged over 100 independent samples.

3. Simulation results

Firstly, it is very interesting to discuss the effect of y on the evolution of cooperation as the interactions of a player with its nearest neighbors (NN) and the next nearest neighbors (NNN) are included.[22] The simulation results are shown in Fig. 1. Under a fixed b, when y is small, the fraction of defectors increases quickly from the random initial conditions, and the system enters into an absorbing state of defection. With y increasing, cooperation is favored and the system may enter into the absorbing state of cooperation easily when the lattice size is large enough and the simulation time t is long enough. In Fig. 1(a) for $b = 1.0$, the system transits very rapidly from defection to cooperation with y increasing. However, in Fig. 1(b) for $b = 1.8$, there is a coexistence state of defection and cooperation between two absorbing states. It is obvious that there are two continuous phase transitions with absorbing states for $b = 1.8$, as shown in Fig. 1(b), but it is very difficult to analyze that the phase transition is discontinuous or continuous for $b = 1.0$, as shown in Fig. 1(a).

In order to analyze the phase transition behavior in Fig. 1(a), we investigate whether there is a hysteresis behavior or not. Because the system may enter into an absorbing state of defection easily for small y, the hysteresis behavior is calculated with a method which was proposed by Bidaux *et al.*:[32] the dynamical rule is modified slightly to calculate the hysteresis loop, when $\sum_{i-2}^{i+2} s_i = -5$ (which means that player i and all of its neighbors take a defection strategy), player i changes its state with a probability p_e ($p_e < 10^{-2}$). Therefore there is a fluctuation for the "pure" absorbing state and the simulation for a hysteresis loop can continue as y is small. We begin the simulation process with a large y and a random initial condition, after δT MCS simulations are performed, decrease y slightly δy and let the evolution continue δT MCS again. When y is small enough and increases y slightly inversely to the initial value.

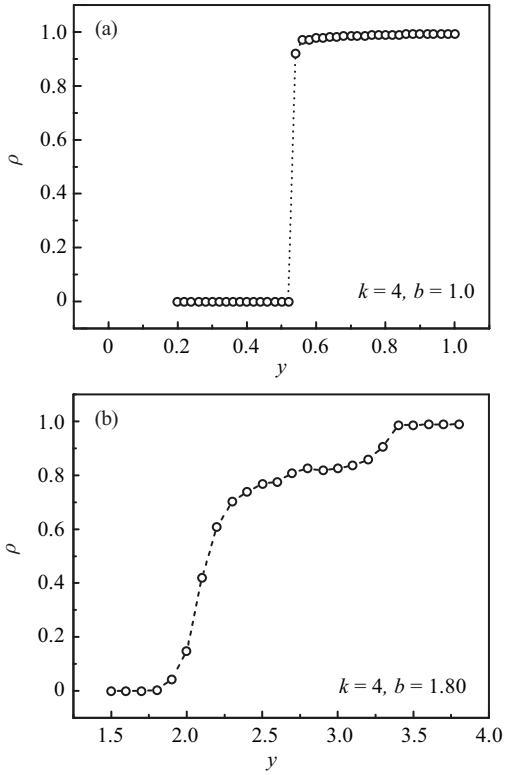


Fig. 1. The emergence of cooperation and the phase transition behavior with y. (a) A phase transition from an absorbing state of defectors to the absorbing state of cooperators. $b = 1.0$, $L = 1 \times 10^4$, $t = 1 \times 10^6$ MCS. (b) The continuous phase transitions between the absorbing state and the coexistence state of defectors and cooperators. $b = 1.8$, $L = 1 \times 10^4$, $t = 5 \times 10^5$ MCS.

As shown in Fig. 2(a), a hysteresis loop is found for $b = 1.0$, and the hysteresis loop disappears for $b = 1.8$, as shown in Fig. 2(b). In Fig. 3(a), we discuss the effect of δT on the hysteresis behavior. When $\delta y = 0.01$ and $L = 10^4$, the $\delta T \geq 2 \times 10^4$ MCS are long enough for the system to relax to the steady state, and the jump points τ_1 and τ_2 do not change nearly. In Fig. 3(b), both τ_1 and τ_2 increase with increasing b, but τ_1 increases more rapidly than τ_2. A key character for the discontinuous phase transition is the existence of the hysteresis phenomenon.

On the other hand, we investigate the fluctuating interface between the cooperation cluster and the defection region at a transition point in Fig. 1(a).[24,28] The transition point $y_c = 0.555 \pm 0.005$ is estimated roughly, and the system is divided into two blocks in the initial configuration. One is fully occupied by m_0 cooperators and the other by $L - m_0$ defectors. The system evolves into an absorbing defection state or an active cooperator state, in which cooperators almost occupy the system (in our simulation, we take cooperators occupying $0.95L$). The probability p_{active} of reaching an active state of cooperators is calculated in 500 independent samples on two rings of $L = 500$ and $L = 1000$, respectively.

The results are shown in Fig. 4. p_{active} is nonlinear against the $\rho_0 (= m_0/L)$ fraction of the cooperators. In particular, the curves are obviously distinct for $L = 500$ and $L = 1000$.

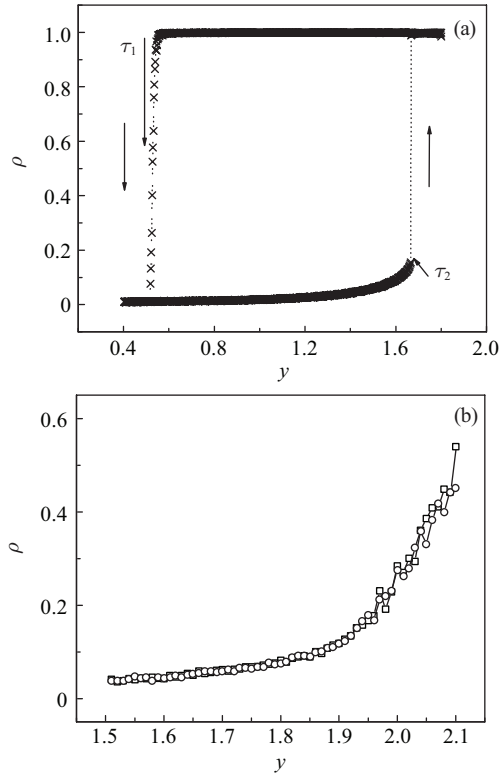


Fig. 2. The ρ fraction of the cooperators when y is scanned. (a) A hysteresis behavior. $b = 1.0$, $L = 1 \times 10^4$, $\delta T = 8 \times 10^4$ MCS, $p_e = 0.005$, and $\delta y = 0.001$. (b) The disappearance of the hysteresis loop. $b = 1.8$. Square symbols: gradual decrease in y; circle symbols: gradual increase in y, $L = 5000$ and $\delta T = 2 \times 10^4$ MCS, $p_e = 0.005$ and $\delta y = 0.01$.

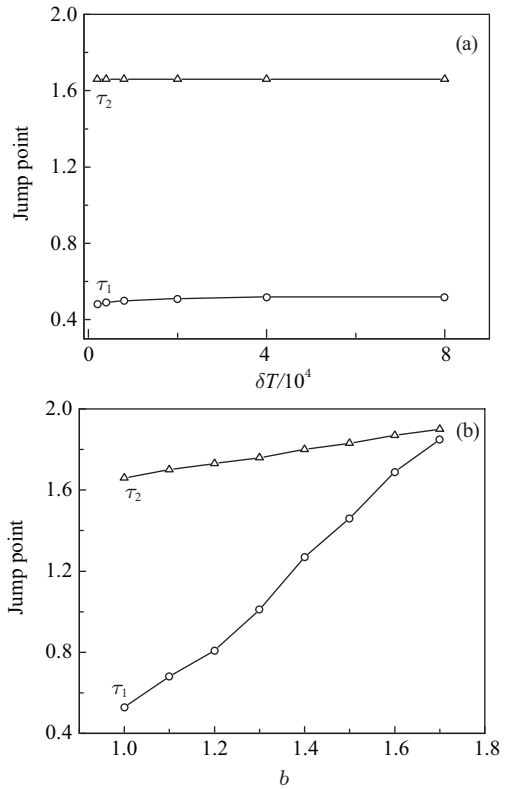


Fig. 3. The effects of some parameters on hysteresis behavior. (a) The effect of scanning rate δT. $b = 1.0$, $L = 1 \times 10^4$, $\delta y = 0.01$, $p_e = 0.005$. (b) The effect of b. $L = 10000$, $\delta y = 0.01$, $p_e = 0.005$, and $\delta T = 2 \times 10^4$ MCS.

In the inset figure of Fig. 4, we re-plot p_{active} versus the initial domain size m_0 of the cooperators, and the curves are almost consistent for $L = 500$ and $L = 1000$. This implies that the probability p_{active} is dependent on the initial cooperator domain size m_0, but not its fraction ρ_0. As shown in the inset figure, if $m_0 \leq 3$, then the domain of cooperation shrinks into the inactive region of defection completely and p_{active} is equal to zero. When $m_0 \geq 4$, p_{active} increases quickly firstly and then to 1.0 slowly with increasing m_0. Consequently, the motion of the interface between the active cluster of cooperation and the inactive region is dependent on the size of the active cluster of cooperation. When the size is small, the shrinking probability of the cluster is larger than its growing probability. However, when $m_0 \geq 12$, a reverse change occurs and then a cooperation cluster grows to reach a compact stable phase of cooperation in the inactive region of defection, with a probability of over 0.5. This is a kind of nucleation phenomenon, which is another important characteristic of the discontinuous phase transition.

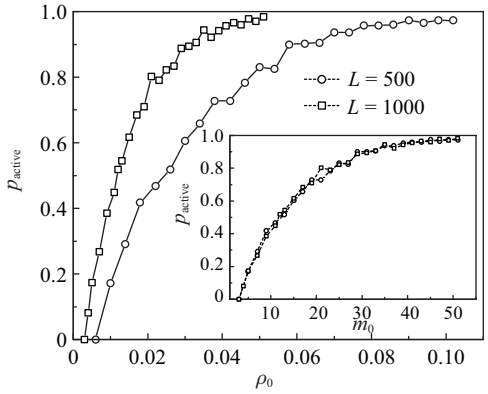


Fig. 4. The dynamical behavior in the critical point neighbor when $b = 1.0$. The probability p_{active} versus $\rho_0 = m_0/L$ with an inhomogeneous initial configuration at transition $y = 0.555$. The inset figure is p_{active} versus m_0; open circles: $L = 500$, open squares: $L = 1000$, $b = 1.0$.

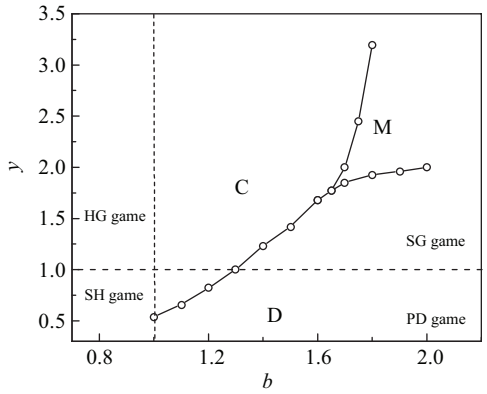


Fig. 5. Phase diagram in the b–y parameter plane. C, D, and M denote an absorbing state of cooperators, an absorbing state of defectors, and a mixed state of cooperators and defectors, respectively. We indicate the boundary with dashed lines of $b = 1.0$ and $y = 1.0$ to separate the prisoner's dilemma game (PD), snowdrift game (SG), harmony game (HG), and stag hunt game (SH).

In Fig. 5, the phase diagram is shown in a b–y plane. From Fig. 2 and Fig. 3(b), it is obvious that there is a tri-critical point

b_c where the phase transition changes from discontinuous to continuous. From Fig. 5, it is estimated roughly that the tri-critical point $b_c = 1.65 \pm 0.05$.

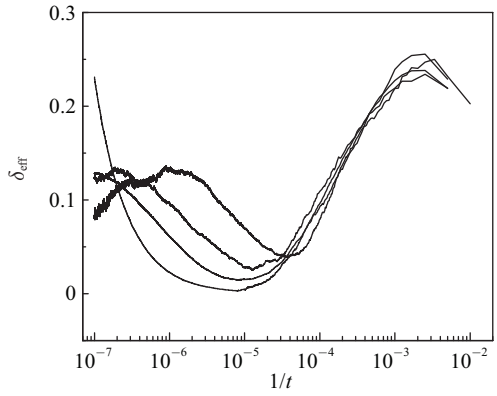


Fig. 6. The estimate of the continuous phase transition point, the local slope of ρ decays, from bottom to up, $y = 1.93339, 1.93335, 1.93334, 1.933333$, respectively. The simulation results are averaged with 40 independent survival samples. $L = 8 \times 10^4$.

The critical behavior of the continuous phase transition in Fig. 1(b) is also investigated by estimating the dynamical exponent. At the critical point y_c, $\rho_{(t)} \propto t^{-\delta}$.[33–35] Through analyzing the local slopes $\delta_{\rm eff}(t)$ of the order parameter: $\delta_{\rm eff}(t) = \{-\ln[\rho(t)/\rho(t/m)]\}/\ln(m)$ where $m = 10$, and both critical point y_c and decay exponent $\delta_{\rm eff}$ can be estimated. In the limit $t \to \infty$, the critical curve tends to the critical exponent $\delta_{\rm eff}$ by a straight line, while the sub(super)-critical curves veer down (up), respectively. In this model, the long time scaling correction brings difficulties for the accurate estimation of the critical point and critical exponent. In Fig. 6, it is found that at $y_c = 1.93335 \pm 0.00001$, we can roughly estimate $\delta_{\rm eff} = 0.13 \pm 0.04$, which is consistent with the ($\delta = 0.159$) value of the directed percolation (DP) class in the error range we expected because there is no long range interaction, no quenched noise, no additional symmetric condition, and only one absorbing state.[22,36]

In the above simulations, k is fixed. It is well known that the average degree may seriously affect the evolution of cooperation. Then we further discuss its effect on the phase transition behavior. In Fig. 7, it is easily proven that as $k = 2$, there is only a discontinuous phase transition from the absorbing state of defectors to the absorbing state of cooperators as $b = 1.0, 1.8$. When $k = 6$, it is shown that the phase transition is discontinuous for $b = 1.0$, and changes to a continuous one for $b = 1.8$ as $k = 4$, as mentioned above.

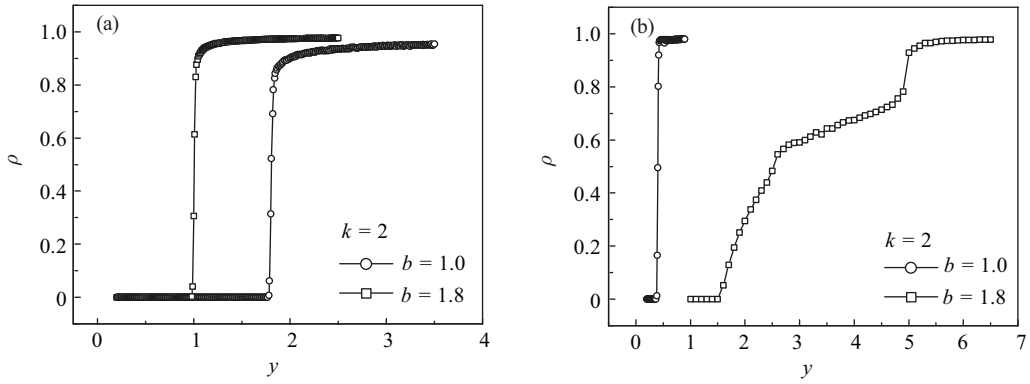


Fig. 7. The fraction of cooperators as a function of y. (a) $k = 2$, $L = 5000$, $t = 1 \times 10^5$ MCS; (b) $k = 6$, $L = 5000$, $t = 1 \times 10^5$ MCS.

4. Conclusion

In conclusion, phase transition behavior is investigated using an evolutionary game model in a one-dimensional lattice. According to the existence of hysteresis behavior and the nucleation phenomenon of cooperators, it is found that the system exhibits a discontinuous phase transition from a defection state to a cooperation state for a small payoff b of a defector exploiting a cooperator as the connectivity $k = 4$. These simulation results provide more details to fully understand the evolutionary stability of cooperation in a spatial game model. When b becomes large enough, the defector and cooperator can coexist. The system exhibits two continuous phase transitions, and the critical behavior of the continuous phase transition with an absorbing state is in the well-known DP universality class. The tri-critical point b_c is estimated roughly. Furthermore, as $k = 2$, it is found that the system exhibits only a discontinuous phase transition in the defined parameter ranges of the PD and SG game models. As $k = 6$, the phase transition behavior is similar to that of the case as $k = 4$.

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