Stress distribution and surface instability of an inclined granular layer*

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Static granular materials may avalanche suddenly under continuous quasi-static drives. This phenomenon, which is important in many engineering applications, can be explained by analyzing the stability of the elastic solutions. We show this for a granular layer driven by its inclination angle in gravity, where the elastic problem can be solved generally and analytically. It is found that a loss of stability may occur only at the free surface of the layer. This result is considered to be relevant for understanding surface avalanches and the flows observed experimentally.

Keywords: granular matter, nonlinear elasticity, stress, surface instability

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1. Introduction

The agglomeration of granulates such as sand and cereal is found frequently in daily life, but the fundamental physics of this phenomenon has only been established in recent years.[1] Most research in this area focuses on experimental,[2,3] and numerical simulations.[4,5] However, for engineering applications a continuum mechanical description of granular materials is also considered important.[6] In Refs. [7]–[9] a continual mechanical theory has been developed for a broad-ranged description of the macroscopic statics and dynamics of granular materials, including their elasticity and plasticity, the onset of mechanical instability, stress distribution, the propagation of sounds, shear band, and fast flows, etc. As the theory is an extension of the classical hydrodynamics of conventional liquids and solids (see for example textbooks[10,11]) to the granular materials, it is termed granular solid hydrodynamics, or GSH. (More details can be found in Ref. [12], or the references listed therein.) Notably the sound mentioned here consists of elastic waves, which are different from the viscous sound waves in the granular suspensions considered in Ref. [13], using the Navier–Stokes equation.

For static cases, the GSH is reduced to a nonlinear elastic theory with the following potential function $w(u_{ij})$:[7,9]

$$w = B\sqrt{\Delta}\left(\frac{2}{5}\Delta^2 + \frac{1}{\xi}u_s^2 - \frac{\chi}{\xi}\frac{u_t^3}{\Delta}\right). \tag{1}$$

Here, B, ξ, χ are material parameters which may depend on mass density ρ, the bulk strain $\Delta \equiv -u_{kk}$ is a negative trace of elastic strain u_{ij} (a summation on the repeating foot index is assumed), and $u_s^2 \equiv u_{ij}^* u_{ij}^*$ and $u_t^3 \equiv u_{ij}^* u_{jk}^* u_{ki}^*$ are the square and cubic of the deviatoric strain defined by $u_{ij}^* \equiv u_{ij} - u_{kk}\delta_{ij}/3$ (namely the traceless part of u_{ij}). An important property of the granular elasticity is that equation (1) is a convex function only in a limited regime of strain space, out of which it becomes concave. This implies mechanical instability, namely an elastic solution of the potential is stable only if it is fully inside the convex regime, and becomes unstable if it or part of it starts to enter into the concave regime. So an elastic analysis gives not only the stress distributions, but also when the analyzed system loses its stability, or begins to yield — a subject of great interest in engineering, for example in studying the avalanche of geological materials. To show this in detail in the present work, we analytically solve the elastic problem for an infinite granular layer inclined from the horizontal by an angle θ, of which the onset to instability will occur as θ increases to a critical value $\theta_{\max}$.

Layer elasticity has been studied in Ref. [8], but only for $\chi = 0$ and a special solution without displacement U_y along the layer direction $\hat{y}$ perpendicular to gravity $\hat{z}$. The present work will abandon these limitations and give general solutions with $\chi > 0$. The existence of the last term of Eq. (1) for granular materials is supported by a number of experiments, such as measurements of yield surface, sound velocities, and stress-strain responding coefficients.[9] It is then worth being included in the analysis. The instability of the special solution considered in Ref. [8] is independent of layer depth z, implying that the whole layer simultaneously becomes unstable at $\theta = \theta_{\max}$. However, for nonvanishing U_y, the instability will depend on z, and is first started at the free surface of layer top $z = 0$, implying a surface flow upon the remaining static layer. The surface instability is supported by the triggering experiment reported in Ref. [14]. The surface flow is also frequently met, for example, on the surface of sand piles.[15,16] Our results show that potential (1) is capable of accounting for this.

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2. Elastic solutions

Consider a static granular layer and the coordinate system shown by Fig. 1. In this section we will find its general elastic solutions under the assumption that stress σ_{ij} and strain u_{ij} vary only with layer depth z. Note that potential (1) is not a quadratic function of strain; this will be a nonlinear elastic problem. For simplicity we neglect mass inhomogeneity (taking the density $\rho = \text{const.}$). In the situation the force balance becomes $\sigma'_{ij} = \rho g_i$ (with the notation $' \equiv \mathrm{d}/\mathrm{d}z$, and g_i is the vector of gravitational acceleration), namely:

$$\sigma_{zz} = z\cos\theta, \tag{2}$$

$$\sigma_{xz} = z\sin\theta, \tag{3}$$

$$\sigma_{yz} = 0. \tag{4}$$

In the above equations and throughout the paper, we will use B as the unit for stress, and $B/\rho g_i$ as the unit for z. This may render all equations simple and dimensionless. Here, the boundary condition is taken as follows: the layer top $z = 0$ is a free surface, on which all stress and strain components are vanishing

$$\sigma_{ij}(z=0) = 0, \tag{5}$$

$$u_{ij}(z=0) = 0. \tag{6}$$

Moreover, if the nonlinear term in expression $2u_{ij} = \nabla_i U_j + \nabla_j U_i - \nabla_i U_k \cdot \nabla_j U_k$ between the strain and displacement vector U_i is neglected, then the strain verifies the so-called compatibility condition: $\varepsilon_{ikl}\varepsilon_{jmn}\nabla_k\nabla_m u_{nl} = 0$, where ε_{ijk} is the third order antisymmetric unit tensor. For the layer considered, it is reduced to $\varepsilon_{izl}\varepsilon_{jzn}u''_{nl} = 0$, namely $u''_{xx} = u''_{yy} = u''_{xy} = 0$. Integrating them under the boundary condition (6), we obtain the following three strain components:

$$u_{xx} = C_1 z, \tag{7}$$

$$u_{yy} = C_2 z, \tag{8}$$

$$u_{xy} = C_3 z, \tag{9}$$

with C_1, C_2, and C_3 being integral constants.

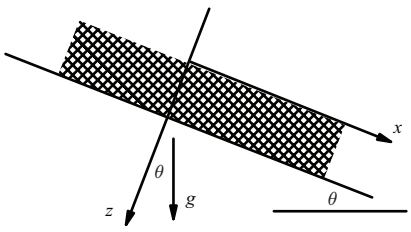


Fig. 1. The granular layer on the inclined plane. The coordinate y is perpendicular to the figure.

Our task is then to clearly determine the other three stress and strain components: σ_{xx}, σ_{yy}, σ_{xy}, and u_{zz}, u_{xz}, u_{yz}. For this end we need the stress–strain relations given by derivating Eq. (1) with strain: $\sigma_{ij} = -\partial w/\partial u_{ij}$. Because they are complicated nonlinear expressions, we will list them in Eqs. (A2)–(A7) of Appendix A. The strain components u_{zz}, u_{xz}, u_{yz} can be calculated as follows. From Eqs. (3), (4), (A3), (A4), and Eqs. (7)–(9), we have

$$3\chi(C_3 u_{yz} - C_2 u_{xz})z - (\chi+2)\Delta u_{xz} = \xi z\sqrt{\Delta}\sin\theta, \tag{10}$$

$$(\chi+2)\Delta u_{yz} - 3\chi(C_3 z u_{xz} - C_1 z u_{yz}) = 0, \tag{11}$$

namely

$$u_{xz} = \frac{-\xi z\sqrt{\Delta}\left[(\chi+2)\Delta + 3C_1\chi z\right]\sin\theta}{\left[(\chi+2)\Delta + 3C_1\chi z\right]\left[(\chi+2)\Delta + 3C_2\chi z\right] - (3C_3\chi z)^2}, \tag{12}$$

$$u_{yz} = \frac{-3C_3\xi\chi z^2\sqrt{\Delta}\sin\theta}{\left[(\chi+2)\Delta + 3C_1\chi z\right]\left[(\chi+2)\Delta + 3C_2\chi z\right] - (3C_3\chi z)^2}. \tag{13}$$

Using Eqs. (7) and (8), the component u_{zz} is

$$u_{zz} = -\Delta - C_1 z - C_2 z. \tag{14}$$

Thus the dependences of u_{zz}, u_{xz}, u_{yz} on z can be known if $\Delta = \Delta(z)$ is known. Inserting Eqs. (2), (7)–(9) into Eq. (A7), we get

$$\begin{aligned} z\cos\theta \;=\; & \frac{9\xi+5\chi+15}{9\xi}\Delta^{3/2} + \frac{3(\chi+2)(C_1+C_2)}{2\xi}\sqrt{\Delta}z \\ & + \frac{\chi z^2}{2\xi\sqrt{\Delta}}\left(C_1^2 + 4C_1C_2 + C_2^2 - 2C_3^2 + \frac{u_{xz}^2+u_{yz}^2}{z^2}\right) \\ & + \frac{z^2}{\sqrt{\Delta}\xi}\left(C_1^2 + C_1C_2 + C_2^2 + C_3^2 + \frac{u_{xz}^2+u_{yz}^2}{z^2}\right) \\ & + \frac{3\chi z^3}{2\Delta^{3/2}\xi}\left[C_1\left(C_3^2 - C_2^2 - \frac{u_{yz}^2}{z^2}\right)\right. \\ & \left. + C_2\left(C_3^2 - C_1^2 - \frac{u_{xz}^2}{z^2}\right) + 2C_3\frac{u_{xz}u_{yz}}{z^2}\right]. \end{aligned} \tag{15}$$

The dependence of Δ on z is given by further inserting Eqs. (12) and (13) into it, and then solving the Δ. As the roots of the nonlinear equation cannot be given analytically, the dependence $\Delta(z)$ will remain implicitly, as given by the complicated Eq. (15).

The stresses σ_{xx}, σ_{yy}, σ_{xy} can be obtained by inserting the above strains into Eqs. (A2), (A5), and (A6). We then have the off-diagonal component

$$\sigma_{xy} = \frac{3\chi\left[u_{xz}u_{yz} + C_3(\Delta + C_1 z + C_2 z)z\right] - C_3(\chi+2)\Delta z}{\sqrt{\Delta}\xi}, \tag{16}$$

and the diagonal components $\sigma_{xx} = z\cos\theta - q$, $\sigma_{yy} = z\cos\theta - q + \widetilde{q}$, with

$$\begin{aligned} q \;=\; & \frac{1-\chi}{\xi/2}\sqrt{\Delta}(\Delta + 2C_1 z + C_2 z) \\ & + \frac{3\chi}{\xi}\frac{u_{yz}^2 - (C_3^2 + C_1^2)z^2 + (\Delta + C_1 z + C_2 z)^2}{\sqrt{\Delta}}, \end{aligned} \tag{17}$$

$$\widetilde{q} = \frac{1-\chi}{\xi/2}\sqrt{\Delta}\,(C_1 - C_2)\,z + \frac{3\chi}{\xi}\frac{u_{yz}^2 - u_{xz}^2 + \left(C_2^2 - C_1^2\right)z^2}{\sqrt{\Delta}}. \quad (18)$$

Here we used u_{xz}, u_{yz} given by Eqs. (12) and (13). Again $\Delta(z)$ is given by Eq. (15).

Now we completely solved the elastic problem of a granular layer analytically and generally, which contains three integral constants C_1, C_2, and C_3. The constants may be determined by boundary conditions at $x, y \to \infty$, which are, however, lacking to shorten the paper. It will be interesting to experimentally measure which values of C_1, C_2, and C_3 a real layer prefers to have. The solutions contain one implicit function, $\Delta = \Delta(z)$.

If $\chi = 0$ and $C_1 = C_2 = C_3 = 0$, we can obtain $\Delta(z)$ explicitly and refind the special solution reported in Ref. [8]. For this situation the equations (12) and (13) are reduced to $u_{xz} = -\xi z \sin\theta / \left(2\sqrt{\Delta}\right)$ and u_{yz}=0. Then equation (15) becomes

$$\frac{3\xi+5}{3\xi\cos\theta}\frac{\Delta^{3/2}}{z} + \frac{\xi\sin^2\theta}{4\cos\theta}\frac{z}{\Delta^{3/2}} = 1, \quad (19)$$

which gives

$$\Delta = \left(\frac{\xi z}{2}\right)^{2/3} \times \left[\frac{3\cos\theta}{3\xi+5} \pm \frac{\sqrt{6\left[(3\xi+8)\cos 2\theta - 2 - 3\xi\right]}}{6\xi+10}\right]^{2/3}. \quad (20)$$

The strain of the special solution in Ref. [8] is

$$u_{ij} = \frac{1}{2\sqrt{\Delta}}\begin{pmatrix} 0 & 0 & -\xi z\sin\theta \\ 0 & 0 & 0 \\ -\xi z\sin\theta & 0 & -2\Delta^{3/2} \end{pmatrix}, \quad (21)$$

which leads to the following two invariant ratios

$$\frac{u_s}{\Delta} = \sqrt{\frac{2}{3} + \frac{1}{2}\xi^2\frac{z^2}{\Delta^3}\sin^2\theta} = \sqrt{\frac{2}{3} + 2\sin^2\theta\left[\frac{3\cos\theta}{3\xi+5} \pm \frac{\sqrt{6\left[(3\xi+8)\cos 2\theta - 2 - 3\xi\right]}}{6\xi+10}\right]^{-2}}, \quad (22)$$

$$\frac{u_t}{\Delta} = -\sqrt[3]{\frac{2}{9} + \frac{1}{4}\xi^2\frac{z^2}{\Delta^3}\sin^2\theta} = -\sqrt[3]{\frac{2}{9} + \sin^2\theta\left[\frac{3\cos\theta}{3\xi+5} \pm \frac{\sqrt{6\left[(3\xi+8)\cos 2\theta - 2 - 3\xi\right]}}{6\xi+10}\right]^{-2}}. \quad (23)$$

The stress tensor for the special case is

$$\sigma_{ij} = z\cos\theta\begin{pmatrix} 1 & 0 & \tan\theta \\ 0 & 1 & 0 \\ \tan\theta & 0 & 1 \end{pmatrix} - \frac{2\Delta^{3/2}}{\xi}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (24)$$

Similar to the case where (Δ, u_s, u_t), we may introduce a triplet of stress invariants (P, σ_s, σ_t), as defined by $P \equiv \sigma_{kk}/3$, $\sigma_s \equiv \sqrt{\sigma_{ij}^*\sigma_{ij}^*}$, $\sigma_t \equiv \sqrt[3]{\sigma_{ij}^*\sigma_{jk}^*\sigma_{ki}^*}$ with deviatoric stress $\sigma_{ij}^* \equiv \sigma_{ij} - \sigma_{kk}\delta_{ij}/3$. From Eq. (24) we have the invariant ratios

$$\frac{\sigma_s}{P} = \frac{\sqrt{18\sin^2\theta + (24/\xi^2)(\Delta^3/z^2)}}{3\cos\theta - (4/\xi)(\Delta^{3/2}/z)} = \left[\frac{8}{3(3\xi+1)^2} + \frac{2}{9}\frac{(3\xi+5)^2}{(3\xi+1)^2}\tan^2\theta\right]^{1/2}, \quad (25)$$

$$\frac{\sigma_t}{P} = 6^{1/3}\left(\frac{\sqrt{\Delta}}{z^{1/3}}\right)\frac{\sqrt[3]{8\Delta^3/z^2 + 9\xi^2\sin^2\theta}}{3\xi\cos\theta - 4\Delta^{3/2}/z} = \left[\frac{16}{9(3\xi+1)^3} + \frac{2}{9}\frac{(3\xi+5)^2}{(3\xi+1)^3}\tan^2\theta\right]^{1/3}. \quad (26)$$

These can also be obtained using Eqs. (A29)–(A30) of Appendix A, which transforms the strain invariants (Δ, u_s, u_t) into the stress ones (P, σ_s, σ_t).

A remarkable property of the special solution is that both the strain and stress invariant ratios are all independent of the layer depth z. Because, as will be seen in the next section, the stability is determined fully by these ratios, this property indicates that the onset to instability is z-independent too, namely as θ increases the whole layer simultaneously becomes unstable everywhere. This situation is inappropriate for understanding surface avalanches or flows. However, if C_1, C_2, and C_3 are not equal to zero, then the invariant ratios are z-dependent, implying that local instability is allowed.

When $\chi = 0$ equation (4) implies $u_{ij} = 0$. In this situation, using $u_{ij} = (\nabla_i U_j + \nabla_j U_i)/2$ and the strains (7)–(9), we can obtain the displacement vector field $U_x = C_1 zx + C_3 zy + X$, $U_y = C_2 zy + C_3 zx$, and $U_z = Z - C_2 y^2/2 - C_3 xy - C_1 x^2/2$, with X and Z being two arbitrary functions of z.

3. Yield loci in strain and stress spaces

For stability analysis we need to determine the convex and concave regimes of the potential (1). Because the potential is isotropic, the regimes can be represented in a three-dimensional (3D) space spanned by the strain invariants (Δ, u_s, u_t) or stress invariants (P, σ_s, σ_t). Moreover, the fact that the potential is a homogeneous function of strain will render the first invariant Δ or P to be an irrelevant proportional factor, and the representation can be further reduced in a two-dimensional (2D) space spanned by the strain ratios

$(u_s/\Delta, u_t/\Delta)$ or stress ratios $(\sigma_s/P, \sigma_t/P)$. The boundary connecting the two regimes is usually a curve in the 2D space, called a yield loci or simply a yield, and will be calculated in this section.

For a general potential function $w(\Delta, u_s, u_t)$, if we denote its six independent strain variables as

$$A_1 = u_{xx}, \quad A_2 = u_{yy}, \quad A_3 = u_{zz},$$
$$A_4 = u_{xy}, \quad A_5 = u_{xz}, \quad A_6 = u_{yz}, \tag{27}$$

it is convex if and only if all eigenvalues of the Hessian matrix

$$H_{\alpha\beta} = -\frac{\partial^2 w}{\partial A_\alpha \partial A_\beta} \tag{28}$$

are positive. Denoting the minimum eigenvalue by $\lambda = \lambda(A_\alpha) = \lambda(\Delta, u_s, u_t)$, the stable or unstable regime in the strain space is then given by those A_α such that $\lambda > 0$ or $\lambda < 0$. The yield is determined by the equation $\lambda(\Delta, u_s, u_t) = 0$, which for the case of a homogeneous potential is further reduced to

$$\lambda(u_s/\Delta, u_t/\Delta) = 0. \tag{29}$$

Converting the loci of Eq. (29) into the stress space with the transformations (A29)–(A30), the yield in the stress space can be easily obtained. Notably because the transformation is nonlinear, the stable and unstable regimes, which are disjointed in the strain space, may however be overlapped in the stress space. Moreover from definitions of the invariants, it can be shown that (see Appendix A)

$$\left|\frac{u_t}{u_s}\right| \le 6^{-1/6}, \quad \left|\frac{\sigma_t}{\sigma_s}\right| \le 6^{-1/6}. \tag{30}$$

The property together with u_s or $\sigma_s \ge 0$ implies that the invariant strain or stress space is in general limited by a triangle of infinite height in the upper semispace, as shown by the dashed straight lines in Fig. 2.

If $\chi = 0$ equation (29) is reduced to $u_s/\Delta = \sqrt{2\xi}$, or in the stress space $\sigma_s/P = \sqrt{2/\xi}$, this is in agreement with the results of Refs. [7] and [8]. The yield of the situation is independent of the third invariant, as shown by the horizontal straight lines in Fig. 2. This cuts the strain space into a lower part and an upper part, corresponding to the stable and unstable regimes, respectively. However, if the two strain regimes are transformed into the stress space, they both go into the lower triangle part, as shown in Fig. 2(b). So the stable and unstable regions are fully overlapped in the stress space. This nonlinear behavior indicates that it is more convenient for the stability to be represented in the strain space than in the stress one because a point in the latter refers to two states (one stable and one unstable).

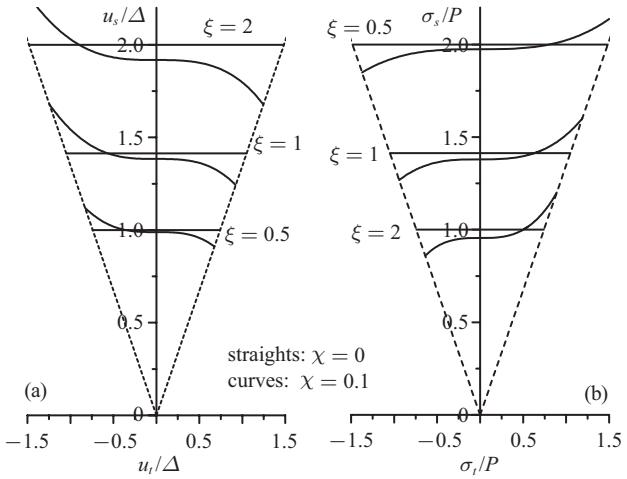


Fig. 2. Yields in the invariant spaces of strain (a) and stress (b). The dashed straight lines are the boundary of the spaces defined by Eq. (30). The horizontal straight lines are for the $\chi = 0$, curves for $\chi = 0.1$.

For $\chi \ne 0$, the yield will vary with the third invariant, displaying as a curve of which the expression cannot be analytically given. In Fig. 2 they are computed numerically for $\chi = 0.1$. Note that the curves are unsymmetric about the u_t or σ_t axis. The behavior is well verified by experiments.[9] In the stress space the stable and unstable regimes are fully overlapped too.

Finally, we note that in the strain space the yield moves up as ξ increases, while in the stress space it moves down. In soil mechanics the yield curve is often also represented in the space spanned by stress eigenvalues instead of invariants. This can be readily obtained by applying the transformations (A11)–(A13) and (A18)–(A20) to Fig. 2. The results are reported in Ref. [9], and agree with the measurements fairly well.

4. Layer stability

The onset from the stable to unstable states of the layer solutions discussed in Section 2 can be visualized by plotting its strain loci at varying θ in Fig. 2(a). For $\chi = 0$ and the special solution with $C_1 = C_2 = C_3 = 0$, the layer loci given by Eqs. (22) and (23) are shown in Fig. 3, where the dashed curve is the solution with the negative sign in Eqs. (22) and (23), which are all unstable. The positive one shown by the full curve intersects with the yield at $\theta = \theta_{\max} \sim 26.2^\circ$, the maximum angle that a layer with the solution can have. These strain loci are independent of z, implying that the yield starts in the whole layer.

When C_1, C_2, and C_3 are not equal to zero, equation (15) also has two roots for Δ, which can be numerically computed. At a given depth z, the corresponding strain loci formed by varying θ have qualitatively similar behaviors to the special case as shown in Fig. 3, with one root being unstable (dashed), and one cutting the yield.

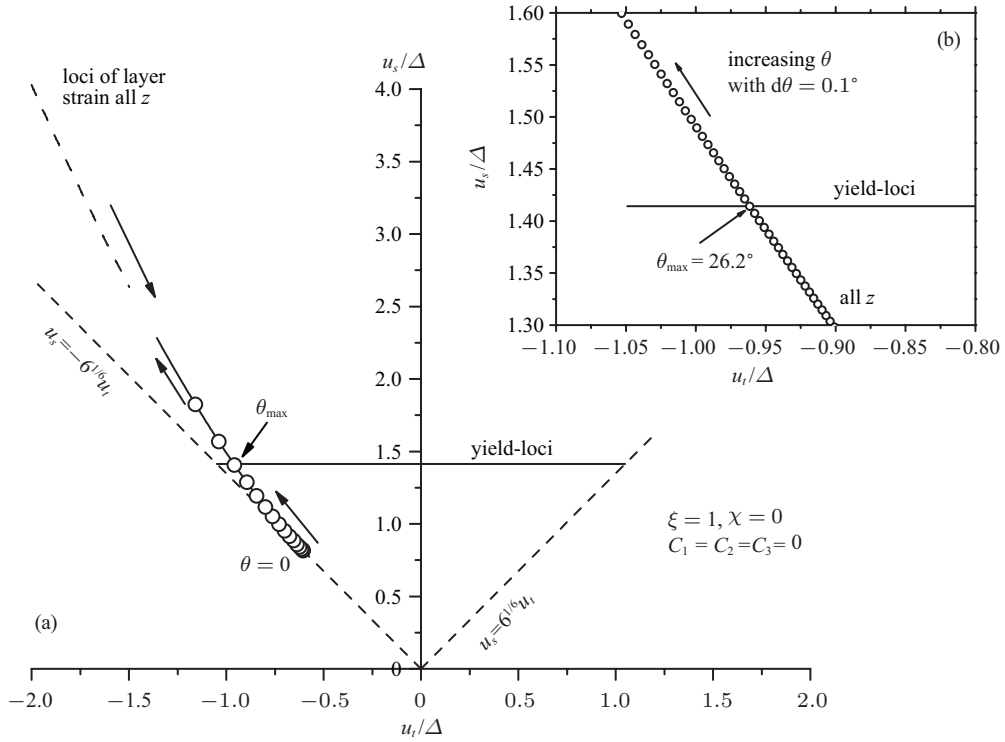


Fig. 3. (a) Loci of elastic strain as θ varies, given by Eqs. (22) and (23). Its intersection with the yield gives the maximum angle θ_{max}. The dashed part is the negative sign of the square root in Eqs. (22) and (23), which is unstable. Panel (b) is an amplification around θ_{max}.

However, the loci will be different for various z, and consequently we have $\theta_{max} = \theta_{max}(z)$, as shown in Fig. 4 for $C_1 = C_2 = -0.1$ and $C_3 = 0$. Note that θ_{max} increases with z (see Fig. 5(a)), implying that the solution starts its instability at the free surface $z = 0$, where θ_{max} has its minimum value, which is always equal to the yield angle of the case with $C_1 = C_2 = C_3 = 0$. In other words, nonvanishing integral constants may stabilize the layer interior, but cannot alter the θ_{max} at the free surface. Such solutions with nonvanishing C_1, C_2, and C_3 could be used to account for the layers that have surface avalanches or flows.

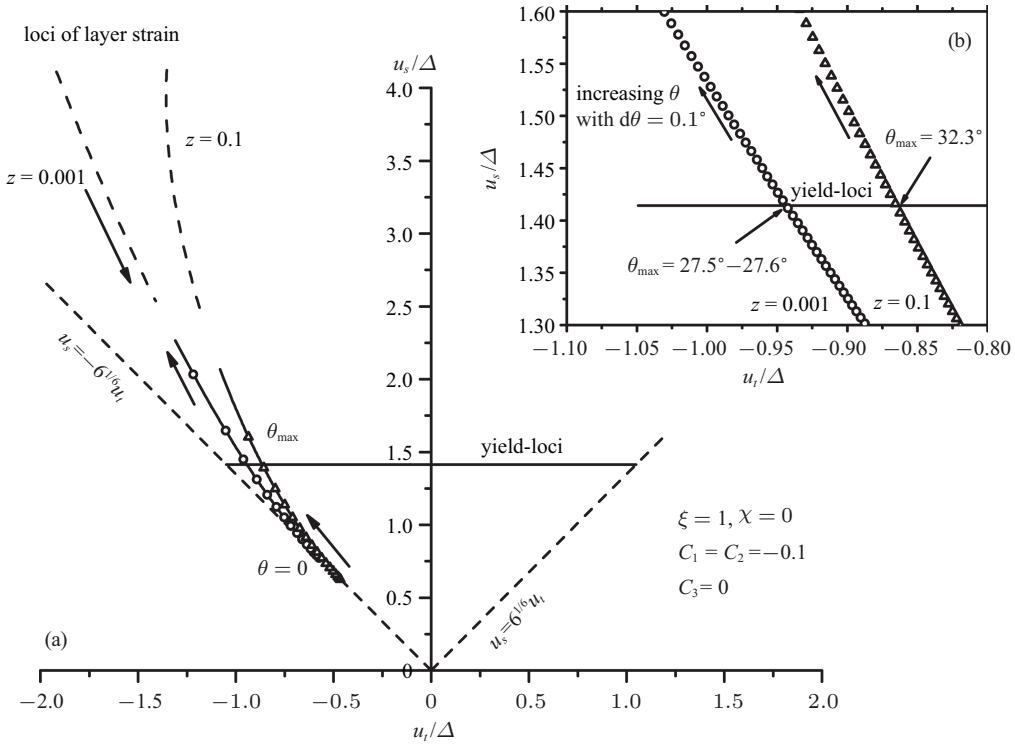


Fig. 4. Same as Fig. 3, except for $C_1 = C_2 = -0.1$. Note that the strain loci are different for $z = 0.001$ and 0.1.

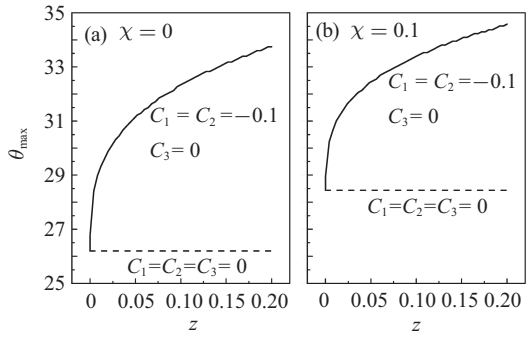


Fig. 5. Variation of θ_{max} with layer depth z for $\chi = 0$ (a) and 0.1 (b).

For the case with $\chi \neq 0$, the behaviors are qualitatively the same as those for $\chi = 0$ as discussed above, except that the yield loci become a curve. Figure 6 shows the layer loci computed for $\chi = 0.1$, of which $\theta_{max}(z)$ is plotted in Fig. 5(b). Note that the surface instability occurs only when C_1, C_2, and C_3 are not equal to zero, irrespective of the value of χ. The above stability analysis can also be transformed into the stress space using the invariant stress–strain relation given by Eqs. (A29)–(A30). As an example we show in Fig. 7 the result of the transformation of Fig. 6. It is interesting that in the stress space the layer loci do not intersect but turn at the yield curve. As the stresses are measurable quantities, this highly nonlinear phenomenon may be considered as a precursor to mechanical instability. Notably the turning occurs in the space spanned by the two invariants σ_s and σ_t, both of which describe the strength of the shearing forces. This probably indicates that details of the shearing stresses are required to foresee an avalanche.

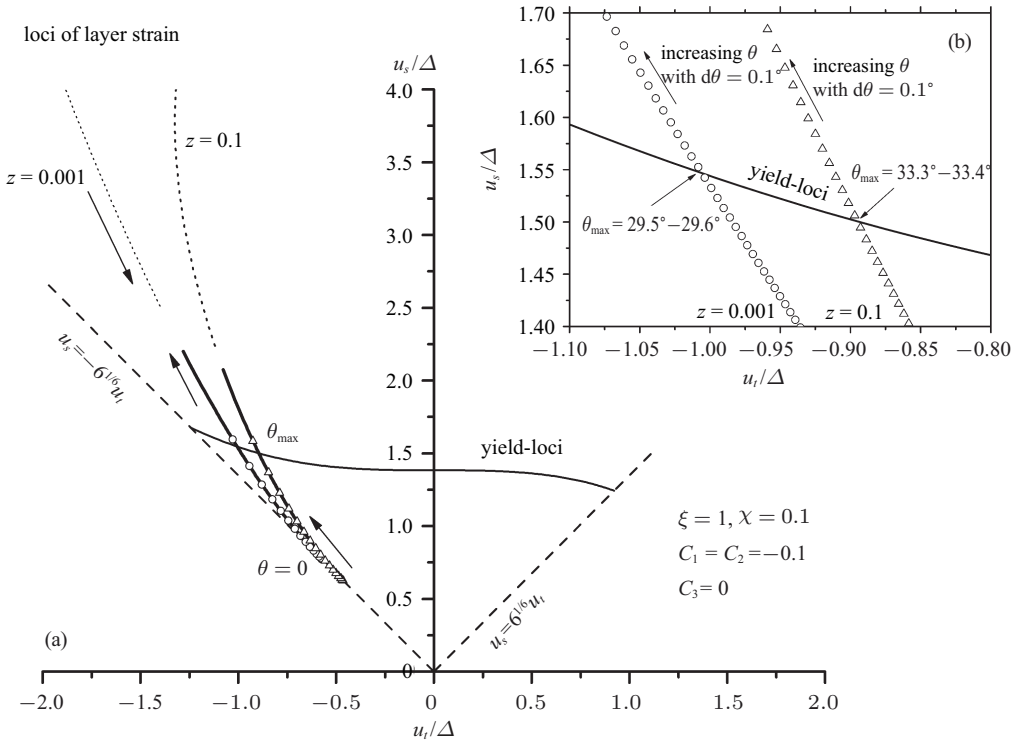


Fig. 6. Same as Fig. 4, but for $\chi = 0.1$.

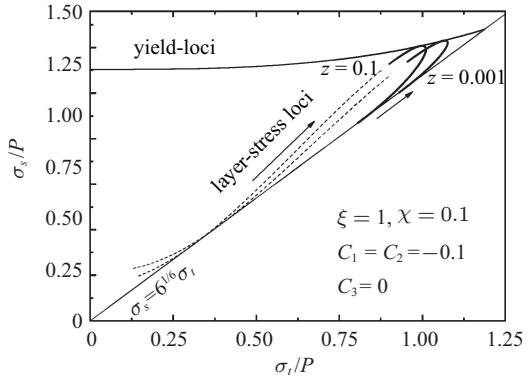


Fig. 7. The yield and layer loci of Fig. 6 in the stress space.

5. Summary and discussion

As a nonlinear elastic body with a limited stable regime in the stress or strain space, the onset to mechanical instability or the avalanche of static granular media can be analyzed by solving their elastic equations given by an appropriate potential model, such as Eq. (1). Using a general basis, the stress distribution and maximum angle of inclination for an inclined layer in gravity is derived in this paper. Because the boundary conditions of the elastic problem are incomplete, the solutions have indeterminate integral constants, and depending on which the instability may occur either globally in the whole

layer or locally at the free surface of the top of the layer. For simplicity we assumed that stress and strain are uniform along the layer plane, which is also considered to be infinite. In this situation the elastic problem can be solved analytically. However, if uniformity is lacking, or the geometric shapes of the media are complicated, numerical computations are needed to solve the nonlinear elastic problem, such as in the recent report[17] for a cone-shape granular pile.

The inclined layer is widely used in experimental studies of surface granular flows, such as complicated flowing patterns and avalanche dynamics,[15,16] etc. The stress distributions in this work can be used in future for further hydrodynamic computations of these dynamics. Recently, special solutions with vanishing C_1, C_2, and C_3 were used in Refs. [18]–[21] for analyzing the properties of surface sound waves and the booming instabilities occurring in natural sand dunes. Our stress distributions can also be used to extend this analysis to cases with nonvanishing C_1, C_2, and C_3.

Appendix A: Stress–strain relation of granular solids

We summarize in the appendix some useful nonlinear relations between stress and strain, given by potential (1).

A1. Stress–strain relation

For an isotropic elastic body, the elastic potential is in general a function of the strain invariants: $w = w(\Delta, u_s, u_t)$. The stress–strain relations $\sigma_{ij} = -\partial w/\partial u_{ij}$ are then

$$\sigma_{ij} = \frac{\partial w}{\partial \Delta}\delta_{ij} - \frac{\partial w}{\partial u_s}\frac{u^*_{ij}}{u_s} - \frac{\partial w}{\partial u_t}\left(\frac{u^*_{ik}u^*_{kj}}{u_t^2} - \frac{u_s^2}{3u_t^2}\delta_{ij}\right). \quad \text{(A1)}$$

Inserting the granular potential (1) into it, we have the off-diagonal stresses

$$\sigma_{xy} = \frac{3\chi\left(u_{xz}u_{yz} - u_{xy}u_{zz}\right) - (\chi+2)\Delta u_{xy}}{\sqrt{\Delta}\xi}, \quad \text{(A2)}$$

$$\sigma_{xz} = \frac{3\chi\left(u_{xy}u_{yz} - u_{xz}u_{yy}\right) - (\chi+2)\Delta u_{xz}}{\sqrt{\Delta}\xi}, \quad \text{(A3)}$$

$$\sigma_{yz} = \frac{3\chi\left(u_{xy}u_{xz} - u_{yz}u_{xx}\right) - (\chi+2)\Delta u_{yz}}{\sqrt{\Delta}\xi}, \quad \text{(A4)}$$

and the stress differences $q \equiv \sigma_{zz} - \sigma_{xx}$ and $\widetilde{q} \equiv \sigma_{yy} - \sigma_{xx}$

$$q = \frac{1-\chi}{\xi/2}\sqrt{\Delta}\left(u_{xx} - u_{zz}\right) + \frac{3\chi}{\xi}\frac{u_{yz}^2 - u_{xy}^2 - u_{xx}^2 + u_{zz}^2}{\sqrt{\Delta}}, \quad \text{(A5)}$$

$$\widetilde{q} = \frac{1-\chi}{\xi/2}\sqrt{\Delta}\left(u_{xx} - u_{yy}\right) + \frac{3\chi}{\xi}\frac{u_{yz}^2 - u_{xz}^2 - u_{xx}^2 + u_{yy}^2}{\sqrt{\Delta}}. \quad \text{(A6)}$$

The stress component σ_{zz} has a more complicated expression:

$$\begin{aligned}\sigma_{zz} = {} & \frac{9\xi + 5\chi + 15}{9\xi}\Delta^{3/2} + \frac{3(\chi+2)}{2\xi}\sqrt{\Delta}\left(u_{xx} + u_{yy}\right) \\ & + \frac{\chi}{2\xi\sqrt{\Delta}}\left(u_{xx}^2 + 4u_{xx}u_{yy} + u_{yy}^2 - 2u_{xy}^2 + u_{xz}^2 + u_{yz}^2\right) \\ & + \frac{1}{\sqrt{\Delta}\xi}\left(u_{xx}^2 + u_{xx}u_{yy} + u_{yy}^2 + u_{xy}^2 + u_{xz}^2 + u_{yz}^2\right) \\ & + \frac{3\chi}{2\Delta^{3/2}\xi}\left[\left(u_{xy}^2 - u_{xx}u_{yy}\right)\left(u_{yy} + u_{xx}\right) - u_{xz}^2 u_{yy}\right. \\ & \left. - u_{yz}^2 u_{xx} + 2u_{xy}u_{xz}u_{yz}\right]. \end{aligned} \quad \text{(A7)}$$

A2. Transformation between eigenvalues and invariants

The strain invariants (Δ, u_s, u_t) are related to its eigenvalues (u_1, u_2, u_3) by

$$\Delta = -u_1 - u_2 - u_3, \quad \text{(A8)}$$

$$u_s = \sqrt{\frac{2}{3}\left(u_1^2 + u_2^2 + u_3^2 - u_1u_2 - u_1u_3 - u_2u_3\right)}, \quad \text{(A9)}$$

$$u_t = \sqrt[3]{\frac{1}{9}\left(u_1 + u_2 - 2u_3\right)\left(2u_1 - u_2 - u_3\right)\left(u_1 - 2u_2 + u_3\right)}. \quad \text{(A10)}$$

Their inverse is

$$u_1 = -\frac{\Delta}{3} + \frac{2u_s}{\sqrt{6}}\sin\left(\frac{1}{3}\arcsin\frac{\sqrt{6}u_t^3}{u_s^3} - \frac{\pi}{3}\right), \quad \text{(A11)}$$

$$u_2 = -\frac{\Delta}{3} - \frac{2u_s}{\sqrt{6}}\sin\left(\frac{1}{3}\arcsin\frac{\sqrt{6}u_t^3}{u_s^3}\right), \quad \text{(A12)}$$

$$u_3 = -\frac{\Delta}{3} + \frac{2u_s}{\sqrt{6}}\sin\left(\frac{1}{3}\arcsin\frac{\sqrt{6}u_t^3}{u_s^3} + \frac{\pi}{3}\right). \quad \text{(A13)}$$

When the arcsine takes values in $[-\pi/2, \pi/2]$, the eigenvalues are ordered as $u_1 \le u_2 \le u_3$, and other values result in permutation of the eigenvalues. Moreover, from Eqs. (A9) and (A10) we have

$$u_s^6 - 6u_t^6 = 2\left(u_2 - u_3\right)^2\left(u_1 - u_3\right)^2\left(u_1 - u_2\right)^2 \ge 0. \quad \text{(A14)}$$

This gives $\sqrt{6}\left|u_t^3/u_s^3\right| \le 1$, so the arcsine in Eqs. (A11)–(A13) is always meaningful.

Similar transformations exist for stress invariants, namely

$$P = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}, \quad \text{(A15)}$$

$$\sigma_s = \sqrt{\frac{2}{3}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_1\sigma_3 - \sigma_2\sigma_3\right)}, \quad \text{(A16)}$$

$$\sigma_s = \sqrt[3]{\frac{1}{9}\left(\sigma_1 + \sigma_2 - 2\sigma_3\right)\left(2\sigma_1 - \sigma_2 - \sigma_3\right)\left(\sigma_1 - 2\sigma_2 + \sigma_3\right)}, \quad \text{(A17)}$$

and

$$\sigma_1 = P + \frac{2\sigma_s}{\sqrt{6}}\sin\left(\frac{1}{3}\arcsin\frac{\sqrt{6}\sigma_t^3}{\sigma_s^3} - \frac{\pi}{3}\right), \quad \text{(A18)}$$

$$\sigma_2 = P - \frac{2\sigma_s}{\sqrt{6}}\sin\left(\frac{1}{3}\arcsin\frac{\sqrt{6}\sigma_t^3}{\sigma_s^3}\right), \quad \text{(A19)}$$

$$\sigma_3 = P + \frac{2\sigma_s}{\sqrt{6}}\sin\left(\frac{1}{3}\arcsin\frac{\sqrt{6}\sigma_t^3}{\sigma_s^3} + \frac{\pi}{3}\right). \quad \text{(A20)}$$

Note that P differs from Δ by a factor $-1/3$.

A3. Invariant stress–strain relation

From the stress–strain relation (A1) we have the deviatoric stress

$$\sigma_{ij}^{*}=-\frac{\partial w}{\partial u_s}\frac{u_{ij}^{*}}{u_s}-\frac{u_s^2}{u_t^2}\frac{\partial w}{\partial u_t}\left(\frac{u_{ik}^{*}u_{kj}^{*}}{u_s^2}-\frac{\delta_{ij}}{3}\right). \tag{A21}$$

Its square and cubic power are

$$\begin{aligned}\sigma_s^2=&\left(\frac{\partial w}{\partial u_s}\right)^2-\frac{u_s^4}{3u_t^4}\left(\frac{\partial w}{\partial u_t}\right)^2\\&+2\frac{\partial w}{\partial u_s}\frac{\partial w}{\partial u_t}\frac{(u_{ij}^{*})^3}{u_s u_t^2}+\left(\frac{\partial w}{\partial u_t}\right)^2\frac{(u_{ij}^{*})^4}{u_t^4},\end{aligned} \tag{A22}$$

and

$$\begin{aligned}-\sigma_t^3=&\left(\frac{\partial w}{\partial u_s}\right)^3\frac{u_t^3}{u_s^3}+\frac{u_s^2}{u_t^2}\left(\frac{\partial w}{\partial u_s}\right)^2\frac{\partial w}{\partial u_t}\left(3\frac{(u_{ij}^{*})^4}{u_s^4}-1\right)\\&+\frac{u_s^4}{u_t^4}\frac{\partial w}{\partial u_s}\left(\frac{\partial w}{\partial u_t}\right)^2\left(3\frac{(u_{ij}^{*})^5}{u_s^5}-2\frac{u_t^3}{u_s^3}\right)\\&+\frac{u_s^6}{u_t^6}\left(\frac{\partial w}{\partial u_t}\right)^3\left(\frac{(u_{ij}^{*})^6}{u_s^6}-\frac{1}{2}+\frac{2}{9}\right).\end{aligned} \tag{A23}$$

Here, $\left(u_{ij}^{*}\right)^n$ is the trace of the deviatoric strain to power n. Using Eqs. (A11)–(A13) we have

$$\begin{aligned}\left(u_{ij}^{*}\right)^n=&\left(\frac{2u_s}{\sqrt{6}}\right)^n\left[\sin^n\left(L-\frac{\pi}{3}\right)\right.\\&\left.+\sin^n(-L)+\sin^n\left(L+\frac{\pi}{3}\right)\right]\end{aligned} \tag{A24}$$

with the notation $L\equiv(1/3)\arcsin\left(\sqrt{6}u_t^3/u_s^3\right)$. So from Eq. (A1) we have the pressure

$$P=\frac{\partial w}{\partial \Delta}. \tag{A25}$$

Inserting Eq. (A24) into Eqs. (22) and (23) we can get the following second and third stress invariants:

$$\sigma_s^2=\left(\frac{\partial w}{\partial u_s}\right)^2+\frac{u_s^4}{6u_t^4}\left(\frac{\partial w}{\partial u_t}\right)^2+2\frac{u_t}{u_s}\frac{\partial w}{\partial u_s}\frac{\partial w}{\partial u_t}, \tag{A26}$$

$$\begin{aligned}-\sigma_t^3=&\left(\frac{\partial w}{\partial u_s}\right)^3\frac{u_t^3}{u_s^3}+\frac{1}{2}\frac{u_s^2}{u_t^2}\left(\frac{\partial w}{\partial u_s}\right)^2\frac{\partial w}{\partial u_t}\\&+\frac{1}{2}\frac{u_s}{u_t}\frac{\partial w}{\partial u_s}\left(\frac{\partial w}{\partial u_t}\right)^2+\frac{1}{18}\left(\frac{\partial w}{\partial u_t}\right)^3\left(6-\frac{1}{2}\frac{u_s^6}{u_t^6}\right).\end{aligned} \tag{A27}$$

For the potential Eq. (1), the stress invariants become

$$P=\Delta^{3/2}+\frac{1}{2\xi\sqrt{\Delta}}\left(u_s^2+\chi\frac{u_t^3}{\Delta}\right), \tag{A28}$$

$$\sigma_s=\frac{1}{\xi}\sqrt{4\Delta u_s^2+\frac{3\chi^2}{2}\frac{u_s^4}{\Delta}-12\chi u_t^3}, \tag{A29}$$

$$\sigma_t=\frac{\Delta^{1/6}}{\xi}\sqrt[3]{6\chi u_s^4-8\Delta u_t^3-9\chi^2\frac{u_s^2u_t^3}{\Delta}+\frac{3\chi^3}{\Delta^2}\left(3u_t^6-\frac{u_s^6}{4}\right)}. \tag{A30}$$

Especially for $\chi=0$ they are reduced to $P=\Delta^{3/2}+u_s^2/\left(2\xi\sqrt{\Delta}\right)$, $\sigma_s=2\sqrt{\Delta}u_s/\xi$, and $\sigma_t=-2\sqrt{\Delta}u_t/\xi$.

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