Lie symmetries and exact solutions for a short-wave model*

Chen Ai-Yong(陈爱永)[a], Zhang Li-Na(章丽娜)[b]†, and Wen Shuang-Quan(温双全)[a]

[a] School of Mathematics and Computing Science, Guilin University of Electronic Technology, Guilin 541004, China
[b] School of Science, Huzhou University, Huzhou 313000, China

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In this paper, the Lie symmetry analysis and generalized symmetry method are performed for a short-wave model. The symmetries for this equation are given, and the phase portraits of the traveling wave systems are analyzed using the bifurcation theory of dynamical systems. The exact parametric representations of four types of traveling wave solutions are obtained.

Keywords: Lie symmetry, short-wave model, bifurcation method, loop solution

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1. Introduction

It is well known that the study of nonlinear wave equations and their solutions are of great importance in many areas of physics.[1–8] The traveling wave solution is an important type of solution for the nonlinear partial differential equation, and many nonlinear partial differential equations (PDEs) have been found to have a variety of traveling wave solutions.[9–16] The short pulse (SP) model equation attracted considerable attention in a recent development of the theory of nonlinear waves. Originally, it was found in an attempt to construct integrable differential equations associated with pseudospherical surfaces. In the context of physical problems, the SP equation was proposed as a model nonlinear equation describing the propagation of ultrashort optical pulses in nonlinear media. Specifically, starting from the Maxwell equation of the electric field in the fiber, Schäfer and Wayne[1] derived it as an alternative of the cubic nonlinear Schrödinger equation. The SP equation may be written in an appropriate dimensionless form as

$$u_{xt} = \alpha u + \frac{1}{3}\beta (u^3)_{xx}, \tag{1}$$

where $u = u(x,t)$ is the unknown real function and subscripts denote differentiation, $x,t \in R, \alpha$ and β are real parameters. Practically, equation (1) can be written as a more usual form as

$$u_{xt} = \alpha u + 2\beta u u_x^2 + \beta u^2 u_{xx}. \tag{2}$$

Several analogous equations to the SP equation exist, and they have been proven to be completely integrable. We write one of them in the form[2]

$$u_{xt} = \alpha u + \frac{1}{2}(1-\beta) u_x^2 - u u_{xx}. \tag{3}$$

When $\beta = 2$, equation (3) becomes the short-wave model for the Camassa–Holm equation, while when $\beta = 4$, it reduces to the short-wave model for the Degasperis–Procesi equation and the Vakhnenko equation. In this paper, we consider the case for $\beta = 3$

$$u_{xt} = \alpha u - u_x^2 - u u_{xx}. \tag{4}$$

The paper is organized as follows. In Section 2, we discuss the Lie symmetries of the short-wave model (4). In Section 3, the generalized symmetry method is employed to investigate the symmetries of Eq. (4). In Section 4, we give exact traveling wave solutions of the short-wave model (4) based on the bifurcation theory of dynamical systems.

2. Lie symmetry analysis for the short-wave model

The Lie group method is sometimes also called symmetry analysis.[17–23] Roughly speaking, a symmetry group of a system of differential equations is a group which transforms the solutions of the system to other solutions. Once one has determined the symmetry group of a system of differential equations, a number of applications become available. To start with, one can directly use the defining property of such a group and construct new solutions to the system from known ones. Firstly, let us consider a one-parameter Lie group of infinitesimal transformation:

$$x \to x + \varepsilon \xi(x,t,u), \quad t \to t + \varepsilon \tau(x,t,u),$$
$$u \to u + \varepsilon \phi(x,t,u)$$

with a small parameter $\varepsilon \ll 1$. The vector field associated with the above group of transformations can be written as

$$V = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}. \tag{5}$$

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†Corresponding author. E-mail: zsdzln@126.com

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The first and second prolongations are

$$pr^{(1)}V = \xi\frac{\partial}{\partial x}+\tau\frac{\partial}{\partial t}+\phi\frac{\partial}{\partial u}+\phi^{x}\frac{\partial}{\partial u_x}+\phi^{t}\frac{\partial}{\partial u_t}, \quad (6)$$

$$pr^{(2)}V = pr^{(1)}V+\phi^{xx}\frac{\partial}{\partial u_{xx}}+\phi^{xt}\frac{\partial}{\partial u_{xt}}+\phi^{tt}\frac{\partial}{\partial u_{tt}}. \quad (7)$$

If the vector field (5) generates a symmetry of Eq. (4), then V must satisfy the Lie's symmetry condition $pr^{(2)}V(\Delta)|_{\Delta=0}=0$, where $\Delta = u_{xt}-\alpha u+u_x^2+uu_{xx}$. Applying the symmetry condition to Eq. (4), we find that the coefficient functions ξ,τ, and ϕ must satisfy the following condition

$$(u_{xx}-\alpha)\phi+2u_x\phi^{x}+u\phi^{xx}+\phi^{xt}=0, \quad (8)$$

where ϕ^{x},ϕ^{xx}, and ϕ^{xt} are the coefficients of $pr^{(2)}V$. Furthermore, we have

$$\phi^{x} = \mathrm{D}_x\phi-u_x\mathrm{D}_x\xi-u_t\mathrm{D}_x\tau, \quad (9)$$

$$\phi^{xx} = \mathrm{D}_x^2(\phi-\xi u_x-\tau u_t)+\xi u_{xxx}+\tau u_{xxt}, \quad (10)$$

$$\phi^{xt} = \mathrm{D}_t\mathrm{D}_x(\phi-\xi u_x-\tau u_t)+\xi u_{xxt}+\tau u_{xtt}, \quad (11)$$

where D_x and D_t are the total derivatives. Application of the symmetry condition (9) to Eq. (4) yields the following determining equations

$$\xi_u=0,\quad \xi=\xi(x,t), \quad (12)$$

$$\tau_x=\tau_u=0,\quad \tau=\tau(t), \quad (13)$$

and

$$\xi_x=c_1,\quad \xi_x+\tau_t=0,\quad \phi=2c_1u. \quad (14)$$

Solving these equations, we get the following forms of the coefficient functions

$$\xi=c_1x+c_2,\quad \tau=-c_1t+c_3,\quad \phi=2c_1u, \quad (15)$$

where c_1 and c_2 are arbitrary constants. The Lie algebra of infinitesimal symmetries of Eq. (4) is spanned by the two vectors

$$V_1=\frac{\partial}{\partial x},\quad V_2=\frac{\partial}{\partial t},\quad V_3=x\frac{\partial}{\partial x}-t\frac{\partial}{\partial t}+2u\frac{\partial}{\partial u}. \quad (16)$$

The commutators are listed in Table 1. Thus, we have the corresponding one-parameter group of symmetries of the negative order KdV equation:

$$G_1:(x,t,u)\to(x+\varepsilon,t,u),$$
$$G_2:(x,t,u)\to(x,t+\varepsilon,u),$$
$$G_3:(x,t,u)\to(x\mathrm{e}^{\varepsilon},t\,\mathrm{e}^{-\varepsilon},u\,\mathrm{e}^{2\varepsilon}).$$

We can see that G_1 is a space translation, G_2 is a time translation, and G_3 is a scaling translation.

Table 1. The commutator table.

$[V_i,V_j]$	V_1	V_2	V_3
V_1	0	0	V_1
V_2	0	0	$-V_2$
V_3	$-V_1$	V_2	0

3. Generalized symmetries for the short-wave model

In this section, we consider the symmetries of Eq. (4) by using the generalized symmetry method. This method is also called the method of undetermined coefficient.[17]

Suppose

$$\sigma(x,t,u)=a(x,t)u_t+b(x,t)u_x+c(x,t)u+d(x,t) \quad (17)$$

is a symmetry of Eq. (4), where $a(x,t),b(x,t),c(x,t)$, and $d(x,t)$ are coefficient functions to be determined. According to the definition of generalized symmetry,[17] it is easy to show that $\sigma(x,t,u)$ is a symmetry of Eq. (4) if and only if

$$\sigma_{xt}-\alpha\sigma+u_{xx}\sigma+2u_x\sigma_x+u\sigma_{xx}=0. \quad (18)$$

It is not difficult to show that the coefficient functions $a(x,t),b(x,t),c(x,t)$, and $d(x,t)$ are as follows:

$$a(x,t)=-c_1t+c_2,\quad b(x,t)=c_1x+c_3,$$
$$c(x,t)=-2c_1,\quad d(x,t)=0, \quad (19)$$

where c_i $(i=1,2,3)$ are arbitrary constants. Substituting Eqs. (19) into Eq. (17), we have

$$\sigma=c_2u_t+c_3u_x+c_1(xu_x-tu_t-2u). \quad (20)$$

Hence, we find that the symmetries for Eq. (4) are of the forms as follows:

$$\sigma_1=u_x,\quad \sigma_2=u_t,\quad \sigma_3=xu_x-tu_t-2u, \quad (21)$$

which coincide with the vector fields V_1,V_2, and V_3 obtained in Section 2. In general, the linear combination of the two generators V_1 and V_2 will generate the traveling wave solutions for a PDE (see Section 4).

4. Exact traveling wave solutions of the short-wave model

In general, the traveling wave solutions to a PDE arise as special group-invariant solutions in which the group under consideration is a translational group on the space of independent variables. In the present case, we consider the translation group $(x,t,u)\mapsto(x+c\varepsilon,t+\varepsilon,u)$ generated by the generator $c\partial/\partial x+\partial/\partial t$, where c is a fixed constant, which will determine the speed of the waves. So we can find that the global invariants of this group are as follows:

$$\xi=x-ct,\quad \varphi=u. \quad (22)$$

In view of Eq. (22), let $u(x,t)=\varphi(x-ct)=\varphi(\xi)$, where c is the propagating wave velocity. Substituting it into the original PDE, we obtain a traveling wave equation. Then, solving this equation by the bifurcation theory method of dynamical systems,[23–28] we can obtain the traveling wave solutions.

Substituting Eq. (22) into Eq. (4), we obtain

$$(\varphi-c)\varphi''=\alpha\varphi-(\varphi')^2. \tag{23}$$

Equation (23) is equivalent to the plane system

$$\begin{cases}\dfrac{\mathrm{d}\varphi}{\mathrm{d}\xi}=y,\\[2mm] \dfrac{\mathrm{d}y}{\mathrm{d}\xi}=\dfrac{\alpha\varphi-y^2}{\varphi-c}\end{cases} \tag{24}$$

with the first integral

$$H(\varphi,y)=(\varphi-c)^2\left(y^2-\frac{2}{3}\alpha\varphi-\frac{1}{3}\alpha c\right)=h, \tag{25}$$

where h is is an integral constant. On the singular line $\varphi=c$, system (24) is discontinuous. Such a system is called a singular traveling wave system, studied by Li and Dai in Ref. [23]. To avoid the line temporarily we make transformation $\mathrm{d}\xi=(\varphi-c)\,\mathrm{d}\zeta$. Under this transformation, system (24) becomes

$$\begin{cases}\dfrac{\mathrm{d}\varphi}{\mathrm{d}\zeta}=(\varphi-c)y,\\[2mm] \dfrac{\mathrm{d}y}{\mathrm{d}\zeta}=\alpha\varphi-y^2.\end{cases} \tag{26}$$

Systems (26) and (24) have the same first integral as Eq. (25). Consequently, system (26) has the same topological phase portraits as system (24), except for the singular line $\varphi=c$. Obviously, $\varphi=c$ is an invariant straight-line solution for system (26).

Clearly, system (26) has three singular points under the condition $\alpha c>0$. We denote them $O(0,0), P_1(c,\sqrt{\alpha c})$, and $P_2(c,-\sqrt{\alpha c})$, respectively, and we have two types of singular points of system (26), as follows (see Fig. 1):

Case I If $\alpha c>0$, then O is a center; P_1 and P_2 are saddle points.

Case II If $\alpha c<0$, then O is a saddle point.

According to the phase portraits of system (26), we obtain the following various types of traveling wave solutions of Eq. (4).

Type 1 Smooth periodic wave solutions

For $\alpha>0$ and $c>0$, a family of smooth periodic wave solutions of Eq. (4) exist, which correspond to a family of periodic orbits defined by $H(\varphi,y)=h$, where $h\in(h_0,0)$, $h_0=-\frac{1}{3}\alpha c^3$. Now, equation (25) becomes

$$y^2=\frac{\frac{2}{3}\alpha z^3+\alpha c z^2+h}{z^2}, \tag{27}$$

where $z=u-c$. By using the first equation of system (24) to do the integration, we have

$$\int_{z_3}^{z}\frac{z\,\mathrm{d}z}{\sqrt{(z_1-z)(z_2-z)(z-z_3)}}=\sqrt{\frac{2}{3}\alpha}\int_0^{\xi}\mathrm{d}\xi, \tag{28}$$

where $(z_1-z)(z_2-z)(z-z_3)=z^3+\frac{3}{2}cz^2+\frac{3}{2\alpha}h$, $z_3<0<z_2<z_1$. Thus, we obtain the following exact parametric representations of smooth periodic wave solutions of Eq. (4) [see Fig. 2(a)]

$$\begin{aligned}&u(\tau)=z_3+c+(z_2-z_3)\mathrm{sn}^2(\tau,k),\\ &\xi(\tau)=\frac{1}{k^2\sqrt{\frac{2}{3}\alpha}}\left[(k^2-\gamma^2)\tau+\gamma^2E(\arcsin(\mathrm{sn}(\tau,k)),k)\right],\end{aligned} \tag{29}$$

where $k^2=(z_2-z_3)/(z_1-z_3)$, $\gamma=(z_3-z_2)/z_3$, and $\mathrm{sn}(\tau,k)$ are Jacobian elliptic functions with the modulus k and E is the elliptic integral of the second kind.

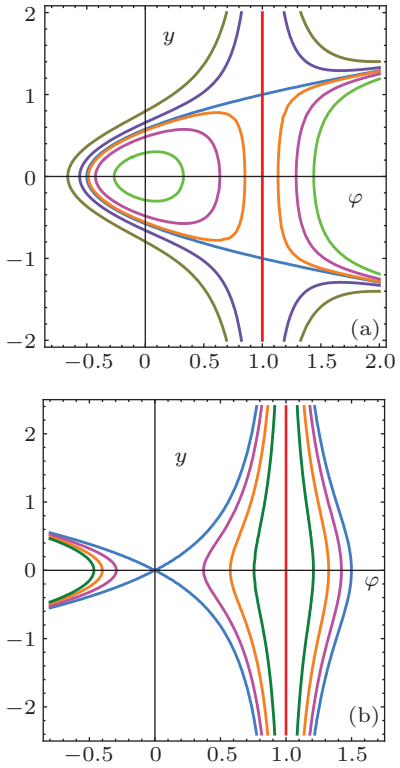


Fig. 1. (color online) Phase portraits of system (26). (a) $\alpha c>0$, (b) $\alpha c<0$.

Type 2 Periodic cuspons

For $\alpha>0$ and $c>0$, a periodic cuspon of Eq. (4) exists, which corresponds to the heteroclinic orbits (arch curves) defined by $H(\varphi,y)=h_s=0$. Now, equation (25) becomes

$$y^2=\frac{2}{3}\alpha\varphi+\frac{1}{3}\alpha c. \tag{30}$$

Thus, we have the following exact parametric representations of the traveling wave solution of Eq. (4)

$$u(x,t)=\frac{\alpha}{6}(x-ct)^2-\frac{c}{2},\quad -3\sqrt{\frac{c}{\alpha}}\le x-ct\le 3\sqrt{\frac{c}{\alpha}}. \tag{31}$$

Let $T=3\sqrt{c/\alpha}$, then we obtain the following periodic cuspon [see Fig. 2(b)]

$$\begin{aligned}&u(x,t)=\frac{\alpha}{6}(x-ct-2nT)^2-\frac{c}{2},\\ &(2n-1)T\le x-ct\le(2n+1)T.\end{aligned} \tag{32}$$

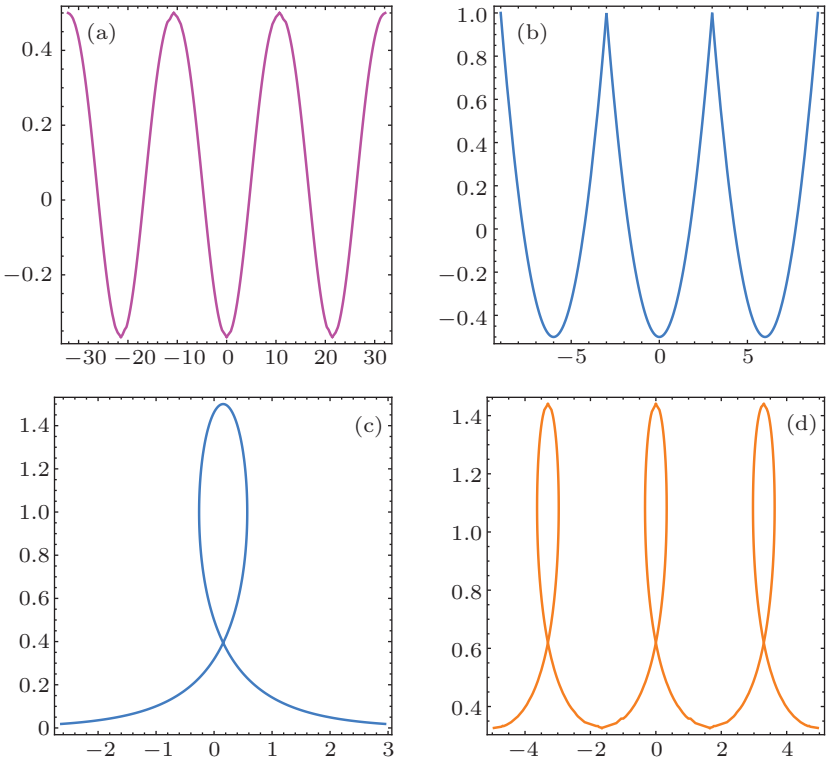


Fig. 2. (color online) Wave profiles. (a) Smooth periodic wave solutions, (b) periodic cuspons, (c) loop-soliton solutions, and (d) periodic loop solutions.

Type 3 Loop-soliton solutions

When $\alpha < 0$ and $c > 0$, the equilibrium point $O(0,0)$ is a saddle point, and we use $l^s_{\varphi_0}$ and $l^u_{\varphi_0}$ to denote the stable and unstable manifolds of saddle point $O(0,0)$. On the other hand, we notice that an integral curve l_{φ_1} passing through the point $(\varphi_1,0)$ for $H(\varphi,y) = h_0$ exists, which lies on the right side of straight line $\varphi = c$ (see Fig. 1(b)). The orbits $l^s_{\varphi_0}, l^u_{\varphi_0}$, and l_{φ_1} are determined by the algebra curve

$$y^2 = \frac{\alpha\varphi^2(2\varphi - 3c)/3}{(\varphi - c)^2}. \tag{33}$$

By using the first equation of system (24) to perform the integration along the three orbits for the initial value $\varphi(0) = c/2$ and $\varphi(0) = 3c/2$, respectively, we obtain the following parametric representations of the traveling wave solutions of Eq. (4) [see Fig. 2(c)]

$$\begin{aligned} u(\tau) &= \frac{3c}{2}\operatorname{sech}^2\left(\frac{\tau}{2\sqrt{3c}}\right), \\ \xi(\tau) &= \frac{\tau}{\sqrt{-3\alpha}} - 3\sqrt{-\frac{c}{\alpha}}\tanh\left(\frac{\tau}{2\sqrt{3c}}\right) - \frac{\eta_0}{\sqrt{-3\alpha}}, \end{aligned} \tag{34}$$

where

$$\eta_0 = 2\sqrt{3c}\tanh^{-1}\sqrt{\frac{2}{3}} - 3\sqrt{2c}. \tag{35}$$

Type 4 Periodic loop solutions

When $\alpha < 0$ and $c > 0$, the graph defined by $H(\varphi,y) = h, h \in (0,h_0)$ consists of two open-end curves, Γ_1 and Γ_2, passing through the points $(\varphi_1,0)$ and $(\varphi_2,0)$, respectively, where $0 < \varphi_2 < c < \varphi_1$. The orbits Γ_1 and Γ_2 are determined by the algebra curve

$$y^2 = \frac{\frac{2}{3}\alpha z^3 + \alpha c z^2 + h}{z^2}, \tag{36}$$

where $z = u - c$. By using the first equation of system (24) to do the integration, we have

$$\int_z^{z_1} \frac{z\,dz}{\sqrt{(z_1 - z)(z - z_2)(z - z_3)}} = \sqrt{-\frac{2}{3}\alpha}\int_0^{\xi} d\xi, \tag{37}$$

where

$$\begin{aligned} &(z_1 - z)(z - z_2)(z - z_3) = -z^3 - \frac{3}{2}cz^2 - \frac{3}{2\alpha}h, \\ &z_3 < 0 < z_2 < z_1. \end{aligned}$$

Thus, we obtain the following exact parametric representations of the periodic loop solutions of Eq. (4) [see Fig. 2(d)]

$$\begin{aligned} u(\tau) &= z_1 + c - (z_1 - z_2)\mathrm{sn}^2(\tau,k), \\ \xi(\tau) &= \sqrt{\frac{-6}{\alpha(z_1 - z_3)}} \\ &\quad \times [z_3\tau + (z_1 - z_3)E(\arcsin(\mathrm{sn}(\tau,k),k))], \end{aligned} \tag{38}$$

where $k^2 = (z_1 - z_2)/(z_1 - z_3)$, $\mathrm{sn}(\tau,k)$ are the Jacobian elliptic functions, with the modulus k and E the elliptic integral of the second kind.

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