Interactions among special embed-solitons for the (3+1)-dimensional Burgers equation*

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With the help of a modified mapping method and a new mapping method, we re-study the (3+1)-dimensional Burgers equation, and derive two families of variable separation solutions. By selecting appropriate functions in the variable separation solution, we discuss the interaction behaviors among taper-like, plateau-type rings, and rectangle-type embed-solitons in the periodic wave background. All the interaction behaviors are completely elastic, and no phase shift appears after interaction.

Keywords: (3+1)-dimensional Burgers equation, modified mapping method, interaction between special embed-solitons

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1. Introduction

In nonlinear mathematical physics, (3+1)-dimensional (3D) localized structures have been extensively investigated. Spatiotemporal solitons,[1] similaritons,[2] rogue waves,[3] paraboloid-type solitons, dipole-type dromions,[4] and embedded solitons[5] play an important role in different branches of physics. Compared with the soliton structures of (1+1) and (2+1) dimensions (1D and 2D),[6–8] 3D structures are more important since the real space is three dimensions. However, 3D structures have been reported less than 1D and 2D structures in the previous literature, and thus there are still many interesting issues worth studying.

Variable separation approaches (VSAs) based on different mapping equations have been established, e.g. the extended tanh-function method,[5] the improved projective approach,[6–8] and the projective Ricatti equation method,[9] etc. In this paper, we intend to answer the following questions: (i) can other mapping equations be used to obtain other forms of variable separation solutions for some 3D nonlinear physics systems, and (ii) can we discuss some new 3D soliton dynamical behaviors? To gain our goal we will study the following well-known 3D Burgers equation[4,5,10]

$$\begin{aligned} &u_t = 2uu_y + 2vu_x + 2wu_z + u_{xx} + u_{yy} + u_{zz}, \\ &u_x = v_y, u_z = w_y, \end{aligned} \tag{1}$$

which describes the propagation processes for nonlinear waves in fluid mechanics, such as diverse nonequilibrium, nonlinear phenomena in turbulence, and inter-face dynamics. If u is z-independent or both z-independent and y-independent, equation (1) will be degenerated to the known 2D or 1D Burgers equation, respectively. This is the Navier–Stokes equation (a model for Newtonian incompressible fluids) without the pressure term and the volume forces. Moreover, it can describe the flow of cars on a highway, and u denotes the density of cars. In Refs. [3] and [4], the authors extended the VSA to this system. More detailed physical backgrounds about this equation can be found in the above mentioned literature.

2. Exact solutions for the 3D Burgers equation

We suppose that equation (1) has the solution

$$U_i = a_{i0} + a_{i1}\phi(R) + a_{i2}/\phi(R), \tag{2}$$

where $i = 1,2,3, (U_1,U_2,U_3) \equiv (u,v,w)$, a_{ij} $(j = 0,1,2)$ and R are functions of $\{x,y,t\}$ to be determined, and ϕ satisfies a new mapping equation[11]

$$\mathrm{d}\phi/\mathrm{d}R = (A\phi - C)(B\phi - D), \tag{3}$$

with the general solution

$$\phi = \frac{D\exp[(BC-AD)R] - C\exp[C_1(AD-BC)]}{B\exp[(BC-AD)R] - A\exp[C_1(AD-BC)]}. \tag{4}$$

Here, C_1 is an integration constant, and further, A,B,C, and D are arbitrary constants. Note that this equation (3) is different from the mapping equations in Refs. [5] and [8], such as the Riccati equation $\mathrm{d}\phi/\mathrm{d}R = l_0 + \phi^2$ with the constant l_0 in Ref. [5] and $\mathrm{d}\phi/\mathrm{d}R = \sigma\phi + \phi^2$ with the constant σ in Refs. [6]– [8].

Inserting Eq. (2) with Eq. (3) into Eq. (1), selecting the variable separation ansätz

$$R = p(x,z,t) + q(y,t), \tag{5}$$

and eliminating all the coefficients of the powers of ϕ, yields a set of partial differential equations. Combining special solutions of these partial differential equations and ansatz (2) with

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Eq. (4), we obtain the following variable separation solutions of the 3D Burgers equation.

Solution 1

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{p_t - p_{xx} - p_{zz} + q_t - q_{yy} + (AD+BC)(p_x^2 + p_z^2 + q_y^2)}{2p_x} \\ 0 \end{pmatrix} - AB \begin{pmatrix} q_y \\ p_x \\ p_z \end{pmatrix}, \tag{6}$$

with $\Theta = D\exp[(BC - AD)(p+q)] - C\exp[C_1(AD - BC)]$, $\Lambda = B\exp[(BC - AD)(p+q)] - A\exp[C_1(AD - BC)]$, $p \equiv p(x,z,t)$, and $q = ky$.

Solution 2

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{p_t - p_{xx} - p_{zz} + q_t - q_{yy}}{2p_x} \\ 0 \end{pmatrix} - AD \left[\frac{\Gamma_+}{\Gamma_-} + \frac{\Gamma_-}{\Gamma_+} \right] \begin{pmatrix} q_y \\ p_x \\ p_z \end{pmatrix}, \tag{7}$$

where $AD + BC = 0$, $\Gamma_\pm = A\exp(2ADC_1) \pm B\exp[-2AD(p+q)]$, $p \equiv p(x,z,t)$, and $q = ky$.

Note that if $A = B = 1, C_1 = 0, D^2 = -\sigma > 0$, solution (7) degenerates into the corresponding solution expressed as Eq. (4) in Ref. [5].

3. 3D novel localized coherent structures

Although taper-like solitons, rectangle solitons, and plateau-type ring solitons have been discussed in Ref. [4], we can re-construct them via different functions from new a solution (7), and here we omit them. Since soliton interaction behaviors in the periodic waves have been investigated in Refs. [5] and [12], here we pay attention to taper-like embed-soliton, rectangle-type embed-soliton, plateau-type ring embed-soliton, and their interaction in the periodic waves.

The taper-like embed-soliton [Fig. 1(a)] can be obtained by choosing $p = -2 - \tanh[\sqrt{x^2 + (z-t)^2}] - \mathrm{sech}[\sqrt{x^2 + (z-t)^2}]$ for the physical quantity u (Eq. (4)), while the rectangle-type embed-soliton [Fig. 1(b)] or plateau-type ring embed-soliton [Fig. 1(c)] can be constructed by setting $p = -2 - \mathrm{sech}[0.2x^m + 0.2(z-t)^m - 4]$ with $m = 4$ or 2, respectively.

Now we focus on the interaction behaviors between these solitons in the periodic wave backgrounds. In the first case, function p is chosen as

$$p = -2 - 0.05[0.2\sin^2(x) - \cos^2(z-t)] - \left\{ \tanh\left[\sqrt{x^2 + (z-t)^2}\right] + \mathrm{sech}\left[\sqrt{x^2 + (z-t)^2}\right] \right\} - 0.5\mathrm{sech}[0.2x^4 + 0.2(z+t)^4 - 4], \tag{8}$$

where the second term represents the periodic wave background, the third term describes the taper-like embed-soliton moving along the positive z direction, and the last term denotes the rectangle-type embed-soliton moving along the negative z direction.

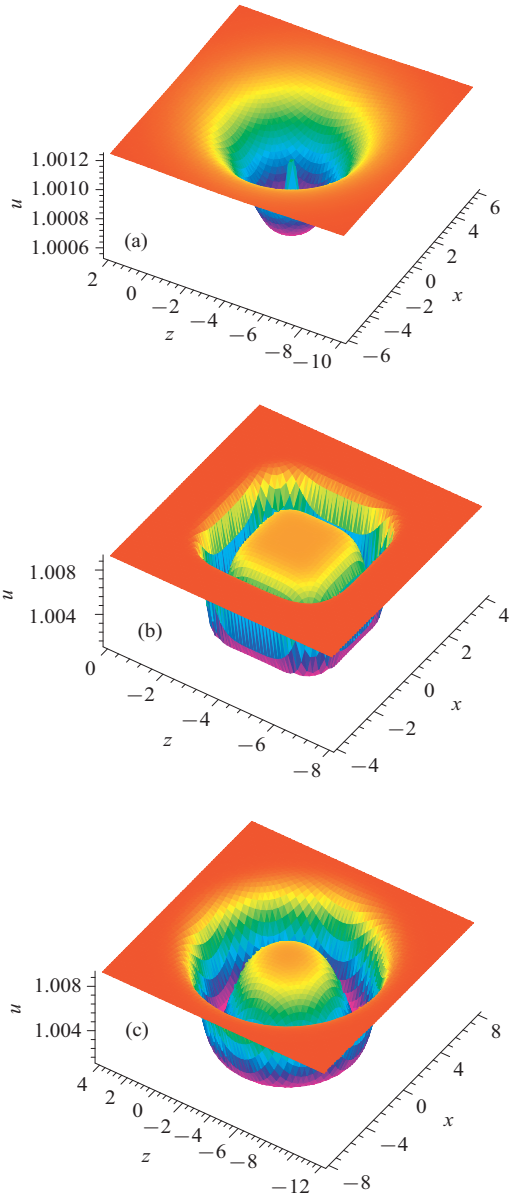


Fig. 1. (color online) Profiles of (a) taper-like, (b) rectangle-type, and (c) plateau-type ring embed-solitons for parameters $A = 2$, $C_1 = B = 1$, $C = 0.5$, $D = -0.25$, $k = -1$, and $y = 1$ at time $t = -4$.

From Fig. 2 and by detailed analysis, one can find that the interaction between taper-like and rectangle-type embed-solitons may exhibit a completely elastic behavior, since solitons' shapes, amplitudes, and velocities do not undergo any change after the nonlinear interaction. Moreover, we can see that this interaction possesses a novel property, that is, during the collision, the taper-like peak of taper-like embed-solitons stack in the top of the flat-top of the rectangle-type embed-

solitons. The phase shifts are not observed, and two solitons exchange the corresponding positions after collision. The taper-like and rectangle-type embed-solitons maintain their initial velocities $\{v_{1x} = v_{01x} = 1\}$ and $\{v_{2x} = v_{02x} = -1\}$.

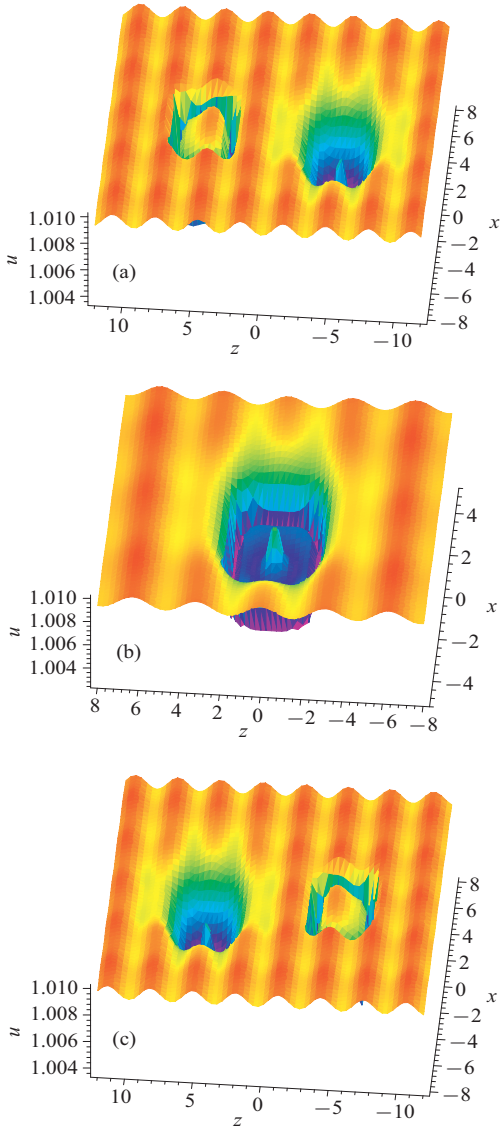


Fig. 2. (color online) Interaction between taper-like and rectangle-type embed-solitons in the periodic wave background at time (a) $t = -5$, (b) $t = 0$, and (c) $t = 5$. The parameters are $A = 2$, $C_1 = B = 1$, $C = 0.5$, $D = -0.25$, $k = -1$, and $y = 1$.

Along with the above line, we consider the second case: the interaction between taper-like and rectangle-type embed-solitons in the periodic wave background, when p is selected to be

$$\begin{aligned} p = & -2 - 0.05[0.2\sin^2(x) - \cos^2(z-t)] \\ & - \left\{\tanh\left[\sqrt{x^2+(z+t)^2}\right] + \mathrm{sech}\left[\sqrt{x^2+(z+t)^2}\right]\right\} \\ & - 0.2\mathrm{sech}[0.2x^2 + 0.2(z-t)^2 - 4], \end{aligned} \tag{9}$$

where the second, third, and fourth terms describe the periodic wave background, the taper-like embed-soliton moving along the negative z direction, and the plateau-type ring embed-soliton moving along the positive z direction, respectively. From Fig. 3, we can find that the interaction between the taper-like and plateau-type ring embed-solitons in the periodic wave background is also completely elastic, because their amplitudes, velocities, and wave shapes are completely maintained after interaction. Note that during the collision, the taper-like peak of taper-like embed-solitons does not stack in the top of the plateau of the plateau-type ring embed-soliton. On the contrary, the taper-like peak inserts into the plateau, which is different from the first case mentioned above. Moreover, we can discuss the interaction between rectangle-type and plateau-type ring embed-solitons by setting function $p = -2 - 0.05[0.2\sin^2(x) - \cos^2(z-t)] - \mathrm{sech}[0.2x^2 + 0.2(z-t)^2 - 4] - 0.2\mathrm{sech}[0.2x^4 + 0.2(z+t)^4 - 4]$. For the length of limit, we omit it.

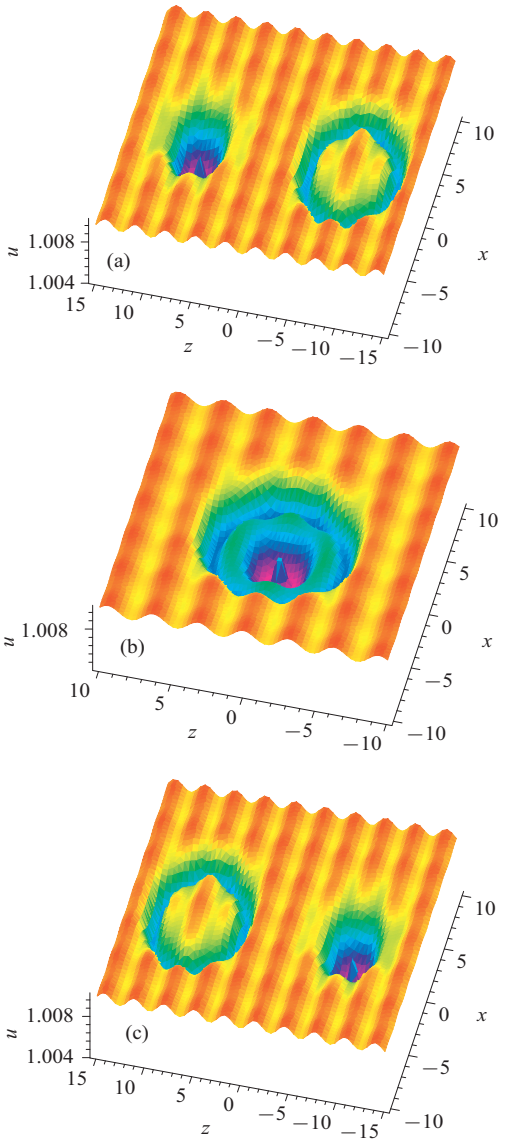


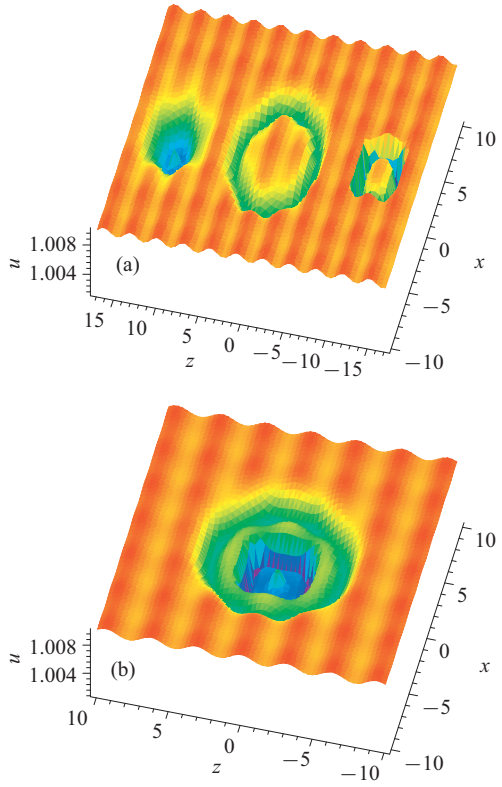
Fig. 3. (color online) Interaction between taper-like and plateau-type ring embed-solitons in the periodic wave background at time (a) $t = -8$, (b) $t = 0$, and (c) $t = 8$. The parameters are $A = 2$, $C_1 = B = 1$, $C = 0.5$, $D = -0.25$, $k = -1$, and $y = 1$.

Besides these interactions between two special solitons,

we can also investigate the interaction among three different solitons by choosing the arbitrary function p. In the following, we investigate the interaction behaviors among three different solitons in the periodic wave background when setting

$$\begin{aligned} p = & -2-0.05[0.2\sin^2(x)-\cos^2(z-t)] \\ & -\left\{\tanh\left[\sqrt{x^2+(z+t)^2}\right]+\operatorname{sech}\left[\sqrt{x^2+(z+t)^2}\right]\right\} \\ & -0.2\operatorname{sech}[0.2x^2+0.2z^2-4] \\ & -\operatorname{sech}[0.2x^4+0.2(z-t)^4-4], \end{aligned} \tag{10}$$

where the second, third, fourth, and last terms represent the periodic wave background, the taper-like embedded-soliton moving along the negative z direction, the static plateau-type ring embed-soliton, and the rectangle-type embed-solitons moving along the positive z direction, respectively. Figure 4 exhibits the completely elastic interaction among taper-like, plateau-type rings, and rectangle-type embed-solitons in the periodic wave background since their amplitudes, velocities, and wave shapes are preserved after their collisions. The static plateau-type ring embed-soliton still locates at $z=0$, and the taper-like and rectangle-type embed-solitons also maintain their initial velocities and exchange their positions after interaction. Phase shifts are still not observed during this interaction.



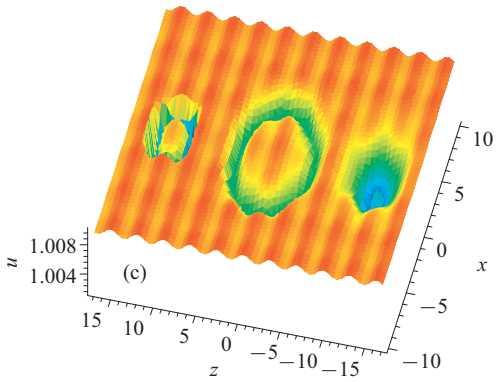


Fig. 4. (color online) Interaction among taper-like, plateau-type rings, and rectangle-type embed-solitons in the periodic wave background at time (a) $t=-12$, (b) $t=0$, and (c) $t=12$. The parameters are $A=2, C_1=B=1, C=0.5, D=-0.25, k=-1$, and $y=1$.

4. Summary and discussion

In this paper, a new mapping equation is used to derive variable separation solutions of the 3D Burgers equation. Two families of variable separation solutions are presented. Based on the selection of appropriate functions in the variable separation solution, we discussed the interaction behaviors among taper-like, plateau-type rings, and rectangle-type embed-solitons in the periodic wave background. The analysis results show that all the interaction behaviors among them are completely elastic, and no phase shift appears after interaction. Of course, there are some pending issues to be further studied, for example, how to quantify the description of complete or non-complete elastic interaction more suitably, and what is the general expression for the distribution of the energy and momentum for these interactions?

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