Synchronization of uncertain fractional-order chaotic systems with disturbance based on a fractional terminal sliding mode controller*

Wang Dong-Feng(王东风), Zhang Jin-Ying(张金营)†, and Wang Xiao-Yan(王晓燕)

Hebei Engineering Research Center of Simulation & Optimized Control for Power Generation (North China Electric Power University), Baoding 071003, *China*

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This paper provides a novel method to synchronize uncertain fractional-order chaotic systems with external disturbance via fractional terminal sliding mode control. Based on Lyapunov stability theory, a new fractional-order switching manifold is proposed, and in order to ensure the occurrence of sliding motion in finite time, a corresponding sliding mode control law is designed. The proposed control scheme is applied to synchronize the fractional-order Lorenz chaotic system and fractional-order Chen chaotic system with uncertainty and external disturbance parameters. The simulation results show the applicability and efficiency of the proposed scheme.

Keywords: fractional-order chaotic system, synchronization, terminal sliding mode control, uncertainty, disturbance

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1. Introduction

Research on complex networks has attracted increasing attention since Watts and Strogatz's work[1] in 1998 and Barabási and Albert's work[2] in 1999. Many real-world networks, such as the Internet, the World Wide Web, electric power grids, cellular networks, and so on, are complex networks. Complex networks[3–7] are considered to be "between the rules and the random",[8] which is also the intrinsic character of chaotic dynamics, according to most physicists at present. Therefore, chaotic dynamics can be considered as a special kind of complex network with a wide range of potential applications, such as image encryption,[9,10] secure communication,[11,12] genetic networks,[13] and so on. Along with the development of chaos, fractional-order calculus has been used to describe the chaos system, and this extends the application field of chaos theory. In this context, this paper tries to research the synchronization problems of fractional order chaotic systems.

The theory of fractional-order derivatives, which was mainly developed in the 19th century, is a generalization of integration and differentiation to fractional (non-integer) order. Nowadays, it is known that many systems in interdisciplinary fields, such as viscoelastic materials,[14] electrode-electrolyte polarization,[15] thermoelectric systems,[16] finance systems,[17] dielectric polarization,[18] etc., can be elegantly described by fractional differential derivatives. However, the application of fractional-order calculus to dynamic control is only a recent focus of interest.[19,20] Moreover, some scholars have demonstrated that many fractional-order differential systems, like the fractional-order Lorenz system,[21] the fractional-order Duffling system,[22] the fractional-order Chua system,[23] and the fractional-order Chen system[24] exhibit chaotic behaviors.

Since Pecora and Carroll established a chaos synchronization scheme for two identical chaotic systems with different initial conditions,[25] chaos synchronization has attracted a great deal of interest.[26–29] Because fractional-order chaotic systems can describe many practical systems more accurately, their synchronization has also attracted the attention of many scholars, and some techniques have been used to synchronize fractional-order chaotic systems.[21,30–33] In practical situations, parameter uncertainty and external disturbance for fractional-order chaotic systems exist widely. Based on fractional-order stability theory, Xu Jian-Qiang[34] presented a novel adaptive controller and parameter updating rule to synchronize the fractional-order unified chaotic system with uncertain parameters. Zhou Ping *et al.*[35] designed an adaptive controller to synchronize two entirely different fractional-order chaotic systems with uncertain parameters. Yang Chi-Ching,[36] associated with system parameter estimated laws, proposed a single state proportional feedback method to synchronize two nearly identical generalized Lorenz systems. Zhang Ruo-Xun *et al.*,[37] based on Lyapunov stability theory and a fractional-order differential inequality, developed a modified adaptive control scheme and adaptive laws of parameters to robustly synchronize coupled fractional-order chaotic systems with unknown parameters and uncertain perturbations. Xiang Wei *et al.*[38] investigated the robust synchronization problem for class systems with external disturbances. How-

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†Corresponding author. E-mail: k.ying_zhang@163.com

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ever, the effects of both unknown parameters and the external disturbance of the fractional-order chaotic systems have seldom been considered in these works.

Sliding mode control (SMC) is a powerful theory for controlling uncertain systems. The main advantage is that a controlled system with sliding mode exhibits robustness properties with respect to both internal parameter uncertainties and external disturbances. Dadras Sara *et al.*[39] introduced a control strategy combining fractional calculus with terminal sliding mode control, called fractional terminal sliding mode control strategy, for a class of dynamical systems subject to uncertainties. Qi Dong-Lian *et al.*[40] designed two different sliding mode controllers for a fractional-order unified chaotic system, and Zhang Hong *et al.*[41] presented a controller based on sliding mode theory to synchronize fractional-order Newton–Leipnik chaotic systems with uncertain parameters. In addition, Yin Chun *et al.*[42] designed a sliding mode control law to control chaos in a class of fractional-order chaotic systems.

Inspired by the above work, this paper designs a novel terminal fractional-order sliding mode controller for the robust synchronization problem of a class of fractional-order chaotic systems in the presence of parameter uncertainties and external disturbances. Then, based on the fractional-order Lyapunov stability theory, a robust sliding control law is derived to guarantee the occurrence of the sliding motion in finite time. Experimental results demonstrate the effectiveness and feasibility of the proposed synchronization method.

This paper is organized as follows. In Section 2, a synchronization law for fractional-order chaotic systems with uncertainty and external disturbance using the fractional terminal sliding mode is designed. Numerical simulation results are given in Section 3 to illustrate the effectiveness of the proposed controller, and the conclusions in Section 4 close the paper.

2. Synchronization of fractional-order chaotic systems with uncertainty and external disturbance using the fractional terminal sliding mode

2.1. Fractional derivatives

Consider a fractional nonautonomous system[19,43]

$$ {}_{t_0}D_t^{\alpha}x(t) = f(x,\beta) \tag{1}$$

with initial condition $x(t_0)$, where ${}_{t_0}D_t^{\alpha}$ denotes either the Caputo or Riemann–Liouville fractional operator, $\alpha \in (0,1)$, $\beta \in R^p$ are the unknown parameter vectors of the systems, $f : [t_0,\infty] \times \Omega \to R^n$ is the piecewise that is continuous in x and locally Lipschitz in x on $[t_0,\infty] \times \Omega$, and $\Omega \in R^n$ is a domain that contains the origin $x = 0$.

Definition 1 The constant x_0 is an equilibrium point of the fractional dynamic system (1), if and only if $f(t,x_0) = {}_{t_0}D_t^{\alpha}x_0$. Without loss of generality, let the equilibrium point be $x = 0$.

In this paper, we use both Riemann–Liouville and Caputo fractional order operators as our main tools, which are shown in Podlubny's work.[19] The uniform formula of a fractional integral with $\alpha \in (0,1)$ is defined as

$$ {}_{t_0}I_t^{\alpha}x(t) = {}_{t_0}D_t^{-\alpha}x(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}\frac{x(\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau, \tag{2}$$

where $x(t)$ is an arbitrary integrable function, ${}_{t_0}I_t^{\alpha}$ is the fractional integral of order α on $[t_0,t]$, and $\Gamma(\cdot)$ denotes the Gamma function. For an arbitrary real number m, the Riemann–Liouville and Caputo fractional derivatives are defined, respectively, as

$$ {}_{t_0}D_t^{m}x(t) = \frac{\mathrm{d}^{[m]+1}}{\mathrm{d}t^{[m]+1}}\left[{}_{t_0}D_t^{-([m]-m+1)}x(t)\right] \tag{3}$$

and

$$ {}_{t_0}^{C}D_t^{m}x(t) = {}_{t_0}D_t^{-([m]-m+1)}\left[\frac{\mathrm{d}^{[m]+1}}{\mathrm{d}t^{[m]+1}}x(t)\right], \tag{4}$$

where $[m]$ stands for the integer part of m, ${}_{t_0}D_t^{\alpha}$ and ${}_{t_0}^{C}D_t^{\alpha}$ denote the Riemann–Liouville and Caputo fractional derivatives, respectively.

Property 1[19] If the fractional derivative ${}_{t_0}D_t^{q}x(t)$ $(0 \le k-1 \le q < k)$ of a function $x(t)$ is integrable, then

$$\begin{aligned} &{}_{t_0}I_t^{p}\left({}_{t_0}D_t^{q}x(t)\right) \\ &= {}_{t_0}D_t^{q-p}x(t) - \sum_{j=1}^{k}\left[{}_{t_0}D_t^{q-j}x(t)\right]_{t=t_0}\frac{(t-t_0)^{p-j}}{\Gamma(1+p-j)}. \end{aligned} \tag{5}$$

Lemma 1[44]

(i) The fractional integration operators I_{a+}^{α} and I_{b-}^{α} with $R(\alpha) > 0$ are bounded in $L_p(a,b)$ $(1 \le p \le \infty)$:

$$\begin{aligned} \left\|I_{a+}^{\alpha}x\right\|_{L_p(a,b)} &\le K\|x\|_{L_p(a,b)}, \left\|I_{b-}^{\alpha}x\right\|_{L_p(a,b)} \\ &\le K\|x\|_{L_p(a,b)}\left(K = \frac{(b-a)^{R(\alpha)}}{R(\alpha)\,|\Gamma(\alpha)|}\right). \end{aligned} \tag{6}$$

(ii) If $0 < \alpha < 1$ and $1 < \mathrm{p} < 1/\alpha$, then the operators I_{a+}^{α} and I_{b-}^{α} are bounded from $L_p(a,b)$ into $L_q(a,b)$, where $q = p/(1-\alpha p)$.

Remark 1 Lemma 1 (i) was proved by Samko *et al.*[45] while Lemma 1 (ii) is known as the Hardy–Littlewood theorem.[45]

2.2. The asymptotic stability of the fractional-order system

Definition 2[46] A continuous function α: $[0,t) \to [0,\infty)$ is said to belong to class-k if it is strictly increasing and $\alpha(0) = 0$.

Theorem 1[47] Let $x = 0$ be an equilibrium point for the nonautonomous fractional-order system (1). Assume that there exist a Lyapunov function $V(t,x(t))$ and class-k functions α_i $(i = 1,2,3)$ satisfying

$$\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|), \tag{7}$$

$${}_0^C D_t^\alpha V(t,x(t)) \le -\alpha_3(\|x\|), \tag{8}$$

where $\alpha \in (0,1)$. Then the equilibrium point of the system (1) is asymptotically stable.

2.3. Brief introduction of integer-order terminal sliding mode control

In this part, the definition and some of the properties of terminal sliding mode control[48] (TSMC) are briefly introduced for the essential preparation of the fractional-order TSMC design.

Definition 3[49,50] Terminal sliding manifold can be defined as the following nonlinear differential equation

$$s_m = \dot{e}(t) + \lambda e(t)^{q/p} = 0, \tag{9}$$

where λ is a designed positive constant, the $e \in R$ stands for error state, and p and q are both positive odd integers which should satisfy the following condition $q < p < 2q$.

Taking the time derivative of Eq. (9), one can launch

$$\dot{s}_m = \ddot{e}(t) + \lambda \frac{q}{p} \dot{e}(t) e(t)^{(q-p)/p}. \tag{10}$$

Remark 2 Based on the finite time stability theory, the equilibrium point $e = 0$ of the error system is globally finite time stable, i.e., the exact tracking occurs, the error original state $e(0) \neq 0$ can converge to $e = 0$ in finite time[51]

$$t_s = \frac{p}{\lambda(p-q)} e(0)^{1-q/p}. \tag{11}$$

Remark 3 The sufficient condition for the existence of the terminal sliding mode is[50]

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} s_m^2(t) < -k_1 |s_m(t)|, \tag{12}$$

where k_1 is a positive constant.

2.4. Design of the fractional-order terminal sliding manifold and synchronization law

Considering the following two dynamic systems

1) driving system

$${}_{t_0}D_t^\alpha x(t) = f(x,\beta) + d_1, \tag{13}$$

2) response system

$${}_{t_0}D_t^\alpha y(t) = h(y,\gamma) + d_2 + u, \tag{14}$$

where $x, y \in R^n$ are the state vectors of the driving system and response system, respectively; f and h are both $n \times 1$ bounded continuous functional matrices; d_1 and d_2 are both $n \times 1$ unknown bounded external disturbance matrices; $\beta \in R^p$ and $\gamma \in R^q$ are the unknown parameter vectors of the driving and response systems; and $u \in R^n$ is a control input vector.

Assume that $\tilde{\beta}$, $\tilde{\gamma}$ are the rational pre-estimated value of β, γ, respectively, that is to say, we will use the approximate model $f(x,\tilde{\beta})$ and $h(y,\tilde{\gamma})$ to design the synchronization control law. Correspondingly, the error between the pre-estimated models and the real systems are as follows, respectively:

$$\Delta f(x,\beta,\tilde{\beta}) = f(x,\beta) - f(x,\tilde{\beta}), \tag{15}$$

$$\Delta h(y,\gamma,\tilde{\gamma}) = h(y,\gamma) - h(y,\tilde{\gamma}). \tag{16}$$

Assumption 1 $L(x,y,t)$ is called the lumped uncertainty and is defined as

$$L(x,y,\beta,\gamma,\tilde{\beta},\tilde{\gamma}) = \Delta f(x,y,\beta,\tilde{\beta}) - \Delta h(x,y,\gamma,\tilde{\gamma}) + d_1 - d_2. \tag{17}$$

The lumped uncertainty is assumed to satisfy the following inequality

$$\left\| L(x,y,\beta,\gamma,\tilde{\beta},\tilde{\gamma}) \right\| \le \delta, \tag{18}$$

where δ is a known positive constant. The synchronization error $(e = y - x)$ dynamic system can be written as

$${}_{t_0}D_t^\alpha e(t) = h(y,\gamma) - f(x,\beta) + u + d_2 - d_1. \tag{19}$$

Our goal is to design a control u which should be robust enough such that the trajectory of the response system (14) with initial conditions y_0 and pre-estimated parameter $\tilde{\gamma}$, as well as unknown external disturbance d_2, can asymptotically approach the trajectory of the driving system (13) with initial conditions x_0 and pre-estimated parameter $\tilde{\beta}$, as well as unknown external disturbance d_1. Finally, we implement synchronization in the sense that $\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|y(t,y_0,d_2) - x(t,x_0,d_1)\| = 0$ where $\|\cdot\|$ is the Euclidean norm.

Theorem 2 Let the sliding surface be defined as follows:

$$s = {}_{t_0}D_t^{\alpha-1} e + \lambda e^{q/p}, \quad \alpha \in (0,1), \tag{20}$$

where p and q are both positive odd integers. Consider the systems (13) and (14) in which the lumped uncertainty satisfy Eq. (18). If the terminal sliding mode control law is designed as follows:

$$u = f(x,\tilde{\beta}) - h(y,\tilde{\gamma}) - \lambda \frac{q}{p} \dot{e} e^{(q-p)/p} - K\mathrm{sign}(s) - Ks - \delta, \tag{21}$$

where K is a positive constant, then the tracking error will converge to zero in finite time.

Proof The Lyapunov function to be defined is

$$V = \frac{1}{2} s^2. \tag{22}$$

Taking the derivative of both sides of Eq. (22) with respect to time and using Eq. (20), we have

$$\begin{aligned}\dot{V} &= s\dot{s}\\ &= s\left({}_{t_0}D_t^{\alpha}e+\lambda\frac{p}{q}e^{q/p-1}\dot{e}\right)\\ &= s\left(h(y,\gamma)-f(x,\beta)+u+d_2-d_1+\lambda\frac{p}{q}e^{q/p-1}\dot{e}\right).\end{aligned}\tag{23}$$

Substituting Eq. (21) into Eq. (23) results in

$$\begin{aligned}\dot{V} &= s\dot{s}\\ &= s[h(y,\gamma)-f(x,\beta)+(f(x,\tilde{\beta})-h(y,\tilde{\gamma})\\ &\quad-\lambda\frac{q}{p}e^{q/p-1}\dot{e}-K\mathrm{sign}(s)-Ks-\delta)+d_2\\ &\quad-d_1+\lambda\frac{p}{q}e^{q/p-1}\dot{e}]\\ &= s[(f(x,\tilde{\beta})-f(x,\beta)+h(y,\gamma)-h(y,\tilde{\gamma})-d_1+d_2)\\ &\quad-\delta-K\mathrm{sign}(s)-Ks]\\ &= s(-\delta-L(x,\beta,\gamma,\tilde{\beta},\tilde{\gamma})-K\mathrm{sign}(s)-Ks).\end{aligned}\tag{24}$$

Substituting Eq. (18) into Eq. (24) results in

$$\dot{V}=s\dot{s}\le-K(s^2+|s|).\tag{25}$$

Consequently, the tracking error converges to zero if the controller parameter K is chosen appropriately. To show that this (exact tracking, $e=0$) occurs in finite time, we can define a stopping time as follows:

$$\tau_s=\inf\{t\ge t_{\mathrm{r}}:e(t)=0\},\tag{26}$$

where t_{r} is the reaching time, which means the time taken for the system states to reach the sliding surface, and can be estimated using Eq. (25), i.e.,

$$s\dot{s}\le-K(s^2+|s|).\tag{27}$$

Apply the integer-order integration operator to both sides of Eq. (27) with final time $t=t_{\mathrm{r}}$, and noting that $s(t)=0$ at $t=t_{\mathrm{r}}$, one gets

$$t_{\mathrm{r}}\le\left|\frac{s(0)}{K}\right|.\tag{28}$$

So, all we need is to prove that there exists $t_{\mathrm{r}}\le t_s<\infty$ such that $\tau_s\le t_s$. Taking the concept of fractional integral and derivative operators into account and using Eq. (20), one obtains

$${}_{t_{\mathrm{r}}}D_t^{\alpha-1}e(t)=-\lambda e^{q/p}(t).\tag{29}$$

From Eq. (29), we can conclude that

$${}_{t_{\mathrm{r}}}D_t^{1-\alpha}[{}_{t_{\mathrm{r}}}D_t^{\alpha-1}e(t)]=-\lambda_{t_{\mathrm{r}}}D_t^{1-\alpha}[e^{q/p}(t)],\tag{30}$$

where ${}_{t_{\mathrm{r}}}D_t^{1-\alpha}[{}_{t_{\mathrm{r}}}D_t^{\alpha-1}e(t)]=e(t)$. So, it can be concluded that

$$e(t)=-\lambda_{t_{\mathrm{r}}}D_t^{1-\alpha}[e^{q/p}(t)].\tag{31}$$

According to Lemma 1, we have

$${}_{t_{\mathrm{r}}}I_t^2[e(t)]\le N\|e(t)\|.\tag{32}$$

According to Property 1, we have

$$\begin{aligned}&{}_{t_{\mathrm{r}}}I_t^2\{-\lambda_{t_{\mathrm{r}}}D_t^{1-\alpha}[e^{q/p}(t)]\}\\ &=-\lambda_{t_{\mathrm{r}}}D_t^{-\alpha-1}[e^{q/p}(t)]+\lambda_{t_{\mathrm{r}}}\{D_t^{-\alpha}[e^{q/p}(t)]\}_{t=t_{\mathrm{r}}}\frac{t-t_{\mathrm{r}}}{\Gamma(2)}\\ &=-\lambda_{t_{\mathrm{r}}}D_t^{-\alpha-1}[e^{q/p}(t)]+\lambda_{t_{\mathrm{r}}}\{D_t^{-\alpha}[e^{q/p}(t)]\}_{t=t_{\mathrm{r}}}(t-t_{\mathrm{r}}).\end{aligned}\tag{33}$$

Substituting Eq. (32) into Eq. (33) results in

$$-\lambda_{t_{\mathrm{r}}}D_t^{-\alpha-1}[e^{q/p}(t)]+\lambda_{t_{\mathrm{r}}}\{D_t^{-\alpha}[e^{q/p}(t)]\}_{t=t_{\mathrm{r}}}(t-t_{\mathrm{r}})\le N\|e(t)\|.\tag{34}$$

According to Lemma 1, we have

$$\lambda_{t_{\mathrm{r}}}D_t^{-\alpha-1}[e^{q/p}(t)]\le\lambda N_1\left\|e^{q/p}(t)\right\|.\tag{35}$$

Substituting Eq. (35) into Eq. (34) results in

$$\begin{aligned}&\lambda\{{}_{t_{\mathrm{r}}}D_t^{-\alpha}[e^{q/p}(t)]\}_{t=t_{\mathrm{r}}}(t-t_{\mathrm{r}})\\ &\le N\|e(t)\|+\lambda_{t_{\mathrm{r}}}D_t^{-\alpha-1}[e^{q/p}(t)]\\ &\le N\|e(t)\|+\lambda N_1\left\|e^{q/p}(t)\right\|.\end{aligned}\tag{36}$$

Noting that $e(t)=0$ at $t=t_s$, it yields

$$\lambda\{{}_{t_{\mathrm{r}}}D_t^{-\alpha}[e^{q/p}(t)]\}_{t=t_{\mathrm{r}}}(t_s-t_{\mathrm{r}})\le0.\tag{37}$$

Consequently, if $\lambda\{{}_{t_{\mathrm{r}}}D_t^{-\alpha}[e^{q/p}(t)]\}_{t=t_{\mathrm{r}}}=0$, then $t_s=t_{\mathrm{r}}$. Else, we have

$$\begin{aligned}t_s &\le \frac{N\|e(t)\|+\lambda N_1\left\|e^{q/p}(t)\right\|}{\lambda\{{}_{t_{\mathrm{r}}}D_t^{-\alpha}[e^{q/p}(t)]\}_{t=t_{\mathrm{r}}}}+t_{\mathrm{r}}\\ &= \frac{0}{\lambda\{{}_{t_{\mathrm{r}}}D_t^{-\alpha}[e^{q/p}(t)]\}_{t=t_{\mathrm{r}}}}+t_{\mathrm{r}}\\ &= t_{\mathrm{r}}.\end{aligned}\tag{38}$$

So, we have $t_s\equiv t_{\mathrm{r}}$. Therefore, it can be concluded that the system trajectories attain the reference trajectory in finite time. Q.E.D.

3. Illustrative example

In this section, we will give one illustrative example to show the applicability and efficiency of the proposed controller. The numerical simulation has been carried out using MATLAB software.

The driving fractional-order Lorenz chaotic system[52,53] is described as follows:

$${}_{t_0}D_t^{\alpha}x(t)=\begin{bmatrix}\beta_1(x_2-x_1)\\ \beta_2x_1-x_1x_3-x_2\\ x_1x_2-\beta_3x_3\end{bmatrix}+d_1.\tag{39}$$

According to Eq. (13), we have

$$f(x,\beta)=\begin{bmatrix}\beta_1(x_2-x_1)\\ \beta_2x_1-x_1x_3-x_2\\ x_1x_2-\beta_3x_3\end{bmatrix},\quad d_1=\begin{bmatrix}d_{11}\\ d_{12}\\ d_{13}\end{bmatrix}.\tag{40}$$

The response fractional-order Chen chaotic system[54,55] is described as follows:

$$ {}_{t_0}D_t^{\alpha} y(t) = \begin{bmatrix} \gamma_1(y_2 - y_1) \\ (\gamma_2 - \gamma_1)y_1 - y_1 y_3 + \gamma_2 y_2 \\ y_1 y_2 - \gamma_3 y_3 \end{bmatrix} + d_2 + u. \quad (41) $$

According to Eq. (14), we have

$$ h(y,\gamma) = \begin{bmatrix} \gamma_1(y_2 - y_1) \\ (\gamma_2 - \gamma_1)y_1 - y_1 y_3 + \gamma_2 y_2 \\ y_1 y_2 - \gamma_3 y_3 \end{bmatrix}, \quad d_2 = \begin{bmatrix} d_{21} \\ d_{22} \\ d_{23} \end{bmatrix}. \quad (42) $$

In the simulation, when $\alpha = 0.993, \beta_1 = 10, \beta_2 = 28, \beta_3 = 8/3, \gamma_1 = 35, \gamma_2 = 28, \gamma_3 = 3$, both the fractional-order Lorenz system and fractional-order Chen system are in chaos.[56,57] We choose $x = [-20, -14, 30]^{\mathrm{T}}$ and $y = [16, 15, -5]^{\mathrm{T}}$ as the initial conditions. The control parameters for the control law (Eq. (21)) are chosen as $K = 5$ and $\delta = 25$, and the sliding surface parameters are selected as $p = 25$ and $q = 23$.

Case 1 Simulation without uncertainty and disturbance

Figure 1 shows the original chaotic curves of x_i and y_i $(i = 1,2,3)$. Figure 2 shows the trajectories of the fractional-order Lorenz and Chen systems when the control law is used, from which we can see that the fractional-order Lorenz and Chen systems are indeed synchronized. Figure 3 demonstrates the curves of x_i and y_i $(i = 1,2,3)$ and the error between them. It is clear that after a finite period of time, the initial error converges to zero and the synchronization between the driving and response systems is achieved.

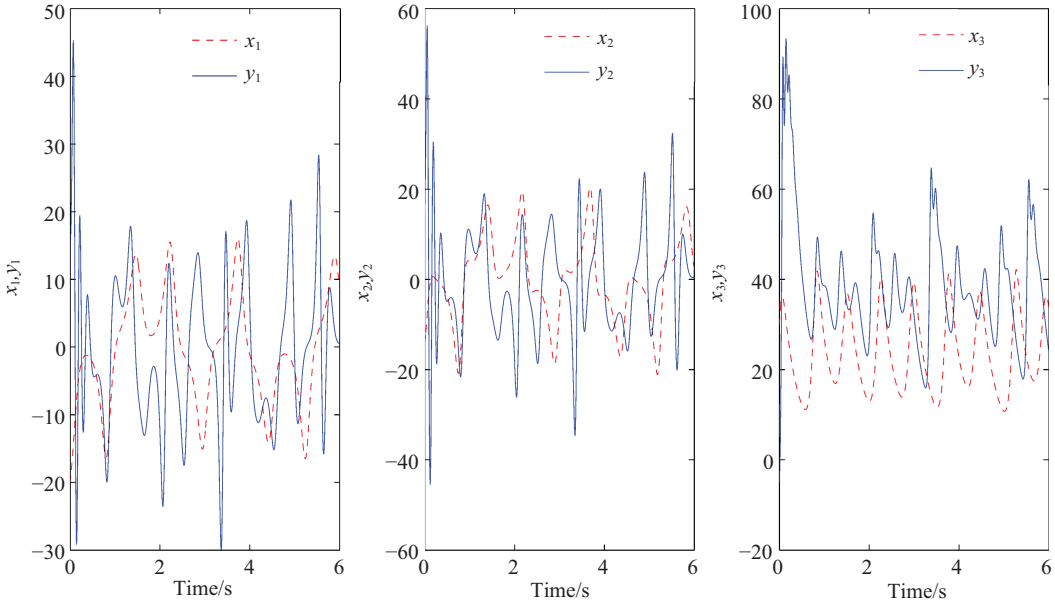


Fig. 1. (color online) The chaotic curves of the driving (x_1, x_2, x_3) and response (y_1, y_2, y_3) systems.

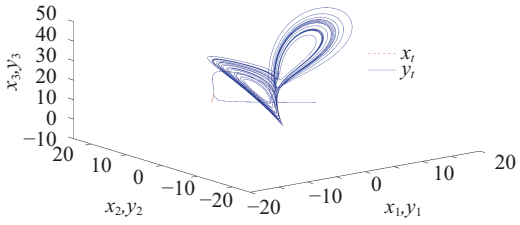


Fig. 2. (color online) The synchronization curves of the driving (x_t) and response (y_t) systems.

Case 2 Simulation with uncertainty and disturbance

Suppose that the pre-estimated model parameters are $\tilde{\beta}_i = 1.05\beta_i$, $\tilde{\gamma}_i = 0.95\gamma_i$ $(i = 1,2,3)$, and for simplicity but without loss of generality, we suppose the external disturbances to the driving system and the response system are $d_1 = [0,0,0]^{\mathrm{T}}$ and $d_2 = [10 + 10\sin(3t), 10 + 10\cos(t/2), 10 + 10\cos(t/4)]^{\mathrm{T}}$, respectively, i.e., there are no external disturbances acting on the driving system and the external disturbances on the response system are time-varying.

Figure 4 shows the chaotic curves of x_i and y_i $(i = 1,2,3)$ under the aforementioned disturbances. Comparing Fig. 4 with Fig. 1, we can see that the curves of the response system (fractional-order Chen chaotic system) are changed because of the parameters' uncertainty and the external disturbances introduced. The trajectories of the fractional-order Lorenz and Chen systems are shown in Fig. 5, which also indicates that the fractional-order Lorenz and Chen systems with uncertainty and disturbance are indeed synchronized. Figure 6 demonstrates that after a finite period of time, the initial error will converge to zero and the synchronization between the driving and response systems can be achieved.

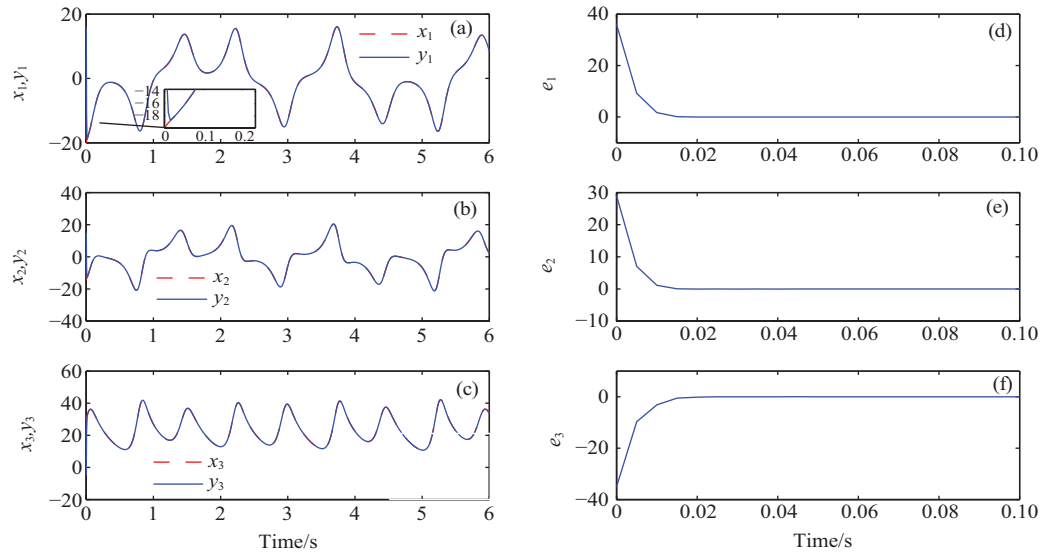


Fig. 3. (color online) Synchronization results of the driving and response systems: (a) x_1, y_1; (b) x_2, y_2; (c) x_3, y_3; (d) e_1; (e) e_2; (f) e_3.

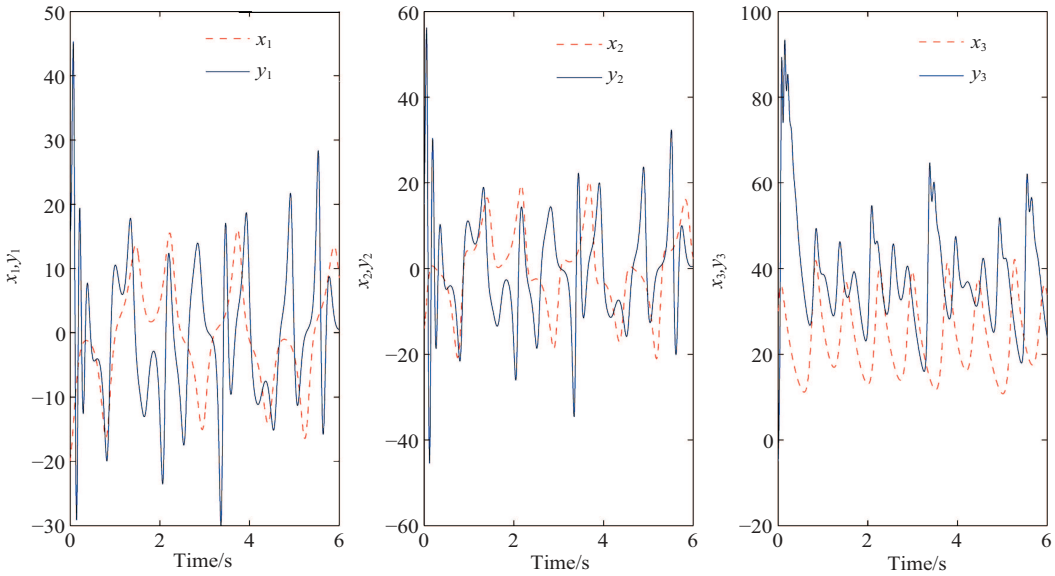


Fig. 4. (color online) Chaotic curves of the driving (x_1, x_2, x_3) and response (y_1, y_2, y_3) systems with uncertainty and disturbance.

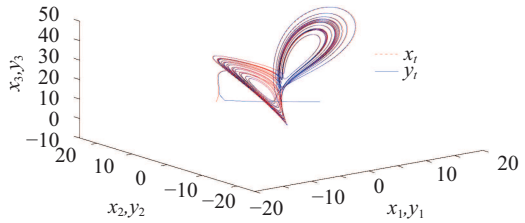
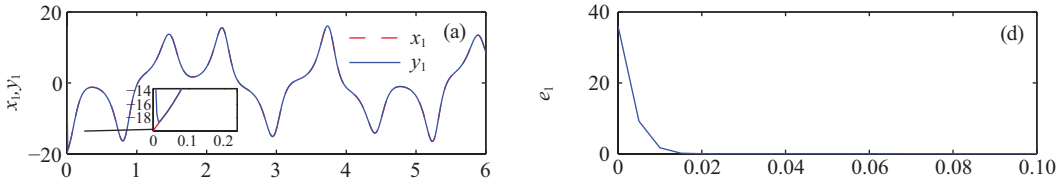


Fig. 5. (color online) Synchronization curves of the driving (x_t) and response (y_t) systems with uncertainty and disturbance.



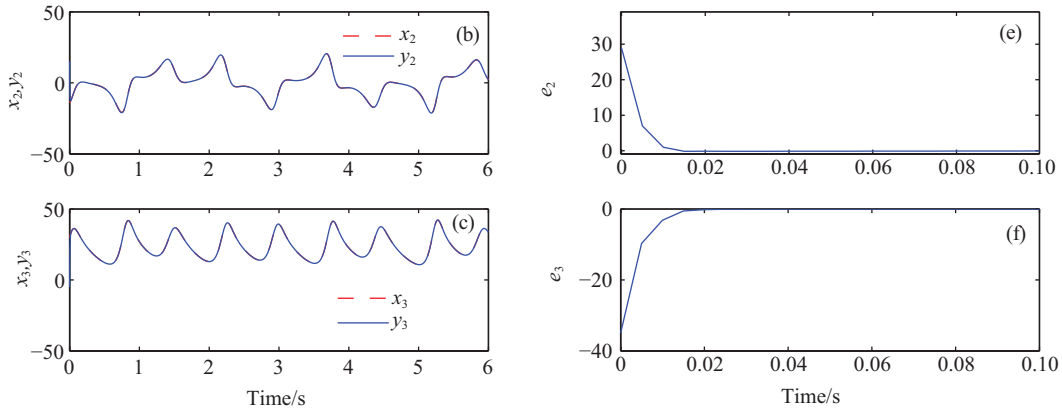


Fig. 6. (color online) Synchronization results of the driving and response systems with uncertainty and disturbance: (a) x_1, y_1; (b) x_2, y_2; (c) x_3, y_3; (d) e_1; (e) e_2; (f) e_3.

4. Conclusions

In this article, a fractional-order terminal sliding mode control technique is proposed to robustly synchronize two fractional-order chaotic systems with parameter uncertainty and external disturbances. Based on fractional-order Lyapunov stability theory, a robust sliding control law is derived to guarantee the occurrence of the sliding motion in finite time. The simulation results for the synchronization of the fractional-order Lorenz and Chen systems show the applicability of the approach. In particular, the introduced fractional-order terminal sliding mode controller is applicable for a large class of fractional-order chaotic systems with parameter uncertainty and external disturbances.

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