Complex dynamical behavior and chaos control in fractional-order Lorenz-like systems*

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In this paper, the complex dynamical behavior of a fractional-order Lorenz-like system with two quadratic terms is investigated. The existence and uniqueness of solutions for this system are proved, and the stabilities of the equilibrium points are analyzed as one of the system parameters changes. The pitchfork bifurcation is discussed for the first time, and the necessary conditions for the commensurate and incommensurate fractional-order systems to remain in chaos are derived. The largest Lyapunov exponents and phase portraits are given to check the existence of chaos. Finally, the sliding mode control law is provided to make the states of the Lorenz-like system asymptotically stable. Numerical simulation results show that the presented approach can effectively guide chaotic trajectories to the unstable equilibrium points.

Keywords: fractional-order Lorenz-like system, stability analysis, pitchfork bifurcation, chaos control

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1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary (non-integer) order. In recent years, numerous studies and applications of fractional-order systems in many areas of science and engineering have been presented.[1–3] This is a result of better understanding of the potential of fractional calculus revealed by problems such as viscoelasticity and damping, chaos, diffusion, wave propagation, percolation, and irreversibility.[4] So far, many systems have been identified as able to display fractional-order dynamics, such as electromagnetism,[5] viscoelastic systems,[6] psychological and life sciences,[7] diffusion processes,[8] mechanics,[9] financial mathematics,[10] and so on. Virtually no area of classical analysis has been left untouched by fractional calculus.[11]

Moreover, in the last decade many researchers have found that chaotic attractors indeed exist in fractional-order systems,[11–18] but due to the essential differences between ordinary differential equations (ODE) and fractional-order differential equations (FODE), most of conclusions of the ODE systems cannot be directly extended to the case of FODE systems. For example, chaos cannot occur in autonomous continuous-time systems of integer-order less than three according to the Poincaré–Bendixon theorem,[19] but this is not the case for fractional-order systems.[11–18] Besides, the stability analysis and numerical algorithm for fractional-order systems are also more complex than for the corresponding integer-order systems.[1–3,11–18] So there are many open problems to be dealt with when discussing the chaotic behavior of fractional-order systems.

In 2006, Liu *et al.*[20] found a new butterfly-shaped attractor in a three-dimensional autonomous system with two square nonlinear terms. A natural question is whether there are chaotic behaviors in the corresponding fractional-order system. If so, we need to know what is the lowest order at which the chaos is present, how the chaos comes into being, and further, how it can be controlled.

In this work we attempt to provide some answers to the above questions. The rest of the paper is organized as follows. In Section 2, basic definitions and lemmas in fractional calculus are presented. In Section 3, the fractional-order Lorenz-like system is introduced, then the stability and bifurcation analysis are presented. In Section 4, the chaotic behavior in the cases of the commensurate order system and incommensurate order are discussed. In Section 5, the fraction-integer integral switching surface is designed to achieve chaos control. Numerical simulations are performed in Sections 3–5, and some concluding remarks are provided in Section 6.

2. The fractional derivative and stability theorem

At present, there are several definitions of fractional derivatives, and the most widely used definitions are the Riemann–Liouville version[1–3]

$$D^q x(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m} J^{m-q} x(t), \quad q > 0,$$

and the Caputo one[1–3]

$$D^q x(t) = J^{m-q} x^{(m)}(t), \quad q > 0,$$

where m is the minimal integer which is not less than q, $x^m(t)$ is a general m-order derivative, and

$$J^p x(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} x(\tau)\,\mathrm{d}\tau, \quad p > 0,$$

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where $\Gamma(\cdot)$ is the gamma function. Due to the fact that the initial values of the integer-order derivatives are measurable in real applications and nonhomogeneous initial conditions are allowed, the Caputo fractional derivative is adopted in this paper.

Lemma 1[21] For the following scalar nonlinear fractional-order system:

$$\frac{\mathrm{d}^q x}{\mathrm{d}t^q} = f(x), \tag{1}$$

where $0 < q < 1$ and $f'(x) \in C[0,+\infty]$. Assume that $\bar{x} = 0$ is the equilibrium points of system (1), then $\bar{x} = 0$ is locally asymptotically stable if $\lambda = f'(0) < 0$.

Remark 1 For the stability analysis of the other equilibrium point $\bar{x}$, one can apply the above lemma by coordinate transformation $\tilde{x} = x - \bar{x}$.

Remark 2 For the following n-dimensional nonlinear fractional-order system:

$$\frac{\mathrm{d}^q x_i}{\mathrm{d}t^q} = f_i(x), \quad i = 1,2,\ldots,n, \tag{2}$$

where $0 < q < 1$ and $f_i'(x) \in C[0,+\infty], i = 1,\ldots n$. Assume that $\bar{x} = 0$ is the equilibrium points of system (2), then $\bar{x} = 0$ is locally asymptotically stable if all the eigenvalues $(\lambda_i, i = 1,\ldots,n)$ of the Jacobian matrix evaluated at the equilibrium points satisfy: $\mathrm{Re}(\lambda_i) < 0$.

Remark 3 For the n-dimensional incommensurate nonlinear fractional-order system, it can be changed into the equivalent commensurate fractional-order system using the technique in Ref. [22].

3. System description and stability analysis

The fractional-order Lorenz-like dynamic system is described as[20]

$$\begin{aligned}
\frac{\mathrm{d}^{q_1} x_1}{\mathrm{d}t^{q_1}} &= a(x_2 - x_1),\\
\frac{\mathrm{d}^{q_2} x_2}{\mathrm{d}t^{q_2}} &= bx_1 - lx_1x_3,\\
\frac{\mathrm{d}^{q_3} x_3}{\mathrm{d}t^{q_3}} &= -cx_3 + hx_1^2 + kx_2^2,
\end{aligned} \tag{3}$$

where $\mathrm{d}^{q_i}/\mathrm{d}t^{q_i} = D^{q_i}$, $q_i \in (0,1)$ and a, c, l, h, k are all positive real parameters and $b \in R$.

3.1. The existence and uniqueness of the solution

Theorem 1 The initial value problem of the commensurate order system can be rewritten as follows:

$$D^q x(t) = Ax(t) + x_1(t)Bx(t) + x_2(t)Cx(t), \quad x(0) = x_0, \tag{4}$$

where $0 < t \le T$, $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}} \in R^3$, $x(0) = (x_{10}, x_{20}, x_{30})^{\mathrm{T}}$,

$$A = \begin{pmatrix} -a & a & 0 \\ b & 0 & 0 \\ 0 & 0 & -c \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -l \\ h & 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & k & 0 \end{pmatrix},$$

then for constant $T = \min\{T^*, (\varepsilon\Gamma(q+1)/||f||_\infty)^{1/q}\}$, $T^* > 0$, fractional-order system (4) has a unique solution.

Proof Consider function $f(x(t)) = Ax(t) + x_1(t)Bx(t) + x_2(t)Cx(t)$, which is continuous and bounded on the interval $x(t) \in [0,T^*] \times [x_0 - \varepsilon, x_0 + \varepsilon]$ for any $T^*, \varepsilon > 0$. Furthermore, one has

$$\begin{aligned}
|f(x) - f(y)| &= |A(x-y) + x_1Bx - y_1By + x_2Cx - y_2Cy| \\
&\le \|A\||(x-y)| + |x_1Bx - y_1By| \\
&\quad + |x_2Cx - y_2Cy|,
\end{aligned} \tag{5}$$

where $y(t) \in [0,T^*] \times [x_0 - \varepsilon, x_0 + \varepsilon]$ and $|\cdot|$ represent the matrix and vector norm, respectively. It is obvious that

$$\begin{aligned}
|x_1Bx - y_1By| &= |x_1Bx - y_1Bx + y_1Bx - y_1By| \\
&= |(x_1 - y_1)Bx + y_1B(x-y)| \\
&\le \|B\|(|x| + |y_1|)(|x-y|).
\end{aligned}$$

In a similar way to the above derivation, one has the following result:

$$|x_2Cx - y_2Cy| \le \|C\|(|x| + |y_2|)(|x-y|).$$

From inequality (5), we can obtain

$$\begin{aligned}
&|f(x) - f(y)| \\
&\le [\|A\| + \|B\|(|x| + |y_1|) + \|C\|(|x| + |y_2|)](|x-y|) \\
&\le [\|A\| + (\|B\| + \|C\|)(2|x_0| + 2\varepsilon)](|x-y|) \\
&\le L(|x-y|),
\end{aligned}$$

where $L = \|A\| + (\|B\| + \|C\|)(2|x_0| + 2\varepsilon) > 0$. In other words, the fractional-order Lorenz-like system meets the Lipschitz condition. Then, according to the existence and uniqueness theorem of the fractional-order system,[23] the initial value problem of the commensurate order system (4) has a unique solution in the interval $T = \min\{T^*, (\varepsilon\Gamma(q+1)/||f||_\infty)^{1/q}\}$.

3.2. The stability conditions of the equilibrium points

The fractional-order system (3) has three equilibrium points, that is,

$$E_0 = (0,0,0), \quad E_1 = \left(\sqrt{\frac{bc}{l(h+k)}}, \sqrt{\frac{bc}{l(h+k)}}, \frac{b}{l}\right),$$

$$E_2 = \left(-\sqrt{\frac{bc}{l(h+k)}}, -\sqrt{\frac{bc}{l(h+k)}}, \frac{b}{l}\right).$$

In order to analyze the local stabilities of the above equilibrium points, the Jacobian matrix is given as

$$J|_{(x,y,z)} = \begin{bmatrix} -a & a & 0 \\ b - lx_3 & 0 & -lx_1 \\ 2hx_1 & 2kx_2 & -c \end{bmatrix}.$$

3.2.1. The stability conditions of equilibrium E_0

First, we consider the case of $b < 0$, where system (3) has only one real equilibrium point E_0 at this moment. The characteristic polynomial of $J(E_0)$ is given by

$$\lambda^3 + (a+c)\lambda^2 + (ac - ab)\lambda - abc = 0. \tag{6}$$

It is obvious that the eigenvalues of Eq. (6) are $\lambda_1 = -c$ and $\lambda_{2,3} = (-a \pm \sqrt{a^2+4ab})/2$. Then according to Lemma 1, one can easily obtain the following conclusion.

Proposition 1 The equilibrium E_0 is stable with $q \in (0,1)$ for $b < 0$.

3.2.2. The stability conditions of equilibrium points $E_{1,2}$

When $b > 0$, E_0 becomes unstable and two other equilibrium points, E_1 and E_2, appear. Both E_1 and E_2 have the same characteristic equation, that is

$$\lambda^3 + (a+c)\lambda^2 + (ac + 2\frac{bck}{h+k})\lambda + 2abc = 0. \tag{7}$$

The eigenvalues of Eq. (7), which are dependent on system parameters a, b, c, l, h, and k, can be obtained by the Cardan formula or Shengjin's formula. Since a, c, l, h, and k are all positive real parameters, one can ensure that equation (7) has at least an eigenvalue with a negative real part as $b > 0$. And the other two eigenvalues have the following four cases:

(I) two negative real roots;

(II) two positive real roots;

(III) two complex-conjugate roots with a negative real part;

(IV) two complex-conjugate roots with a positive real part.

In the cases of (I)–(III), the stabilities of equilibrium points $E_{1,2}$ of the fractional-order system are in accordance with the corresponding integer-order system. But in the case of (IV), the stabilities of $E_{1,2}$ rely on differential order q.

To determine the stability conditions for equilibrium points $E_{1,2}$, we first consider the integer-order case. Based on the Routh–Hurwitz criterion, it is easy to deduce that the equilibrium points are locally asymptotically stable if and only if $b < b_c$, where

$$b_c = \frac{a(a+c)(h+k)}{2(ah-ck)}$$

and $ah - ck > 0$. It is well known that the fractional-order system is at least as stable as their integer-order counterpart, so one can obtain the following conclusion.

Proposition 2 The equilibrium $E_{1,2}$ is stable with $q \in (0,1)$ for $b < b_c$.

When $b > b_c$, the equilibrium points $E_{1,2}$ become unstable in an integer-order system but may be stable in the fractional-order case. Then, one can obtain the stability condition according to Lemma 1, and a detailed analysis is given below by numerical simulation.

3.3. Numerical methods and results

The aim of this section is to numerically seek the effect of system parameter b and differential order q on the dynamic behavior of the fractional-order system. However, the numerical calculation of a fractional differential equation is not so simple as that of an ordinary differential equation. An Adams-type predictor–corrector method was introduced in Ref. [24] and investigated further in Refs. [25]–[27]. In the present paper, the improved Adams–Bashforth–Moulton algorithm[27] will be adopted for the numerical solution of the fractional integral equation. In the simulations, the rest of the system parameters are fixed as $(a,c,l,h,k) = (10.0, 2.5, 1.0, 2.0, 2.0)$ and the initial conditions are taken as $x(0) = (2.2, 2.4, 28.0)$, unless otherwise stated.

Setting $q = 0.98$ and $b = -5.0$, the phase portraits in the x_1–x_2 plane with different values of parameters b and q are plotted in Figs. 1 and 2, respectively. From Figs. 1 and 2, one can find that the trajectories of the fractional-order system converge to the equilibrium point E_0 with $b < 0$ and $q \in (0,1)$, which tests the validity of Proposition 1. For fixed value q, the convergence speed is faster as b increases; but for fixed value b, the convergence speed is slower as q increases.

To test the validity of Proposition 2, a series of analogous numerical experiments are carried out, but the results are more complex. Setting $q = 0.98$, from Figs. 3 and 4, one can see that the trajectories of the fractional-order system converge to the equilibrium point E_1 or E_2 with $b < b_c = 50/3$. And the orbits take more time to converge with b approaching critical value b_c. Setting $b = 12.0$, we also find that the convergence speed is slower as q increases, as shown in Fig. 5.

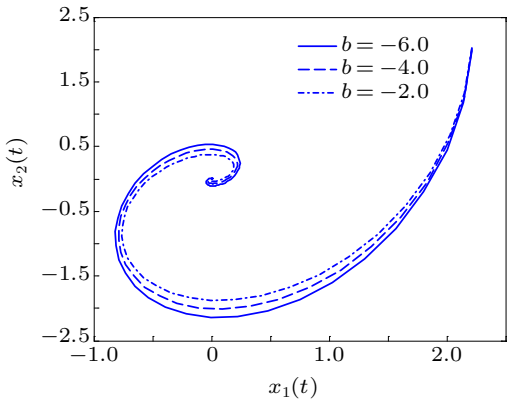


Fig. 1. State trajectories of the fractional-order system converging to the equilibrium E_0 for different values of parameter b with $q = 0.98$.

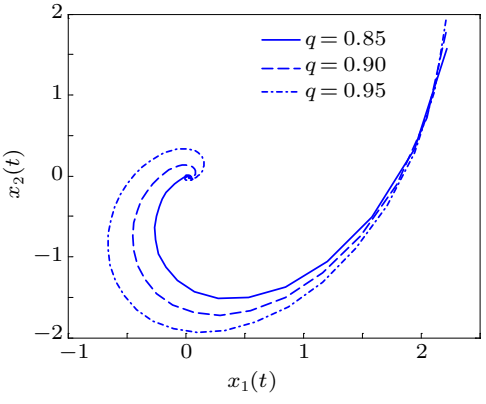


Fig. 2. State trajectories of the fractional-order system converging to the equilibrium E_0 for different values of q with $b = -5.0$.

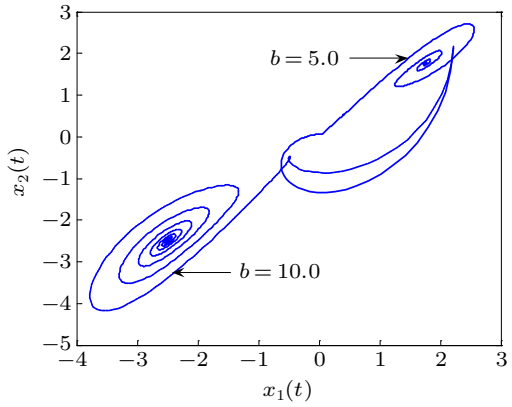


Fig. 3. State trajectories of the fractional-order system converging to the equilibrium E_1 or E_2 for $b = 5.0$ and $b = 10.0$ with $q = 0.98$.

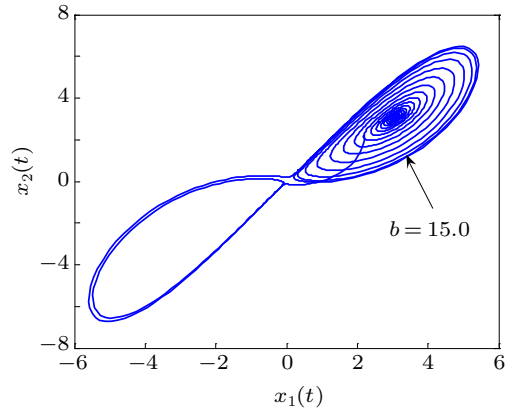


Fig. 4. State trajectories of the fractional-order system converging to the equilibrium E_1 for $b = 15.0$ with $q = 0.98$.

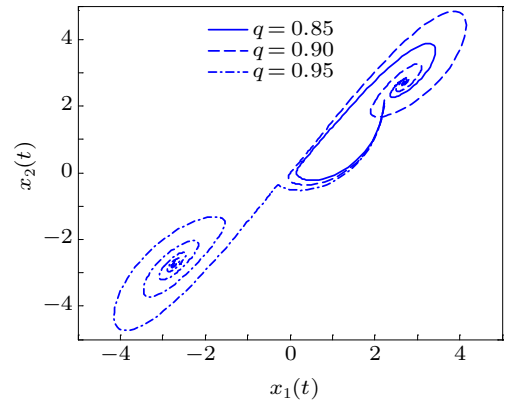


Fig. 5. State trajectories of the fractional-order system converging to the equilibrium E_1 or E_2 for different values of q with $b = 12.0$.

3.4. Bifurcation analysis

By changing the bifurcation parameter b, from the above analysis, it is clear that system (3) has a unique locally asymptotically stable equilibrium point E_0 as $b < 0$. When $b > 0$, the equilibrium point E_0 becomes unstable and two other equilibrium points, $E_{1,2}$, appear and are locally asymptotically stable as $b < b_c$. Thus, we have the following conclusion.

Proposition 3 The fractional-order system undergoes a pitchfork bifurcation[28] at $b = 0$ for positive parameters a, c, l, h, and k.

4. Chaotic behavior of the fractional-order Lorenz-like system

It is known that the necessary condition for the fractional-order system to exhibit a chaotic attractor is the instability of the equilibrium points. Otherwise, one of these equilibrium points will become asymptotically stable and attract the nearby trajectories. Then, we will continue to explore the stabilities of equilibrium points as parameter b increases. Let b vary from 10 to 300. The stable region for equilibrium point E_1 is provided based on Lemma 1, which has been drawn in Fig. 6. It is not hard to find that the curve descends quickly before about $b = 150$ and then ascends slowly. That is to say, the effect of parameter b on equilibrium point stability becomes small as b increases.

Next, we will identify the condition of differential order q for system (3) to show chaotic behavior with fixed parameter b. Consider $b = 40$, then system (3) will have three equilibrium points: $E_0(0,0,0)$, $E_{1,2}(\pm 5, \pm 5, 40)$, among which E_0 is a saddle node and $E_{1,2}$ are saddle focus nodes. But for the fractional-order system, the saddle focus node may be stable with suitable differential order q. While the saddle focus node becomes unstable, system (3) may show a double-scroll chaotic attractor.

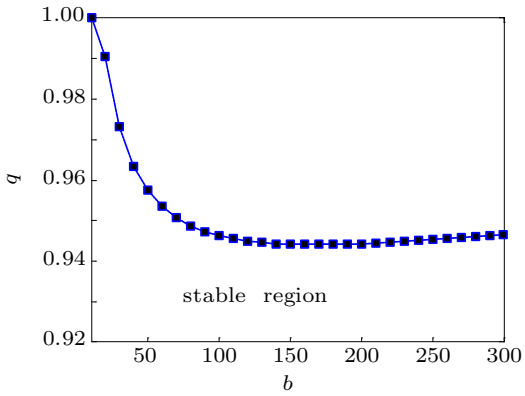


Fig. 6. The stable region of equilibrium E_1 as parameter b increases.

4.1. Commensurate order system ($q_1 = q_2 = q_3 = q$)

The largest Lyapunov exponents[29] for system (3) with distinct differential orders are shown in Fig. 7, from which one can see that when $q = 0.6$, system (3) shows chaotic behavior.

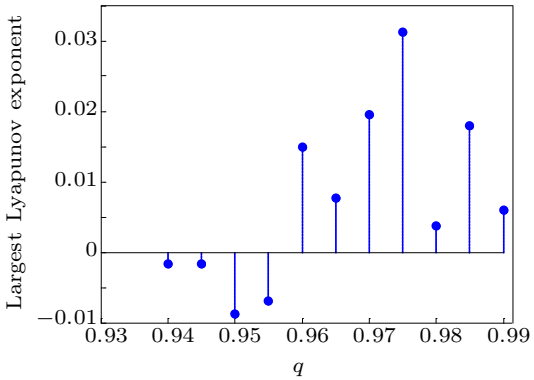


Fig. 7. The largest Lyapunov exponent of the fractional-order system versus q.

Furthermore, the phase portraits in the xyz-plane for specific values of q are provided in Figs. 8(a)–8(c). For $q = 0.94$, the orbit of system (3) converges asymptotically to the equilibrium point $E_2(5,5,40)$. For $q = 0.96$ and $q = 0.98$, the orbits of system (3) form chaotic attractors.

4.2. The incommensurate order system

Due to the fact that the stability analysis for the incommensurate order system is more complex than the commensurate case, here we only consider six groups of typical differential order values, which are given in Table 1. At the same time, the largest Lyapunov exponent is computed to determine the dynamical behavior of the incommensurate order system. Then, according to Lemma 1, one can easily find the following conclusion.

Proposition 4 For the incommensurate order system, when $q_{1,2,3} < 0.96$, the equilibrium points of system (3) are locally asymptotically stable and when $q_{1,2,3} > 0.96$, the equilibrium points are unstable and system (3) ultimately forms the chaotic attractor.

The numerical results for the first case (i) and the last case (vi) in Table 1 are indeed in keeping with the above conclusion. Instead, when $q_i < 0.96 < q_j$, $(i \neq j, i, j = 1,2,3)$, one may judge the dynamic behavior of system (3) according to the numerical study. The remaining cases (ii)–(v) in Table 1 present the final results. In addition, figures 9(a)–9(f) display the phase portraits in the xyz-plane for several cases corresponding to Table 1.

Table 1. The largest Lyapunov exponents for the incommensurate order system with different values of q.

Differential order (q_1, q_2, q_3)	The largest Lyapunov exponent
(i) $q_1 = 0.92, q_2 = 0.93, q_3 = 0.94$	−0.0017
(ii) $q_1 = 0.93, q_2 = 0.95, q_3 = 0.97$	−0.0056
(iii) $q_1 = 0.95, q_2 = 0.97, q_3 = 0.96$	0.0343
(iv) $q_1 = 0.97, q_2 = 0.98, q_3 = 0.94$	−0.0032
(v) $q_1 = 0.97, q_2 = 0.98, q_3 = 0.95$	0.0096
(vi) $q_1 = 0.97, q_2 = 0.98, q_3 = 0.99$	0.0134

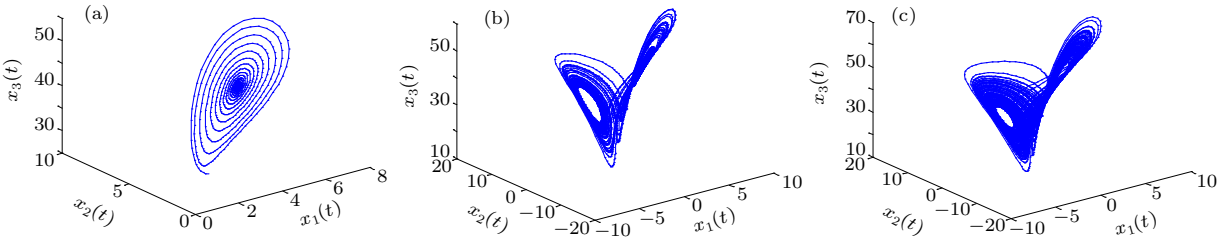


Fig. 8. Variations in the phase portraits of the commensurate order system with values of q. (a) $q = 0.94$; (b) $q = 0.96$; (c) $q = 0.98$.

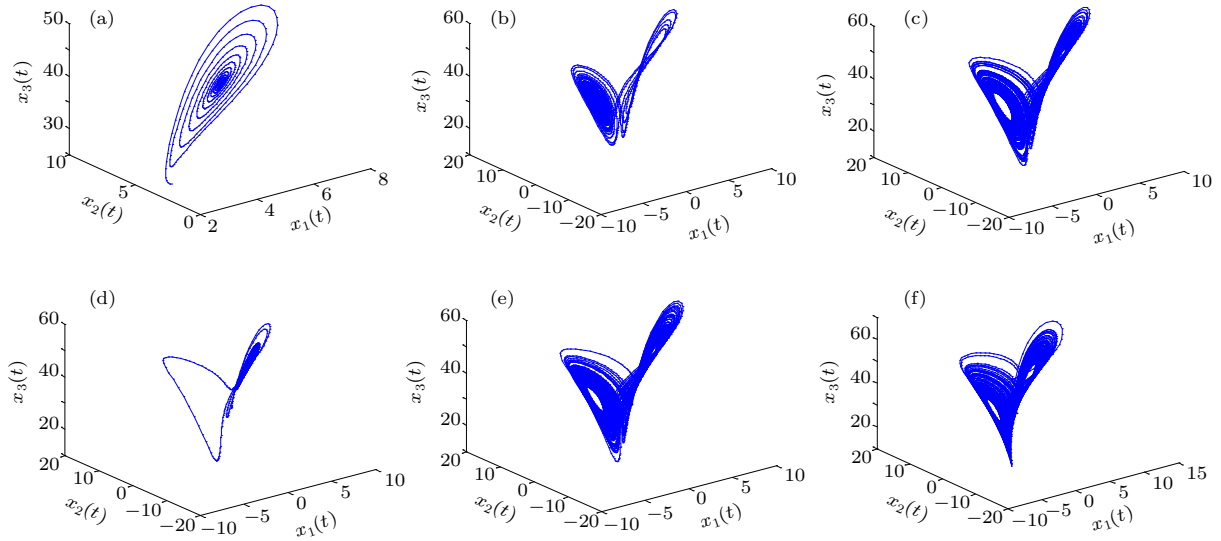


Fig. 9. Phase portraits of the incommensurate order system with different values of q.

5. Chaos control for the fractional-order Lorenz-like system

The feature that the chaotic system is extremely sensitive to the initial conditions makes chaos unwanted in the ordinary course of events. So, in this section we will consider the problem of controlling chaotic trajectories in the fractional-order Lorenz-like system.

5.1. The sliding mode control design

Now, assume that the control target is the unstable equilibrium point; here we denote it as $\bar{x}$. Define the state errors between the fractional-order system (3) and the control target as $y_i = x_i - \bar{x}_i$ $(i = 1,2,3)$, then the error dynamical system is

easily obtained as follows:

$$\begin{aligned}
\frac{\mathrm{d}^{q_1}y_1}{\mathrm{d}t^{q_1}} &= a(y_2+\bar{x}_2)-a(y_1+\bar{x}_1),\\
\frac{\mathrm{d}^{q_2}y_2}{\mathrm{d}t^{q_2}} &= b(y_1+\bar{x}_1)-l(y_1+\bar{x}_1)(y_3+\bar{x}_3)+u(t),\\
\frac{\mathrm{d}^{q_3}y_3}{\mathrm{d}t^{q_3}} &= -c(y_3+\bar{x}_3)+h(y_1+\bar{x}_1)^2+k(y_2+\bar{x}_2)^2. \quad (8)
\end{aligned}$$

At present, a variety of nonlinear control methods have been put forward to achieve chaos control.[30–33] Here, sliding mode control will be adopted to achieve chaos control, and the control input is added to the second state equation. The sliding mode control design procedure[34–36] has two steps: first, a sliding surface that represents the desired system dynamics should be constructed. Second, a switch control law should be developed such that the sliding mode exists on every point in the sliding surface, and any state outside the surface is driven to reach the surface in a finite time. As a choice for the sliding surface, one has

$$\begin{aligned}
s(t) &= D^{q_2-1}y_2(t)\\
&\quad +\int_0^t l\left[(y_1(\tau)y_3(\tau)+y_1(\tau)\bar{x}_3+y_3(\tau)\bar{x}_1)+ky_2(\tau)\right]\mathrm{d}\tau, \quad (9)
\end{aligned}$$

where $D^{q_2}=(\mathrm{d}^{q_2})/(\mathrm{d}t^{q_2})$ and k is an arbitrary positive constant. For the sliding mode approach, the sliding surface and its derivative must satisfy

$$s(t)=0, \quad \frac{\mathrm{d}s(t)}{\mathrm{d}t}=0. \quad (10)$$

So, the following equation can be obtained

$$D^{q_2}y_2(t) = -l(y_1(t)y_3(t)+y_1(t)\bar{x}_3+y_3(t)\bar{x}_1)-ky_2(t). \quad (11)$$

According to Eqs. (8) and (11), the equivalent control law is derived as

$$u_{eq}(t) = -b(y_1(t)+\bar{x}_1)+l\bar{x}_1\bar{x}_3-ky_2(t).$$

To satisfy the sliding condition, the discontinuous reaching law is chosen as follows:

$$u_r(t) = -\rho\,\mathrm{sign(s)}-\gamma s,$$

where

$$\mathrm{sign(s)} = \begin{cases} +1, & s>0\\ 0, & s=0\\ -1, & s>0 \end{cases},$$

ρ and γ are all positive constants. Finally, the total control law can be determined as follows:

$$\begin{aligned}
u(t) &= u_{eq}(t)+u_r(t)\\
&= -b(y_1(t)+\bar{x}_1)+l\bar{x}_1\bar{x}_3-ky_2(t)-\rho\,\mathrm{sign(s)}-\gamma s. \quad (12)
\end{aligned}$$

Theorem 2 The chaotic fractional-order system (3) can be controlled to the unstable equilibrium points asymptotically with controller (12).

Proof Consider a Lyapunov candidate as follows:

$$V_{\mathrm{SMC}} = \frac{1}{2}s^2(t),$$

then the time derivative of the Lyapunov function will be given by

$$\begin{aligned}
\dot{V}_{\mathrm{SMC}}(t) &= s(t)\dot{s}(t)\\
&= D^{q_2}y_2(t)+l(y_1(t)y_3(t)+y_1(t)\bar{x}_3\\
&\quad +y_3(t)\bar{x}_1)+ky_2(t)\\
&= s\left[b(y_1+\bar{x}_1)-l(y_1+\bar{x}_1)(y_3+\bar{x}_3)+u(t)\right.\\
&\quad \left.+l(y_1(t)y_3(t)+y_1(t)\bar{x}_3+y_3(t)\bar{x}_1)+ky_2(t)\right]\\
&= s[-\rho\,\mathrm{sign(s)}-\lambda s]\\
&= -\rho|s|-\lambda s^2<0.
\end{aligned}$$

It is easy to see that the sufficient condition $s(t)\dot{s}(t)<0$ is satisfied for the states to move from the reaching phase to the sliding phase, which can guarantee the existence of the sliding mode dynamics. Moreover, by the Lyapunov theorem, the closed loop system (8) is asymptotically stable and the chaotic behavior of the fractional-order system can be controlled to the assigned target.

5.2. Numerical simulations

In this section, we take some examples to demonstrate the effectiveness of the proposed control schemes. Numerical simulations are made with the initial value $x(0)=(2.2,2.4,28.0)^{\mathrm{T}}$ and the parameters $(a,b,c,l,h,k)=(10.0,40.0,2.5,1.0,2.0,2.0)$. The parameters of controller (12) are chosen as $k=3.0$, $\rho=2.0$, $\gamma=1.0$ and the control inputs are activated at $t=20$.

5.2.1. Commensurate case

Set $q_1=q_2=q_3=0.98$, then one will obtain a commensurate order system, which can exhibit the chaotic behavior as shown in Fig. 8(c). Assume that the control target is $E_1(5,5,40)$, then the simulation results will be displayed in Fig. 10.

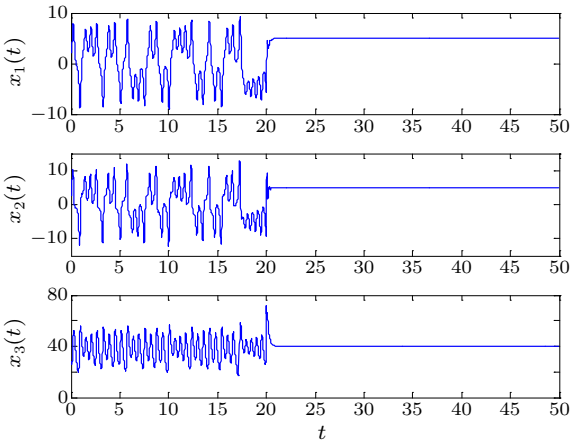


Fig. 10. The time histories of the controlled commensurate order system converging to the unstable equilibrium point E_1.

5.2.2. Incommensurate case

Set $q_1=0.97$, $q_2=0.98$, and $q_3=0.99$, then one will obtain an incommensurate order system, which can also exhibit the chaotic behavior as shown in Fig. 9(f). Assume that

the control target is $E_2(-5,-5,40)$, then the simulation results will be displayed in Fig. 11.

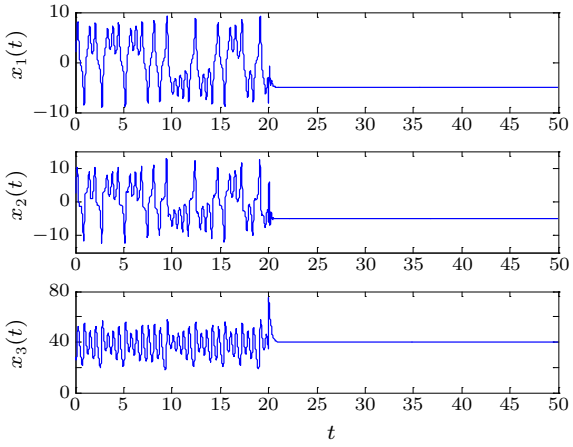


Fig. 11. The time histories of the controlled incommensurate order system converging to the unstable equilibrium point E_2.

From the above simulation results, one can see that once the controller comes into play, the chaotic trajectories of the system will be guided to the control target quickly, which shows that the theoretical estimates obtained in this paper are feasible to achieving good performance of either the commensurate order system or the incommensurate case.

6. Conclusions

In this work, we mainly discussed the complex dynamical behavior of the fractional-order Lorenz-like system and its control. Firstly, the existence and uniqueness of solutions to this fractional-order system are studied. Special attention is paid to the stability analysis of equilibrium points with different system parameters. When b varies from negative to positive, the number of equilibrium points in the system is changed. As b continues to increase to a critical value, the stabilities of the equilibrium points are changed. The above results are supported by the Routh–Hurwitz criterion and numerical simulations.

The necessary condition of chaos occurrence is obtained in the event of the commensurate order system and the incommensurate case. In fact, the fractional-order Lorenz-like system is chaotic, with an order value of less than 2.88. Furthermore, sufficient conditions for controlling the chaotic behavior of the system are obtained using the sliding mode technique. Numerical simulation results show the effectiveness of the proposed method.

Although the control method in this paper is fast and effective, the chattering phenomenon is the inherent vice. Hence, how to reduce or avoid chattering in the control process is a key issue. In particular, when the system is subjected to an external disturbance, the above problem seems to be extremely important. In a word, how to propose a more effective new approach and improve the control performance of the chaotic fractional-order system are our future research topics.

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