Incomplete nonextensive statistics and the zeroth law of thermodynamics*

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On the basis of the entropy of incomplete statistics (IS) and the joint probability factorization condition, two controversial problems existing in IS are investigated: one is what expression of the internal energy is reasonable for a composite system and the other is whether the traditional zeroth law of thermodynamics is suitable for IS. Some new equivalent expressions of the internal energy of a composite system are derived through accurate mathematical calculation. Moreover, a self-consistent calculation is used to expound that the zeroth law of thermodynamics is also suitable for IS, but it cannot be proven theoretically. Finally, it is pointed out that the generalized zeroth law of thermodynamics for incomplete nonextensive statistics is unnecessary and the nonextensive assumptions for the composite internal energy will lead to mathematical contradiction.

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1. Introduction

As is well known, in spite of its great success, Boltzman–Gibbs statistical mechanics are actually not completely universal. Nonextensive statistical mechanics[1–3] pioneered by Tsallis[4] offers a consistent theoretical framework for the studies of complex systems with long-range interactions, long-time memories, multifractal and self-similar structures, or anomalous diffusion phenomena. On the basis of Tsallis' statistics, Wang[5] put forward the concept of incomplete statistics (IS). At present Tsallis' statistics and IS have become two important branches of nonextensive statistical mechanics.[6–20] Recently, IS has been used to research the thermostatistic properties of a variety of physical systems with long-range interacting and/or long-duration memory, and many significant results have been obtained.[9–19] For example, it has been found that for some chaotic systems evolving in fractal phase space,[10,11] the entropy change in time due to the fractal geometry is assimilated to the information growth through the scale refinement; and that the generalized fermion distributions based on incomplete information hypothesis can be useful for describing correlated electron systems.[12,13] However, when investigating the thermostatistic properties of a composite system, especially for nonextensive systems with different values of q indices,[11,18] one needs to deal with two fundamental problems which are similar to those solved recently in Tsallis' statistics,[21] one of which is how to give a reasonable expression of the internal energy of a composite system and the other is how to expound whether the zeroth law of thermodynamics is true. Although these two problems have been discussed for many years,[5,9,11,14] they have not been solved up to now, and consequently, have affected the development and improvement of incomplete nonextensive statistics. Recently, a concomitant definition of the physical temperature in IS is given,[19] which may shed light on the further investigation of the problems existing in IS. In the present paper, with the help of the results obtained in Refs. [5] and [19] and the joint probability factorization condition, we will discuss the two important problems mentioned above and give some useful conclusions.

2. Temperature in IS

According to the results of IS proposed by Wang,[5] the entropy for a nonextensive system may be expressed as

$$S_q = k\frac{\sum_{i=1}^{w} p_i - 1}{q-1}, \tag{1}$$

with the incomplete normalization

$$\sum_{i}^{w} p_i^q = 1, \tag{2}$$

where k is the Boltzmann constant, p_i is the probability of the state i among W possible states that are acceptable to the calculation, ε_i is the energy of the system in state i, and q is a parameter which may be used to describe the nonextensility of

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the system. By the way, the so-called nonextensibility of the system means that $X \neq \sum_l X_l$, where X and X_l indicate the corresponding thermodynamic quantities of the composite system and subsystems, respectively. When the nonextensibility of the system is negligible, $X = \sum_l X_l$ is true. For convenience, $\sum_{i=1}^{w}$ is replaced by $\sum_i$ below.

With Eqs. (1) and (2) and the expression of internal energy

$$U_q = \sum_i^w p_i^q \varepsilon_i, \tag{3}$$

Wang[5] used the Lagrange equation

$$\delta\left(S_q + \frac{\alpha}{1-q}\sum_i p_i^q - \alpha\beta U_q\right) = 0$$

with the Lagrange multiplier $\alpha\beta$ of the average energy to derive the distribution function as[5,7,9,14,16,17]

$$p_i = \frac{[1-(1-q)\beta\varepsilon_i]^{1/(1-q)}}{Z_q}, \tag{4}$$

and the expression of entropy as[5,7,9,14]

$$S_q = k\frac{Z_q^{q-1}-1}{q-1} + k\beta Z_q^{q-1}U_q \tag{5}$$

with

$$Z_q = \left\{\sum_i [1-(1-q)\beta\varepsilon_i]^{q/(1-q)}\right\}^{1/q}. \tag{6}$$

On the basis of the above results, it has been strictly proven that[19]

$$\sum_i p_i = Z_q^{q-1}[1-(1-q)\beta U_q] \tag{7}$$

and

$$\partial S_q/\partial U_q = k\beta Z_q^{q-1}/q = k\beta' = \frac{1}{T}, \tag{8}$$

where $\beta' = \beta Z_q^{q-1}/q = 1/(kT)$ and T is the physical temperature of the system in equilibrium. This T definition is different from the original one of IS in any one of Refs. [5], [7], [9], [11], [14], [16], and [18] where β is defined as the physical (measurable) temperature. It shows[19] that β in Refs. [5], [7], [9], [11], [14], [16], and [18] is not equal to $1/(kT)$.

It is interesting to note that when the fundamental relationship of thermodynamics $\delta Q = T\,\mathrm{d}S$ is demanded to hold for all the local Rindler causal horizons through each spacetime point, the Einstein equation can be derived from the proportionality of entropy and the horizon area together with this relationship,[22] where δQ and T are interpreted as the energy flux and Unruh temperature seen by an accelerated observer just inside the horizon, respectively. It is worthwhile to point out that the fundamental relationship mentioned above also holds for the incomplete statistics, and consequently, it is expected that the IS may be used to discuss the Unruh temperature[23,24] of the Rindler spacetime.[25] This Unruh effect may be of great relevance to the theory of a new form of Tsallis distribution,[26] fractal spacetime,[27] and ultrarelativistic plasmas.[28]

3. Nonextensive expressions of the internal energy

For an independent system C composed of two subsystems A and B, of which the distributions satisfy [4–7,9,11,14,18,21]

$$p_{ij}(C) = p_i(A)p_j(B) \tag{9}$$

or

$$p_{ij}^q(C) = p_i^q(A)p_j^q(B), \tag{10}$$

one can derive the pseudo-additivity entropy rule[5,7,9,11,14,18]

$$S_q(C) = S_q(A) + S_q(B) + [(q-1)/k]S_q(A)S_q(B) \tag{11}$$

from Eqs. (1) and (9). Using the law of entropy conservation $\delta S_q(C) = 0$ and Eq. (11), one can obtain[9,11,13,14,18]

$$[1+(q-1)S_q(B)/k]\frac{\partial S_q(A)}{\partial U_q(A)}\delta U_q(A) + [1+(q-1)S_q(A)/k]\frac{\partial S_q(B)}{\partial U_q(B)}\delta U_q(B) = 0. \tag{12}$$

On the other hand, from Eqs. (4) and (7)–(9), we can obtain

$$\frac{[1-(1-q)\beta(C)\varepsilon_{ij}(C)]^{1/(1-q)}}{Z_q(C)} = \frac{[1-(1-q)\beta(A)\varepsilon_i(A)]^{1/(1-q)}[1-(1-q)\beta(B)\varepsilon_j(B)]^{1/(1-q)}}{Z_q(A)Z_q(B)}, \tag{13}$$

$$\frac{1-(1-q)\beta(C)U_q(C)}{Z_q^{1-q}(C)} = \frac{[1-(1-q)\beta(A)U_q(A)][1-(1-q)\beta(B)U_q(B)]}{Z_q^{1-q}(A)Z_q^{1-q}(B)}, \tag{14}$$

or

$$\frac{\beta'(C)}{\beta(C)} - (1-q)\beta'(C)U_q(C) = q\left[\frac{\beta'(A)}{\beta(A)} - (1-q)\beta'(A)U_q(A)\right]\left[\frac{\beta'(B)}{\beta(B)} - (1-q)\beta'(B)U_q(B)\right]. \tag{15}$$

Equations (13)–(15) are equivalent to each other. It is seen from Eqs. (13)–(15) that the internal energy in IS is nonextensive.

It is significant to note that in the derivative process of Eqs. (13)–(15), we did not add any assumption except Eqs. (1)–(3) and (9) which had been adopted in IS, and that equations (13)–(15) are different from the nonextensive expressions of the composite internal energy, derived in Refs. [5], [7], [9], [11], and [14], because some empirical assumptions have been employed and we will see the mathematical contradiction below.

4. The zeroth law of thermodynamics

When systems A and B are in equilibrium, one important condition

$$T(A) = T(B) = T(C) \tag{16}$$

may be adopted, and consequently, equations (15) may be simplified into

$$\begin{aligned} &\frac{1}{\beta(C)} - (1-q)U_q(C) \\ &= q\beta'\left[\frac{1}{\beta(A)\beta(B)} - \frac{1-q}{\beta(A)}U_q(B) - \frac{1-q}{\beta(B)}U_q(A)\right. \\ &\quad \left. + (1-q)^2 U_q(A)U_q(B)\right]. \end{aligned} \tag{17}$$

Using the law of energy conservation $\delta U_q(C) = 0$ and Eq. (17), one can obtain

$$\begin{aligned} &\left[(1-q)U_q(B) - \frac{1}{\beta(B)}\right]\frac{\partial}{\partial U_q(A)} \\ &\times \left[\frac{\beta'}{\beta(A)(q-1)} + \beta' U_q(A)\right]\delta U_q(A) \\ &+ \left[(1-q)U_q(A) - \frac{1}{\beta(A)}\right]\frac{\partial}{\partial U_q(B)} \\ &\times \left[\frac{\beta'}{\beta(B)(q-1)} + \beta' U_q(B)\right]\delta U_q(B) = 0. \end{aligned} \tag{18}$$

Substituting Eqs. (5), (7), (8), and (16) into Eq. (18), one has

$$\sum_j p_j(B)\delta U_q(A) + \sum_i p_i(A)\delta U_q(B) = 0. \tag{19}$$

From Eqs. (1), (12), and (19), one obtains

$$\frac{\partial S_q(A)}{\partial U_q(A)} = \frac{\partial S_q(B)}{\partial U_q(B)} \quad \text{or} \quad \beta'(A) = \beta'(B), \tag{20}$$

which is nothing but the zeroth law of thermodynamics. Obviously, the physical essence of Eq. (20) is completely identical with that of Eq. (16). This implies that starting from Eq. (16), one obtains Eq. (20), which is the same result as Eq. (16). Thus, it is clear that the derivative process of Eq. (20) is of a self-consistent calculation only, but is not a proof for the zeroth law of thermodynamics in IS. Like in Tsallis' statistics,[21] the zeroth law of thermodynamics still holds in IS, but it cannot be proved theoretically. The conclusion conforms to Abe's standpoint,[20] i.e., statistical mechanics may be modified but the thermodynamics should remain unchanged.

5. Discussion

When the following assumption[7]

$$Z_q(C) = Z_q(A)Z_q(B) \tag{21}$$

is adopted, equations (13) and (14) may be, respectively, simplified into

$$\begin{aligned} \beta(C)\varepsilon_{ij}(C) &= \beta(A)\varepsilon_i(A) + \beta(B)\varepsilon_j(B) \\ &\quad + (q-1)\beta(A)\beta(B)\varepsilon_i(A)\varepsilon_j(B) \end{aligned} \tag{22}$$

and

$$\begin{aligned} \beta(C)U_q(C) &= \beta(A)U_q(A) + \beta(B)U_q(B) \\ &\quad + (q-1)\beta(A)\beta(B)U_q(A)U_q(B). \end{aligned} \tag{23}$$

From Eqs. (7), (8), and (20), one can obtain

$$\begin{aligned} &\sum_{ij} p_{ij}(C) + (1-q)q\beta'(C)U_q(C) \\ &= \left[\sum_i p_i(A) + (1-q)q\beta'(A)U_q(A)\right] \\ &\quad \times \left[\sum_j p_j(B) + (1-q)q\beta'(B)U_q(B)\right]. \end{aligned} \tag{24}$$

Substituting Eqs. (9) and (16) into Eq. (24) gives

$$\begin{aligned} U_q(C) &= \sum_i p_i(A)U_q(B) + \sum_j p_j(B)U_q(A) \\ &\quad + (1-q)q\beta' U_q(A)U_q(B). \end{aligned} \tag{25}$$

Using Eqs. (7) and (25) and the law of energy conservation, we obtain

$$\begin{aligned} &\left[\frac{\partial Z_q^{q-1}(A)}{\partial U_q(A)}U_q(B) + \sum_j p_j(B)\right]\delta U_q(A) \\ &+ \left[\frac{\partial Z_q^{q-1}(B)}{\partial U_q(B)}U_q(A) + \sum_i p_i(A)\right]\delta U_q(B) = 0, \end{aligned} \tag{26}$$

where[19]

$$\frac{\partial Z_q}{\partial U_q} = \frac{Z_q}{q}\frac{\sum_i e_q^{2q-1}(-\beta\varepsilon_i)\varepsilon_i}{\sum_i e_q^{2q-1}(-\beta\varepsilon_i)(\varepsilon_i^2 - U_q\varepsilon_i)}, \tag{27}$$

and $e_q(x) = [1+(1-q)x]^{1/(1-q)}$. Equation (26) is obviously in contradiction with Eq. (12) because $\partial Z_q/\partial U_q \neq 0$, so that equation (20) may not be derived. This means that the calculation process is not self-consistent and that the assumption given above, i.e., equation (21), does not hold true.

If the other assumption

$$\beta(C) = \beta(A) = \beta(B) \tag{28}$$

is also adopted, equations (22) and (23) may be further simplified into

$$\varepsilon_{ij}(C) = \varepsilon_i(A) + \varepsilon_j(B) + (q-1)\beta\varepsilon_i(A)\varepsilon_j(B) \tag{29}$$

and

$$U_q(C) = U_q(A) + U_q(B) + (q-1)\beta U_q(A)U_q(B), \tag{30}$$

respectively. Equations (29) and (30) are simply the main results obtained in Refs. [5], [7], [9], [11], and [14] and have been used to discuss the zeroth law of thermodynamics.

In Refs. [7] and [11], equation (30) was used to calculate the variation of the internal energy

$$\begin{aligned}\delta U_q(C) \;=\; & [1+(q-1)\beta U_q(B)]\delta U_q(A) \\ & +[1+(q-1)\beta U_q(A)]\delta U_q(B)\end{aligned} \tag{31}$$

and to derive the generalized zeroth law of thermodynamics[7,9,14]

$$Z_q^{1-q}(A)\frac{\partial S(A)}{\partial U(A)} = Z_q^{1-q}(B)\frac{\partial S(B)}{\partial U(B)} \;\text{ or }\; \beta(A) = \beta(B). \tag{32}$$

This seems to be a self-consistent calculation. However, it can be clearly seen from the analysis of Eqs. (30) and (31) that equation (21) does not hold true so equation (32) cannot be derived. The cause may be explained as follows: when the systems are in equilibrium, the rational assumption should be described by Eq. (16) rather than Eq. (28), because $\partial S_q/\partial U_q = 1/T \neq k\beta$.[19]

If equation (30) is directly used, one will obtain the expression of the variation of the internal energy as

$$\begin{aligned}\delta U_q(C) \;=\; & \{1+(q-1)[\beta U_q(B) + U_q(A)U_q(B)\partial\beta/\partial U_q(A)]\} \\ & \times \delta U_q(A) + \{1+(q-1)[\beta U_q(A) \\ & + U_q(A)U_q(B)\partial\beta/\partial U_q(B)]\}\delta U_q(B)\end{aligned} \tag{33}$$

with[19]

$$\frac{\partial U_q}{\partial \beta} = \frac{q\sum_i e_q^{2q-1}(-\beta\varepsilon_i)(U_q\varepsilon_i - \varepsilon_i^2)}{Z_q^q}. \tag{34}$$

Using the law of energy conservation $\delta U_q(C) = 0$ and Eqs. (12) and (33), one only obtains the following equation

$$\begin{aligned}& \sum_j p_j(B)\{1+(q-1)[\beta U_q(A) \\ & + U_q(A)U_q(B)\partial\beta/\partial U_q(B)]\}\frac{\partial S_q(A)}{\partial U_q(A)} \\ = & \sum_i p_i(A)\{1+(q-1)[\beta U_q(B) \\ & + U_q(A)U_q(B)\partial\beta/\partial U_q(A)]\}\frac{\partial S_q(B)}{\partial U_q(B)}.\end{aligned} \tag{35}$$

Equation (35) is obviously different from Eq. (32) and is in contradiction with Eq. (28). It indicates that it is unnecessary to introduce the generalized zeroth law of thermodynamics.

The above discussion clearly shows that the two empirical assumptions described above will lead to mathematical contradiction, so neither equation (29) nor equation (30) is the correct expression of the internal energy of a composite system in IS.

6. Conclusions

With the help of the entropy expression proposed by Wang[5] and the results obtained in Ref. [19], we solve two important problems in IS. The internal energy in IS is nonextensive. The reasonable expressions should be given by Eqs. (13)–(15) rather than Eqs. (29) and (30). The zeroth law of thermodynamics cannot be proved theoretically, but it still holds true in IS, while the so-called generalized zeroth law of thermodynamics is unnecessary. The results obtained here shows clearly that the empirical assumptions like Eqs. (21) and (28) lead to inconsistent results, so they should not be employed.

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