# Nonautonomous solitary-wave solutions of the generalized nonautonomous cubic–quintic nonlinear Schrödinger equation with time- and space-modulated coefficients*

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A class of analytical solitary-wave solutions to the generalized nonautonomous cubic–quintic nonlinear Schrödinger equation with time- and space-modulated coefficients and potentials are constructed using the similarity transformation technique. Constraints for the dispersion coefficient, the cubic and quintic nonlinearities, the external potential, and the gain (loss) coefficient are presented at the same time. Various shapes of analytical solitary-wave solutions which have important applications of physical interest are studied in detail, such as the solutions in Feshbach resonance management with harmonic potentials, Faraday-type waves in the optical lattice potentials, and localized solutions supported by the Gaussian-shaped nonlinearity. The stability analysis of the solutions is discussed numerically.

**Keywords:** generalized nonautonomous cubic–quintic nonlinear Schrödinger equation, similarity reduction, Faraday-type waves, solitary wave solution

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## 1. Introduction

During the past several years, different aspects of soliton dynamics described by the nonautonomous nonlinear Schrödinger equation (NLSE) models have been investigated intensively because of their potential applications in nonlinear optics[1–4] and Bose–Einstein condensates (BECs).[5,6] Such situations support temporal or spatial solitons, soliton lasers, and ultrafast soliton switches for experiments.[7–11] Solutions to the nonautonomous NLSE are usually called nonautonomous waves or nonautonomous solitons. In addition, the generalized nonautonomous NLSE models with distributed dispersion, nonlinearity, potential, and dissipation or gain were studied in Refs. [1]–[4], [12]–[19], where the soliton evolution control can be achieved by the nonuniform distribution of the nonlinear media.

Nonlinear interactions are usually of a cubic nature, but there are systems which engender cubic and quintic (CQ) nonlinearities, when the intensity of the optical pulse propagating inside a nonlinear medium exceeds a certain value or the two- and three-body interactions in BECs are considered. Furthermore, if the amplification or absorption of pulses propagating in optical fibers or the interaction of atomic cloud are considered as well, the governing equation should still include the dissipation or gain term. Thus, in nonautonomous systems, a generalized nonautonomous NLSE with CQ nonlinearities as well as the gain (loss) is presented, which will be studied in this paper.

In Ref. [20] we discovered a class of analytical solitary-wave solutions under some interesting external potentials, but most of the solutions were found to be decay, which may be difficult to observe in experiments. Exact bright multi-soliton solutions of the generalized cubic nonautonomous nonlinear Schrödinger equation with time- and space-dependent distributed coefficients and an external potential were reported in Ref. [21]. Here we present analytical solutions to the generalized nonautonomous cubic–quintic nonlinear Schrödinger equation (CQNLSE) with time- and space-modulated coefficients, and most of them no longer decay with time. Such a situation is believed to improve the stability of the solutions. In addition, we extend the analysis to contain new solutions, including solutions in the harmonic potentials, Faraday-type waves in the optical lattice potentials, localized solutions supported by the Gaussian-shaped nonlinearity, and for a number of physically relevant choices the cubic nonlinearity and the potential functions. It is noted that the most interesting feature in this paper is that the Faraday-type waves are constructed analytically in the optical lattice potential, in contrast to the previous work in Ref. [22].

The paper is organized as follows. In Section 2, the model and the solution procedure are presented. In Section 3, a class of analytical solutions under some interesting potentials are reported. In Section 4, the family of solutions supported by the Gaussian-shaped nonlinearity are studied. Section 5 discusses the stability of the solutions, and Section 6 brings our conclusions.

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## 2. General similarity reduction

In Ref. [20] we considered the generalized nonautonomous CQNLSE with time- and space-modulated coefficients for complex function $\psi(x,t)$:

$$\mathrm{i}\frac{\partial\psi}{\partial t}+f(x,t)\frac{\partial^2\psi}{\partial x^2}+g(x,t)|\psi|^2\psi+G(x,t)|\psi|^4\psi +V(x,t)\psi+\mathrm{i}\gamma(x,t)\psi=0, \tag{1}$$

which may be found in the nonlinear media when its transverse and longitudinal directions are non-uniform distribution. The functions $f(x,t),g(x,t),G(x,t),V(x,t),\gamma(x,t)$ are, respectively, the dispersion coefficient, the cubic nonlinearity, the quintic nonlinearity, the external potential, and the gain (loss) coefficient. In nonlinear optics, $\psi(x,t)$ is the complex envelope of the electric field, and $x$ and $t$ represent the propagation distance and the retarded time, respectively. In BECs, $\psi(x,t)$ is the wave function, $t$ and $x$ represent time and spatial coordinates, respectively, and the soliton management can be realized by adjusting the related control parameters via the technique of Feshbach resonance (FR).[23–25]

According to the similarity transformation technique,[26] the solution of Eq. (1) is searched for

$$\psi(x,t)=\rho(x,t)\Phi(X(x,t))\,\mathrm{e}^{\mathrm{i}\phi(x,t)}, \tag{2}$$

and then equation (1) can be reduced to the autonomous CQNLSE

$$E\Phi=-\Phi_{XX}+g_0|\Phi|^2\Phi+G_0|\Phi|^4\Phi, \tag{3}$$

where $\rho(x,t),\phi(x,t),X(x,t)$ are real functions to be determined, $\Phi(X)$ is the solution of Eq. (3), $E$ is the corresponding eigenvalue, and $g_0$ and $G_0$ are real constants.

Substituting Eq. (2) into Eq. (1) and requiring $\Phi(X)$ to satisfy Eq. (3), one can find that $\rho(x,t)$, $X(x,t)$, $\phi(x,t)$, $f(x,t)$, $g(x,t)$, $G(x,t)$, $V(x,t)$, and $\gamma(x,t)$ obey the following equations:

$$\begin{aligned}&\rho_t+f(\rho\phi_{xx}+2\phi_x\rho_x)+\gamma\rho=0,\\&f(\rho_{xx}/\rho-\phi_x^2)+V-\phi_t-EfX_x^2=0,\\&X_t+2f\phi_xX_x=0,\ \ (\rho^2X_x)_x=0,\\&g\rho^2+g_0fX_x^2=0,\ \ G\rho^4+G_0fX_x^2=0.\end{aligned} \tag{4}$$

Next we assume $X(x,t)=\int_0^\xi F[\xi'(x,t)]\mathrm{d}\xi'$, $g(x,t)=\alpha(t)F(\xi)$ with $\xi(x,t)=\alpha(t)x$, where $\alpha(t)$ is a function of time. After some algebra one finds

$$\rho(x,t)=\sqrt{\frac{a}{\alpha F(\xi)}}, \tag{5a}$$

$$\gamma=\frac{\alpha_tF'(\xi)}{F(\xi)}x+\frac{\alpha_t}{\alpha}-\frac{a_t}{2a}, \tag{5b}$$

$$f=-\frac{a}{g_0\alpha^2F(\xi)^2},\ \ G=\frac{G_0\alpha^2F(\xi)^2}{g_0a}, \tag{5c}$$

$$\phi(x,t)=\frac{1}{2}g_0a^{-1}\alpha\alpha_t\int_0^xF^2(\xi)s\,\mathrm{d}s, \tag{5d}$$

and

$$V(x,t)=q_1(\xi,t)x^2+\int_0^x[q_2(\xi,t)s+q_3(\xi,t)s^2]\mathrm{d}s +q_4(\xi,t)+q_5(t), \tag{6}$$

where

$$\begin{aligned}q_1(\xi,t)&=-\frac{g_0\alpha_t^2F^2(\xi)}{4a},\\q_2(\xi,t)&=\frac{g_0(a\alpha\alpha_{tt}+a\alpha_t^2-\alpha\alpha_ta_t)F^2(\xi)}{2a^2},\\q_3(\xi,t)&=\frac{g_0\alpha\alpha_t^2F(\xi)F'(\xi)}{a},\\q_4(\xi,t)&=\frac{a[3F'^2(\xi)-2F(\xi)F''(\xi)]}{4g_0F^4(\xi)},\\q_5(t)&=-\frac{Ea}{g_0}.\end{aligned}$$

Here $a=a(t)$ is an arbitrary function of time and $F'(\xi)=\mathrm{d}F/\mathrm{d}\xi, F''(\xi)=\mathrm{d}^2F/\mathrm{d}\xi^2$. Note that the integral function of time in Eq. (5d) is chosen to be zero. From the above equations one can see that the external potential $V(x,t)$ is related to $f(x,t),g(x,t),G(x,t)$, and $\gamma(x,t)$ via the parameters $a(t),\alpha(t)$, and $F(\xi)$. Thus, equations (5) and (6) can be understood as integrability conditions on Eq. (1) for analytical solutions by the method used in this paper.

As explained in Ref. [20], expression (6) can give rise to many sets of external potentials and CQ nonlinearities by approriate choices of independent parameters $a(t),\alpha(t),F(\xi)$. One can also obtain the potentials reported in Ref. [20]. Here, we mainly focus on some interesting potentials that have important applications in the experiments.

There are many solutions to Eq. (3) of different forms,[27] we will use the following solutions in this paper, but many other solutions can be used to construct analytical solutions as well.

**Solution I** (soliton solution) We obtain a soliton solution for the choice of $g_0=\pm8E/5A, G_0=-3E/5A^2$,

$$\Phi(X)=\sqrt{A[\mathrm{sech}\left(\sqrt{\frac{4E}{5}}X\right)\pm1]}, \tag{7}$$

where $A$ is an arbitrary constant. This solution represents a bright soliton for the upper sign (for positive $A$) and a dark soliton for the lower sign (for negative $A$).

**Solution II** (periodic snoidal solution) If we choose $E=0, g_0=-2\mu^2(k^4-k^2+1)/3, G_0=-\mu^2(k^2-2)(2k^2-1)(k^2+1)/9$, we obtain

$$\Phi(X)=\frac{3\mathrm{sn}(\mu X,k)}{\sqrt{3[3-(1+k^2)\mathrm{sn}(\mu X,k)^2]}}, \tag{8}$$

where $k$ is the modulus of the Jacobian elliptic function sn.

## 3. Nonautonomous solitary-wave solutions in the systems with different external potentials

Next we discuss some example of interesting solutions for the applications to the field in Bose–Einstein condensates.

### 3.1. Solutions in FR management with harmonic potentials

First, we consider the systems with the harmonic potential, which is $V \propto x^2$. We take $F(\xi) = 1, E = 0$, then the harmonic potential is given by

$$V(x,t) = \theta(t)x^2, \tag{9}$$

where $\theta(t) = g_0\alpha(a\alpha_{tt} - a_t\alpha_t)/4a^2$. In this case, equation (1) becomes the generalized nonautonomous CQNLSE with time-dependent coefficients

$$\mathrm{i}\frac{\partial\psi}{\partial t} + f(t)\frac{\partial^2\psi}{\partial x^2} + g(t)|\psi|^2\psi + G(t)|\psi|^4\psi + \theta(t)x^2\psi + \mathrm{i}\gamma(t)\psi = 0, \tag{10}$$

where $f(t) = -a/(g_0\alpha^2), g(t) = \alpha, G(t) = G_0\alpha^2/(g_0 a), \gamma(t) = \alpha_t/\alpha - a_t/2a$. One can see that, by employing the similarity reduction technique, equation (10) is obtained easily here in comparison with the previous work.[12–14] Moreover, expression (9) can stimulate many types of harmonic potentials by appropriately choosing $a(t)$ and $\alpha(t)$, as explained in Ref. [20].

Here, we consider the solution under the harmonic potential (9) for choosing the cubic nonlinearity coefficient in the form of FR function, which is important in BECs experiments. It is shown that various types of solutions can be obtained for the proper choice of $a(t)$.

We choose $g(t) = 1 + \Delta/(B_0 - B_1 t)$, which is in the form of the FR function. Here, $\Delta$ is the width of the resonance, and $B_0$ and $B_1$ are the resonant values of the magnetic field. Then, the harmonic potential may have different forms for the choice of $a(t)$. For example, we can obtain $V(x,t) = \lambda x^2$ by letting $\theta(t) = \lambda$, which leads to

$$a(t) = \frac{g_0 B_1^2}{4\lambda p(t)[B_1 t/\Delta + \ln(B_1 t - B_0 - \Delta)]}$$

with $p(t) = B_1^2 t^2 - 2B_0 B_1 t + B_0^2$ and $\lambda$ being a constant. In addition, if we let $a = \alpha^2$, then we obtain the potential

$$V(x,t) = \frac{\Delta g_0 B_1^2}{2(B_0 - B_1 t)(B_0 - B_1 t + \Delta)^2}x^2,$$

which is modulated by a time function. In this situation, the dispersion management parameter $f(t)$ is a constant, and the gain $\gamma$ vanishes. The corresponding solutions based on the above two situations are given by Eqs. (2), (5a), (5d), and (8).

In Fig. 1, we display the behavior of the solutions and the FR management of cubic nonlinearity. Figures 1(a) and 1(b)

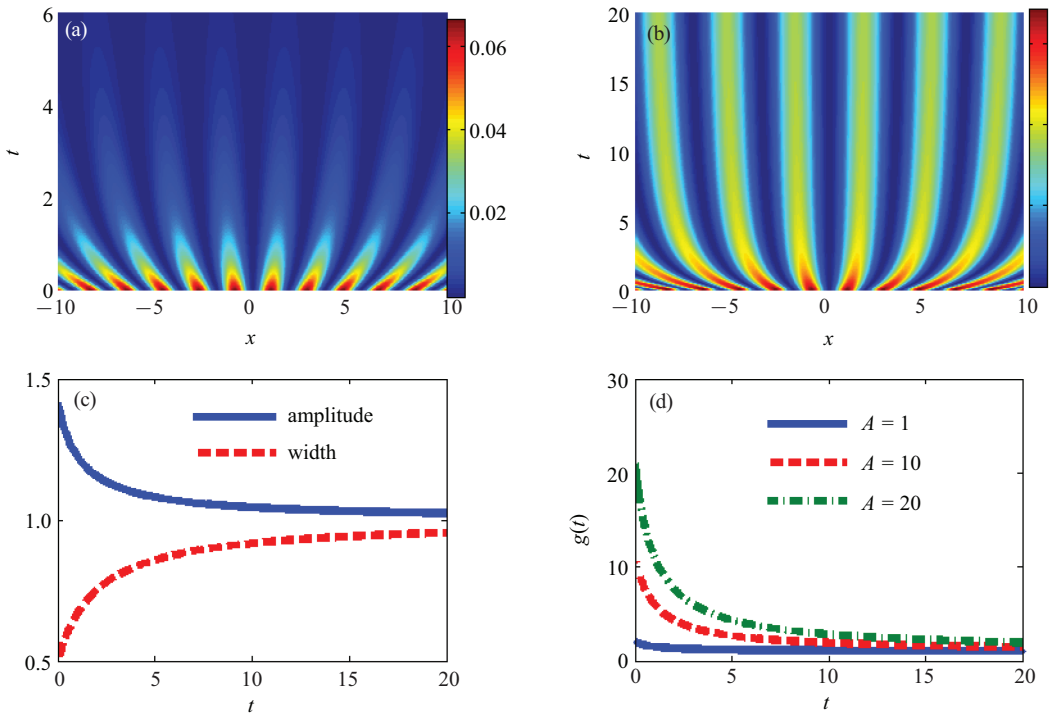



**Fig. 1.** (color online) (a) and (b) Periodic snoidal solution under the harmonic potential, when the cubic nonlinearity is in the form of FR function $1 + \Delta/(B_0 - B_1 t)$. In panel (a) the potential is described as $V(x,t) = \lambda x^2$, while in panel (b) it is described as $V(x,t) = \Delta g_0 B_1^2 x^2/[2(B_0 - B_1 t)(B_0 - B_1 t + \Delta)^2]$. (c) The width $1/\alpha(t)$ and amplitude $\sqrt{\alpha(t)}$ of the solution depicted in panel (b). The resonance width adopted in panels (a)–(c) is $\Delta = 1$. (d) FR management of the cubic nonlinearity for different values of the resonance width $\Delta$. The other parameters are $\lambda = -1, B_0 = 1, B_1 = -1, \mu = 1, k = 1/2$.

demonstrate the intensity profiles of the periodic snoidal solution under the harmonic potential, when the cubic nonlinearity is in the form of FR function $1+\Delta/(B_0-B_1t)$. Figure 1(c) shows the width $1/\alpha(t)$ and amplitude $\sqrt{\alpha(t)}$ of the solution depicted in Fig. 1(b). Figure 1(d) shows the FR management of the cubic nonlinearity for different values of the resonance width $\Delta$. It is observed that the width of the solution becomes wide but the amplitude becomes small as time goes on. Thus, we provide a direct and simple way to design solutions in a predictable fashion by FR management.

### 3.2. Faraday-type waves in optical lattice potentials

Next we consider the systems with optical lattice potentials. To do this, we take $\alpha=1$, then we get $q_1(\xi,t)=q_2(\xi,t)=q_3(\xi,t)=0$. Thus, $F(\xi)=1/[2\mathrm{cn}(\mu_1\xi,m)]$ corresponds to the optical lattice potential

$$V(x,t)=[V_0+V_1\mathrm{sn}(\mu_1\xi,m)^2+V_2\mathrm{sn}(\mu_1\xi,m)^4]a(t), \quad (11)$$

where $V_0=-(2+E)/g_0$, $V_1=(1+4m^2)/g_0$, $V_2=-3m^2/g_0$ with $\mu_1$ and $m$ being the arbitrary constant and the modulus of the Jacobian elliptic sine function, respectively. The real constants $V_0$, $V_1$, and $V_2$ denote the depths of the lattice potential controlled by the intensity of the laser beam. As the elliptic modulus $m\to 0$, $\mathrm{sn}(\mu_1\xi,m)\to\sin(\mu_1\xi)$, thus the Jacobian elliptic sine function potential is more general than the sine or cose function potential, compared with the results of Refs. [20] and [28]. The freedom in choosing $m$ allows one to consider much more general potentials and allow for great flexibility in considering a wide variety of physical realized potentials in the BECs. We choose $a(t)=1+a_0\cos(\omega_1 t)$, which leads to the potential varying periodically both in transverse and longitudinal directions. From Eqs. (5a) and (5b) we obtain that $\rho(x,t)=\sqrt{2a\mathrm{cn}(\mu_1\xi,m)}$ and $\gamma=-a_t/2a$ are time-dependent. Then, the corresponding soliton and periodic snoidal solutions can be given by combining Eqs. (2), (5a), (5d), (7), and (8).

In Fig. 2, we plot the dynamics of the soliton and periodic snoidal solutions under the optical lattice potential (11). Figures 2(a) and 2(c) demonstrate the intensity profiles of the soliton solution with $m=0$ and $m=1$, $a_0=0$, respectively. Figures 2(b) and 2(d) demonstrate the intensity profiles of the periodic snoidal solution with $m=0$ and $m=1$, respectively. It is observed that, from Figs. 2(a) and 2(b), the Faraday-type waves are found, where their excitation in low-dimensional BECs was predicted in Ref. [29]. In addition, these waves are constructed analytically here in the optical lattice potential, in contrast to the work in Ref. [22]. In fact, when one chooses the elliptic modulus $m\in[0,1)$ in Figs. 2(a) and 2(b), the Faraday-type waves are always obtained. We hope this result will be useful for the study in BECs and stimulate novel experiments in the relevant field.

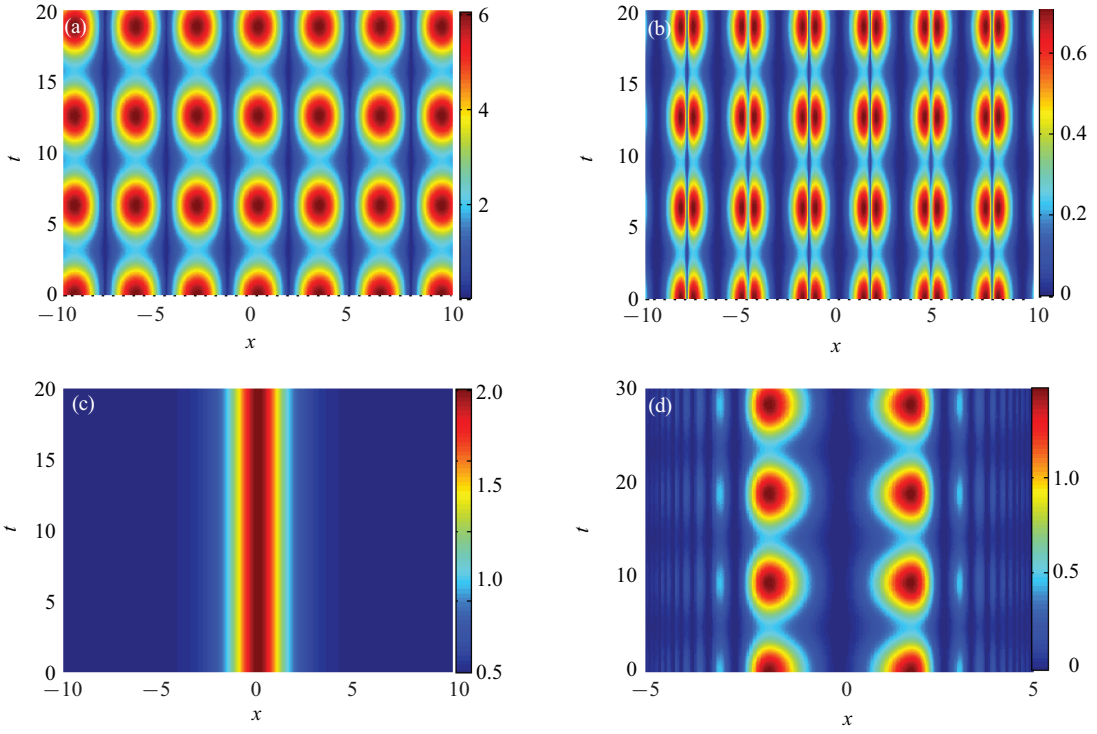



**Fig. 2.** (color online) Soliton solution under the optical lattice potential (11) with $m=0$ (a) and $m=1, a_0=0$ (c). Periodic snoidal solution under the optical lattice potential (11) with $m=0$ (b) and $m=1$ (d). The other parameters are $E=1, A=1, \omega_1=1, \mu=1, \mu_1=1, k=1/2, a_0=1/2$.

## 4. Solutions in the systems supported by the Gaussian-shaped nonlinearity

In this section, we will present analytical localized solutions of Eq. (1) with the Gaussian-shaped nonlinearity. We suppose the Gaussian-shaped nonlinearity is $g(x,t) = \alpha\, \mathrm{e}^{-\xi^2}$, which may be designed in the nonuniform-distributed nonlinear media. This choice leads to $X(x,t) = \int_0^{\xi} \mathrm{e}^{-\xi'^2}\mathrm{d}\xi'$, which takes values in $(-\sqrt{\pi}/2, \sqrt{\pi}/2)$. We consider the special case $a = 1$ and $\alpha$ being the form in the second case of Subsection 3.1 (i.e., $\alpha(t) = 1 + \varepsilon\cos(\omega_0 t)$), then the trapping potential is obtained

$$V(x,t) = l(t) + h_1(\xi,t)\,\mathrm{e}^{2\xi^2} + h_2(\xi,t)\,\mathrm{e}^{-2\xi^2} \tag{12}$$

with

$$l(t) = \frac{g_0}{8\alpha^2}(\alpha\alpha_{tt} - \alpha_t^2) - \frac{E}{g_0},$$
$$h_1(\xi,t) = \frac{1+\xi^2}{g_0},$$
$$h_2(\xi,t) = \frac{g_0[\alpha_t^2(1+2\xi^2) - \alpha\alpha_{tt}]}{8\alpha^2}.$$

This is a trapping potential in the form of a combination of a time-modulation function and Gaussian terms varying in space and time. Such a situation has been explored in Ref. [30], where a class of analytical localized wave solutions were found. It will be shown that the analytical localized solutions of Eq. (1) with the Gaussian-shaped nonlinearity can also be constructed using the same idea. To meet the boundary conditions $\psi(x \to \pm\infty) = 0$, constant $\mu$ in Eq. (8) should satisfy $\mu[X(+\infty) - X(-\infty)] = 2nK(k)$, where $n = 1,2,3,\ldots$ denotes the order of the localized solutions and $K(k)$ is the complete elliptic integral of the first kind. Thus, the family of analytical localized solutions to Eq. (1) is

$$\psi_n = \sqrt{\frac{\mathrm{e}^{\xi^2}}{\alpha}} \times \frac{3\mathrm{sn}(\mu X,k)}{\sqrt{3[3-(1+k^2)\mathrm{sn}(\mu X,k)^2]}}\,\mathrm{e}^{\mathrm{i}\phi(x,t)}, \tag{13}$$

where $\phi(x,t) = g_0\alpha^{-1}\alpha_t(1 - \mathrm{e}^{-2\xi^2})/8$. Therefore, equation (13) describes localized solutions possessing $n-1$ zeros. In Figs. 3(a) and 3(b) we plot examples of such solutions for $n = 2$ and 4, respectively, which displays periodic changes with time. The amplitude and width of the solution are shown in Fig. 3(c). It is found that the width and amplitude of the solution vary periodically with increasing time, which leads to the periodical changes of the localized solution. Note that the intensity profiles of $\psi$ for order $n = 1,3,5,\ldots$ can also be plotted in the same way.

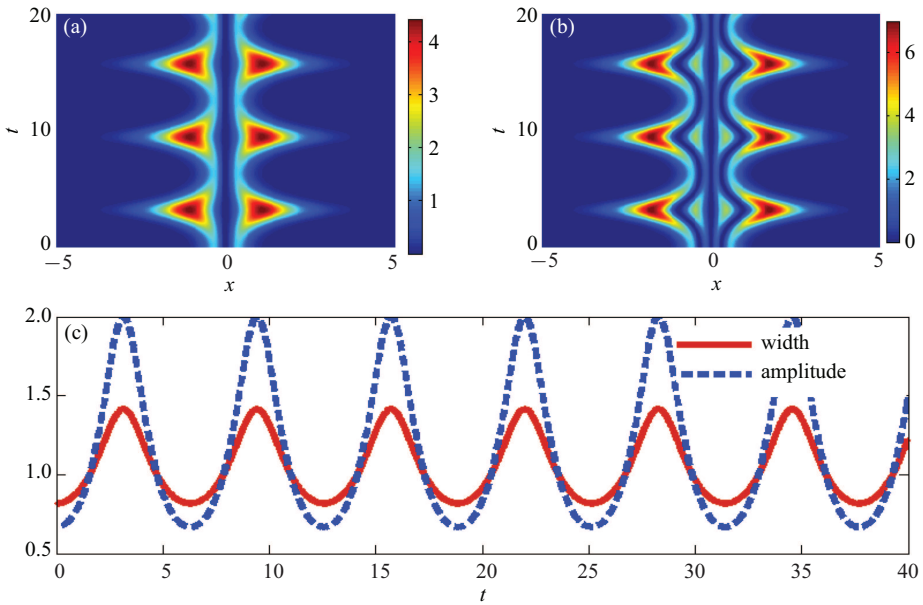



**Fig. 3.** (color online) Examples of the localized solutions under the trapping potential (12) with (a) $n = 2$ and (b) $n = 4$. Panel (c) describes the amplitude $\sqrt{\alpha}$ (red solid line), width $1/\alpha$ (blue dashed line) of the localized solution. The parameters are $\omega_0 = 1, k = 1/2, \varepsilon = 1/2$.

We would like to point out that although one type of the localized solution is displayed here, other types of localized solutions can also be derived by appropriately selecting parameter $a(t)$. For example, one can obtain breathing solutions for the choice of $a = \alpha^2$, similar to the situation in Ref. [30].

## 5. Stability of the solutions

Thus far, we have constructed analytical solutions to the generalized nonautonomous CQNLSE (1) under constraint conditions (5) and (6), where some important potentials of physical interest were generated. However, a study for the

perturbations in the constraint conditions is still needed so as to check the possibility of the soliton parametric control in the CQ model, since it may be difficult to exactly produce such constraint conditions in real applications. An example of such a stability calculation is chosen from Fig. 2(c), where the soliton solution is found under the optical lattice potential. We perturb the constraint conditions in the way: $f = -0.9 \times 4g_0^{-1}\mathrm{cn}^2(\mu_1\xi, m)$, and the other conditions do not change. The result is displayed in Fig. 4(a). It is observed that the soliton enters a stable state with time. In addition, we have made more numerical calculations for the constraint conditions by perturbing other management coefficients, and the results show that the soliton still propagates in a stable way.

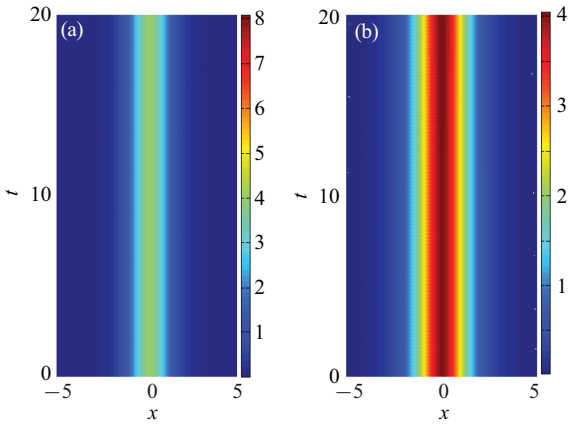



**Fig. 4.** (a) Numerical evolution of the bright soliton from Fig. 2(c) under the perturbation in the constraint conditions for parameters $f = -0.9 \times 4g_0^{-1}\mathrm{cn}^2(\mu_1\xi, m)$. (b) Numerical evolution of the bright soliton from Fig. 2(c) under the initial perturbation of white noise at the 1% level. The parameters are the same as those in Fig. 2(c).

Furthermore, the stability of the solutions can also be checked by adding white noise in the initial pulse $\psi(x,0)$. Such an example of stability calculation is still chosen from Fig. 2(c). The numerical result is shown in Fig. 4(b), where the soliton is found to be stable as well.

We have also studied the stability of the localized solutions found in Section 4, and the results show that the solutions are stable only for $n = 1$, while higher solutions ($n > 2$) are unstable, similar to the case in Ref. [30].

## 6. Conclusions

In summary, within the framework of the generalized nonautonomous CQNLSE including the dispersion coefficient, the CQ nonlinearities, the external potentials, and the gain (loss) coefficient, we have found a large number of analytical solitary-wave solutions (including the solutions in FR management with harmonic potentials, Faraday-type waves in the optical lattice potentials and localized solutions supported by the Gaussian-shaped nonlinearity) in the nonuniform-distributed nonlinear media by using a similarity transformation technique. These solutions require some restrictive conditions depicting dispersion, CQ nonlinearities, gain (loss) coefficients, and the external potentials. The stability of the solutions is studied briefly by numerical simulation. We hope that this research will stimulate further research on the CQ model and help us to study nonautonomous systems.

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