The extended cubic B-spline algorithm for a modified regularized long wave equation

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A collocation method based on an extended cubic B-spline function is introduced for the numerical solution of the modified regularized long wave equation. The accuracy of the method is illustrated by studying the single solitary wave propagation and the interaction of two solitary waves of the modified regularized long wave equation.

Keywords: collocation methods, solitary waves

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1. Introduction

Nonlinear dispersive wave equations exhibit types of solutions such as solitary waves and solitons. The existence of such solutions has been a source of interest for modelling many physical phenomena. Solutions of the nonlinear dispersive wave equation are not analytically available for every boundary and initial conditions in general. Solutions of those equations are of interest in order to understand some nonlinear wave behavior. One such nonlinear dispersive wave equation is the modified regularized long wave (MRLW) equation used to model the shallow water waves and plasma waves. The main properties of this equation are that solitary waves propagate in one direction with constant speed without changing its shape, pass through one another and emerge unaltered in shape. Numerical methods have been proposed to get the numerical solutions of the MRLW equation and its alternative form known as the modified equal wave equation, which can be obtained by using the transformation $U_{\mathrm{MRLW}} = U_{\mathrm{MEW}} - 1$ for the unknown variable of the MRLW equation. But the corresponding solutions of the MRLW equation cannot be obtained from those of the MEW equation using this transformation as the boundary conditions in the two cases are different. The MRLW equation has been solved numerically using the finite difference method in the study.[1] The meshless method of MQ, G, IMQ, and IQ radial basis functions has been used to obtain a numerical solution for the MRLW equation.[2] A mesh free technique based on a global collocation method using sinc basis functions is presented for the numerical solutions of the MRLW equation.[3] The collocation method based on quadratic, cubic, quartic, and quintic B-splines are set up for getting the numerical solutions of the MRLW equation.[4–7]

The extended cubic B-spline, which is a generalization of the B-spline, has been proposed by Han and Liu.[8] The three extended B-splines of degree 4, 5, and 6 were presented in the study of Xu and Wang.[9] For the extended B-splines, one free parameter is introduced within the cubic B-spline base functions so that the shape of the cubic B-spline curve can be changed. So it is constructed in a way that one-free parameter is included and the degree of the piecewise polynomials is increased but the continuity of the extended cubic B-splines remains in the order of 3. This motivates us to build up a numerical method including the extended cubic B-splines as a trial function. Design of the numerical schemes with extended B-splines to solve the numerical solution of the differential equations is not as widespread as the B-splines. There are a few studies in which extended B-splines are used to construct numerical methods. A homogeneous second order linear differential equation,[10] linear two-point boundary value problem,[11] and second order linear two-point boundary value problems[12] have been solved using the collocation method accompanying the extended cubic B-splines. The method of the collocation based on the extended cubic B-spline is described for solving a one-dimensional heat equation with a nonlocal initial condition.[13] But as far as we know, no paper has been found the numerical solutions of the nonlinear partial differential equation using extended B-splines.

In this paper, the MRLW equation is solved using the extended cubic B-splines collocation method. In Section 3, the extended cubic B-spline collocation algorithm is introduced for the numerical solution of the MEW equation. Single solitary wave simulation and two-soliton interaction are studied in Section 5 to illustrate the robustness of the numerical method.

2. Governing equation

The MRLW equation is given by

$$U_t + U_x + \varepsilon U^2 U_x - \mu U_{xxt} = 0 \tag{1}$$

subject to the boundary conditions $U \to 0$ as $x \to \pm\infty$.

For computational purposes, on the restricted domain

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$[a,b]$ the physical boundary conditions are changed to artificial boundary conditions

$$U(a,t)=U_0,\ U(b,t)=U_N,$$
$$U_x(a,0)=0,\ U_x(b,0)=0,$$

and initial conditions $U(x,0)=f(x), x\in(a,b)$.

3. Cubic B-spline collocation method

The problem domain $[a,b]$ is divided by the knots

$$a=x_0<x_1<\cdots<x_N=b$$

into elements $[x_m,x_{m+1}], m=0,\ldots,N-1$, and let $h=x_{m+1}-x_m=(b-a)/N,\ m=0,\ldots,N-1$. On these partitions we need the following extended cubic B-splines $Q_m(x)$ at the knot points $x_m,\ m=0,\ldots,N$.

The extended cubic B-splines $Q_m,\ m=-1,\ldots,N+1$, of degree 4, has the following the form

$$\begin{aligned}&Q_m(x)\\&=\frac{1}{24h^4}\begin{cases}4h(1-\lambda)(x-x_{m-2})^3\\\quad+3\lambda(x-x_{m-2})^4, & [x_{m-2},x_{m-1}],\\(4-\lambda)h^4+12h^3(x-x_{m-1})\\\quad+6h^2(2+\lambda)(x-x_{m-1})^2\\\quad-12h(x-x_{m-1})^3\\\quad-3\lambda(x-x_{m-1})^4, & [x_{m-1},x_m],\\(4-\lambda)h^4+12h^3(x_{m-1}-x)\\\quad+6h^2(2+\lambda)(x_{m+1}-x)^2\\\quad-12h(x_{m+1}-x)^3\\\quad-3\lambda(x_{m+1}-x)^4, & [x_m,x_{m+1}],\\4h(1-\lambda)(x_{m+2}-x)^3\\\quad+3\lambda(x_{m+2}-x)^4, & [x_{m+1},x_{m+2}],\\0, & \text{otherwise.}\end{cases}\end{aligned}\tag{2}$$

$Q_m(x)$ degenerates into the cubic B-spline when $\lambda=0$. Analogous to the cubic B-spline,[14] the set of the extended cubic B-splines $\{Q_{-1},Q_0,\ldots,Q_N,Q_{N+1}\}$ forms a basis over $[a,b]$. Assume that the approximation U_N to the exact solution $U(x,t)$ at the point (x,t_n) is expressed in terms of combinations of the cubic B-splines and unknown time-dependent parameters as

$$U_N(x,t)=\sum_{j=-1}^{N+1}\delta_jQ_j,\tag{3}$$

where δ_j are time-dependent parameters to be determined from the collocation method, the boundary and initial conditions.

The extended cubic B-splines and its four principle derivatives vanish outside of the region $[x_{m-2},x_{m+2}]$. Principle four spline functions cover interval $[x_m,x_{m+1}]$. So over the element, the variation of the $U_N(x,t)$ can be expressed as

$$U_N(x,t)=\sum_{j=m-1}^{m+2}\delta_jQ_j,\tag{4}$$

where $\delta_{m-1},\delta_m,\delta_{m+1},\delta_{m+2}$ act as element parameters. The values of the cubic B-spline $Q_m(x)$ and its successive derivatives $Q(x)'$, $Q(x)''$ at the knots can be calculated from Eq. (2) and these are given in Table 1.

Table 1. Values of $Q_m(x)$ and its principle two derivatives at the node points.

x	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}
Q_m	0	$(4-\lambda)/24$	$(8+\lambda)/12$	$(4-\lambda)/24$	0
Q'_m	0	$1/(2h)$	0	$-1/(2h)$	0
Q''_m	0	$(2+\lambda)/(2h^2)$	$-(2+\lambda)/h^2$	$(2+\lambda)/(2h^2)$	0

Using Table 1, evaluation of Eq. (4) and its first and second derivatives at knots x_i can be obtained as

$$\begin{aligned}U_m&=U(x_m)\\&=(4-\lambda)/24\delta_{m-1}+(8+\lambda)/12\delta_m+(4-\lambda)/24\delta_{m+1},\\U'_m&=U'(x_m)=-1/(2h)\delta_{m-1}+1/(2h)\delta_{m+1},\\U''_m&=U''(x_m)\\&=(2+\lambda)/(2h^2)\delta_{m-1}-(2+\lambda)/(h^2)\delta_m\\&\quad+(2+\lambda)/(2h^2)\delta_{m+1}.\end{aligned}\tag{5}$$

Application of the Crank–Nicholson method to the MRLW equation yields for time variable in the unknown solution U

$$\begin{aligned}&U^{n+1}-U^n+\frac{\Delta t}{2}\left[U_x^{n+1}+U_x^n+\varepsilon\left(U^2\right)^{n+1}U_x^{n+1}+\varepsilon\left(U^2\right)^nU_x^n\right]\\&\quad-\mu(U_{xx}^{n+1}-U_{xx}^n)=0.\end{aligned}\tag{6}$$

The nonlinear term in Eq. (6) is approximated by the following formula based on the Taylor series:[7]

$$\left(U^2\right)^{n+1}U_x^{n+1}\approx\left(U^2\right)^nU_x^{n+1}+2U^nU_x^nU^{n+1}-2\left(U^2\right)^nU_x^n.$$

Then using Eqs. (5) to evaluate the nodal values U, the first and the second derivatives U_x, U_{xx} at the knots, we obtain a fully-discretized form of the MRLW equation

$$\begin{aligned}&\left[\left(1+\varepsilon\Delta tU_m^n\left(U_x\right)_m^n\right)\alpha_1+\frac{\Delta t}{2}\left[1+\varepsilon\left(U^2\right)_m^n\right]\gamma_1\right.\\&\left.-\mu\beta_1\right]\delta_{m-1}^{n+1}+\left[\left(1+\varepsilon\Delta tU_m^n\left(U_x\right)_m^n\right)\alpha_2-\mu\beta_2\right]\delta_m^{n+1}\\&+\left[\left(1+\varepsilon\Delta tU_m^n\left(U_x\right)_m^n\right)\alpha_1\right.\\&\left.+\frac{\Delta t}{2}\left[1+\varepsilon\left(U^2\right)_m^n\right]\gamma_2-\mu\beta_1\right]\delta_{m+1}^{n+1}\\&=\left[\alpha_1+\frac{\Delta t}{2}\left[-1+\varepsilon\left(U^2\right)_m^n\right]\gamma_1-\mu\beta_1\right]\delta_{m-1}^n\\&+\left[\alpha_2-\mu\beta_2\right]\delta_m^n\\&+\left[\alpha_1+\frac{\Delta t}{2}\left[-1+\varepsilon\left(U^2\right)_m^n\right]\gamma_2-\mu\beta_1\right]\delta_{m+1}^n,\end{aligned}\tag{7}$$

where

$$\alpha_1 = \frac{4-\lambda}{24},\ \ \alpha_2 = -\frac{8+\lambda}{12},\ \ \gamma_1 = -\frac{1}{2h},$$
$$\gamma_2 = \frac{1}{2h},\ \ \beta_1 = \frac{2+\lambda}{2h^2},\ \ \beta_2 = -\frac{2+\lambda}{h^2}.$$

The system of $N+1$ equations is obtained with $N+3$ unknown element parameters $\boldsymbol{d}^n = (\delta_{-1}, \delta_0, \ldots, \delta_{N+1})$ at time levels $t = t_n$. To make the system solvable, boundary conditions $U(a,t) = U_0, U(b,t) = U_N$ give two additional linear equations

$$\delta_{-1} = \frac{4-\lambda}{24}\left(U_0 - \frac{(8+\lambda)}{12}\delta_0 - \frac{(4-\lambda)}{24}\delta_1\right),$$
$$\delta_{N+1} = \frac{4-\lambda}{24}\left(U_n - \frac{(4-\lambda)}{24}\delta_N - \frac{(8+\lambda)}{12}\delta_{N-1}\right). \quad (8)$$

Elimination of the variables $\delta_{-1}, \delta_{N+1}$ using the above two equations from the matrix system (7) reduces to $(N+1)\times(N+1)$ matrix system, which can be solved by using the Thomas algorithm.

4. The initial state

To start iterations of the matrix system (7), initial parameters $\delta^0_{-1}, \delta^0_0, \ldots, \delta^0_{N+1}$ must be determined from the initial condition and the first derivative using the initial condition as follows:

(i) $U_N(x_m, 0) = U(x_m, 0), m = 0, \ldots, N,$
(ii) $(U_x)_N(x_0, 0) = U'(x_0),$
(iii) $(U_x)_N(x_N, 0) = U'(x_N).$

The above conditions lead to a tridiagonal matrix system of the form

$$\begin{bmatrix} -\frac{1}{2h} & 0 & \frac{1}{2h} & & & \\ \frac{(4-\lambda)}{24} & \frac{(8+\lambda)}{12} & \frac{(4-\lambda)}{24} & & & \\ & \ddots & & & & \\ & & & \frac{(4-\lambda)}{24} & \frac{(8+\lambda)}{12} & \frac{(4-\lambda)}{24} \\ & & & -\frac{1}{2h} & 0 & \frac{1}{2h} \end{bmatrix} \begin{bmatrix} \delta^0_{-1} \\ \\ \delta^0_0 \\ \vdots \\ \delta^0_N \\ \delta^0_{N+1} \end{bmatrix} = \begin{bmatrix} 0 \\ U(x_0) \\ \\ \vdots \\ U(x_N) \\ 0 \end{bmatrix},$$

which can be solved using a variant of the Thomas algorithm.

5. Numerical tests

In this section, two test problems are studied for the MRLW equation. For the first test problem, the maximum norm

$$L_\infty = \max_m \left|U_m^{\text{analytic}} - U_m\right| \quad (9)$$

is calculated to measure the error between the analytical and numerical results.

Since an accurate numerical scheme must keep the conservation properties of evolution equations, we will monitor the three invariants of numerical solutions for the MRLW equation corresponding to conservation of mass, momentum, and energy given by the following integrals

$$I_1 = \int_{-\infty}^{\infty} u\,\mathrm{d}x \approx \frac{h}{2}\sum_{m=0}^{N-1}(U_m + U_{m+1}),$$
$$I_2 = \int_{-\infty}^{\infty}(u^2 + \mu(u_x)^2)\,\mathrm{d}x$$
$$\approx \frac{h}{2}\sum_{m=0}^{N-1}\left(U_m^2 + U_{m+1}^2 + \mu(U_x)_m^2 + \mu(U_x)_{m+1}^2\right),$$
$$I_3 = \int_{-\infty}^{\infty}\left(u^4 - \frac{6\mu}{\varepsilon}(u_x)^2\right)\mathrm{d}x$$
$$\approx \frac{h}{2}\sum_{m=0}^{N-1}\left(U_m^4 + U_{m+1}^4 - \frac{6\mu}{\varepsilon}(U_x)_m^2 - \frac{6\mu}{\varepsilon}(U_x)_{m+1}^2\right). \quad (10)$$

5.1. Motion of single solitary wave

Equation (1) has a solitary wave solution of the form

$$U(x,t) = \sqrt{6c/\varepsilon}\,\mathrm{sech}(k[x - x_0 - (c+1)t]), \quad (11)$$

where $k = \sqrt{c/\mu(c+1)}$. This solution corresponds to a solitary wave of magnitude $A = \sqrt{6c/\varepsilon}$ whose peak position is initially centered on x_0, propagating towards the right without a change of shape at a steady velocity $c+1$ as time advances.

The analytical values of the invariants for the problem can be found as

$$I_1 = \frac{\pi A}{k},\ \ I_2 = \frac{2A^2}{k} + \frac{2\mu k A^2}{3},\ \ I_3 = \frac{4A^2}{3k\varepsilon}\left(A^2\varepsilon - 3\mu k^2\right).$$

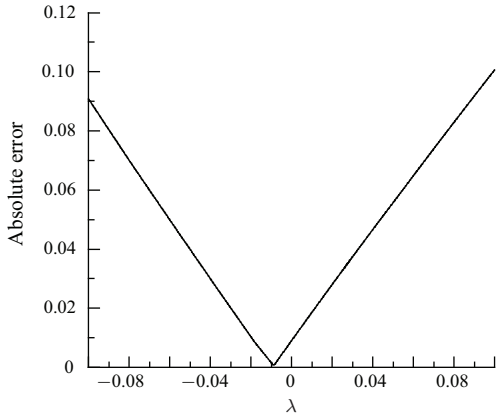


Fig. 1. L_∞ error norm with $-1 \le \lambda \le 1$.

Numerical experiments of solitary wave propagation are performed for the parameter $\varepsilon = 6$, $\mu = 1$, $c = 1$, $x_0 = 40$, and $A = 1$ together with the initial conditions $U_0 = U_N = 0$. We have also used space step $h = 0.2$, time step $\Delta t = 0.025$ to make the comparison with the previous published results possible over the region $0 \le x \le 100$. The program was run until time $t = 10$ to find the best λ over the interval $[-8, 1]$ for numerical solutions of the MRLW equation. So the error norm L_∞ against values λ in the interval $[-1, 1]$ is drawn at time $t = 10$ in Fig. 1.

From the figure, the least absolute error is obtained when parameter $\lambda = -0.0089$. Error norm L_∞ and numerical invariants I_1, I_2, I_3 are documented in Table 2. According to the table, it is clearly seen that the result obtained by the present method is more accurate than in the previous studies.

The initial solution and solitary wave profile with amplitude $A = 1$ at time $t = 10$ are shown in Fig. 2. Since the amplitude of the solitary waves at $t = 10$ is measured as 0.999394 when $\lambda = -0.0089$, we can say that during the propagation of the solitary wave, the amplitudes of the solitary wave remain almost constant. Absolute error (analytical–numerical) distributions for the method are drawn at time $t = 10$ as shown in Fig. 3.

Table 2. Invariants and error norms for a single solitary wave at time $t = 10$. $\varepsilon = 6$, $\mu = 1$, $x_0 = 40$, $h = 0.2$, $\Delta t = 0.025$, $[a, b] = [0, 100]$.

	$L_\infty \times 10^3$	I_1	I_2	I_3
Present	9.0618935	4.4428821	3.2997861	1.4141511
Present ($\lambda = -0.0089$)	0.7764	4.4428806	3.2997842	1.4141568
Ref. [6]	1.24757	4.445176	3.302476	1.417411
Ref. [4]	5.43718	4.44288	3.29983	1.41420
Exact		4.4428829	3.2998316	1.4142136

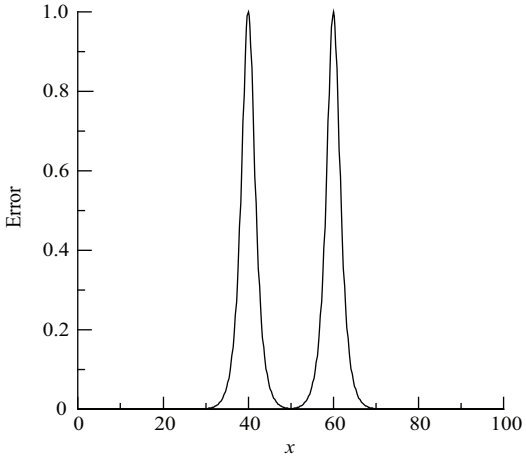


Fig. 2. A single solitary wave at $t = 0$ and $t = 10$. Other parameters are taken as follows: $c = 1, x_0 = 40, h = 0.2, \Delta t = 0.025, [a, b] = [0, 100], \lambda = -0.0089$.

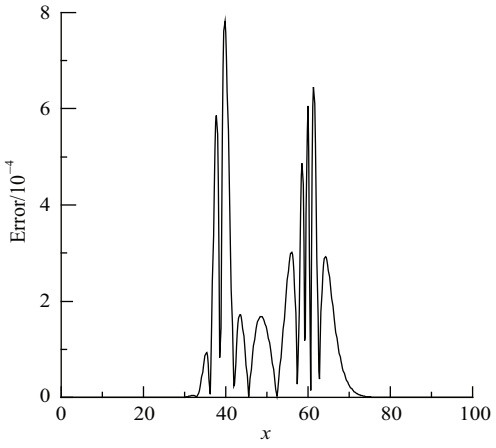


Fig. 3. Absolute error distribution at $t = 10$.

5.2. Interaction of two solitary waves

In this subsection, the interaction of two positive solitary waves for the MRLW equation is studied using the initial condition given by the linear sum of two separate solitary waves of various amplitudes

$$U(x,0) = A_1 \mathrm{sech}(k_1[x - x_1]) + A_2 \mathrm{sech}(k_2[x - x_2]), \quad (12)$$

where

$$A_i = \sqrt{6c_i/\varepsilon}, \quad k_i = \sqrt{c_i/\mu(c_i+1)}, \quad i = 1, 2,$$

and x_i, c_i are arbitrary constants.

In our computational work, we choose the parameters $\varepsilon = 6$, $\mu = 1$, $c_1 = 1$, $c_2 = 1/4$, $x_1 = 25$, $x_2 = 55$, and the computation was carried out from time $t = 0$ to $t = 60$ over the finite space range $[0, 200]$ when time step $\Delta t = 0.01$ and space step $h = 0.1$. These parameters provide two solitary waves of magnitudes 1 and 0.5 and peak positions of them are located at $x = 20$ and 50 respectively. The analytical invariants can be found as

$$\begin{aligned} I_1 &= \frac{\pi}{k_1 k_2}(k_2 A_1 + k_1 A_2) \simeq 7.955290304, \\ I_2 &= \frac{2}{k_1 k_2}\left(k_2 A_1^2 + k_1 A_2^2\right) + \frac{2\mu}{3k_1 k_2}\left(k_1^2 k_2 A_1^2 + k_1 k_2^2 A_2^2\right) \\ &\simeq 4.492401234, \\ I_3 &= \frac{4}{3k_1 k_2 \varepsilon}\left(\varepsilon k_1 A_2^4 - 3\mu k_1 k_2^2 A_2^2 + \varepsilon k_2 A_1^4 - 3\mu k_1^2 k_2 A_1^2\right) \\ &\simeq 1.526016961. \end{aligned}$$

A graph of the two-solitary wave collision, plotted for three-dimensional views at some discrete times, is shown in

Fig. 4 for the proposed algorithm. It can be seen from the figure that an initially larger wave with amplitude 1 is located at $x = 20$ to the left side of the smaller wave with amplitude 0.5 at $x = 50$. These solitary waves propagate to the right with velocities depending on their amplitudes. The larger wave catches up with the smaller one as time increases. Then, two solitary waves regain their original shape after the interaction. At time $t = 60$, the smaller solitary wave of amplitude 0.50444, whose peak point is located at $x = 120.5$, is recorded, and the larger solitary wave of amplitude 0.99989 is obtained and its peak position is located at the point $x = 143.7$. The absolute values of the differences between exact and numerical invariants I_1 and I_3 are found to be smaller than those of the corresponding invariants of the cubic B-spline collocation method, whereas analytical invariant I_2 is not as seen in Figs. 5–7, the dash line for $\lambda = -0.012$ and the solid line for $\lambda = 0$.

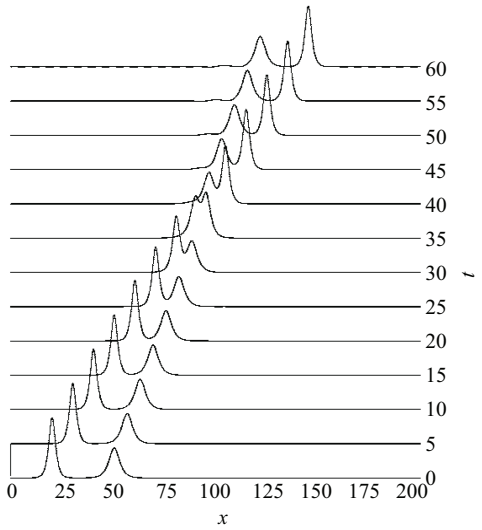


Fig. 4. Interaction of two solitary waves.

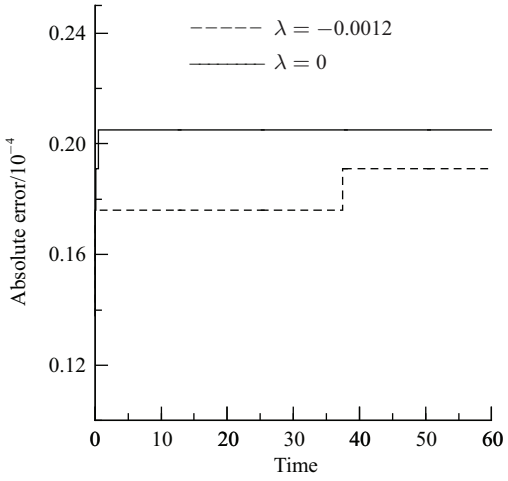


Fig. 5. Absolute error for the first invariant.

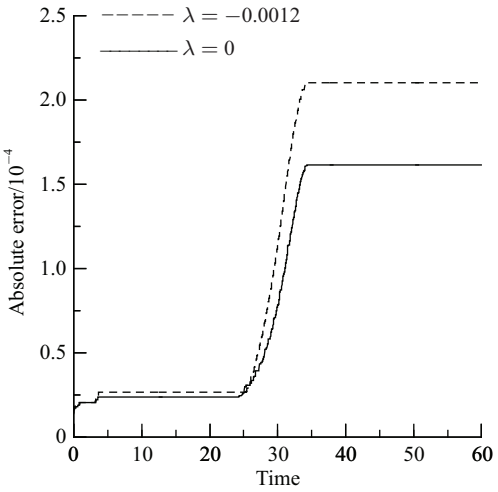


Fig. 6. Absolute error for the second invariant.

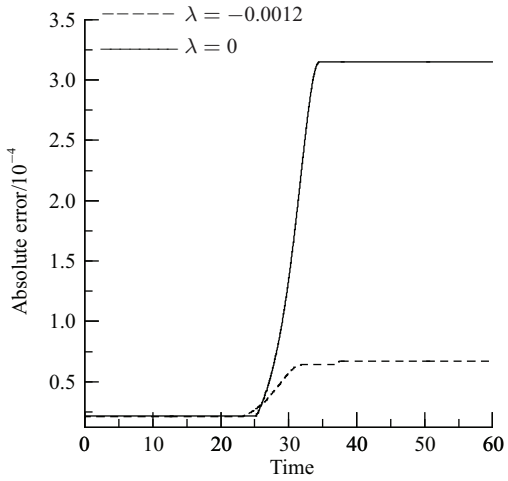


Fig. 7. Absolute error for the third invariant.

6. Conclusion

This paper introduces the use of the extended cubic B-spline to set up a numerical method to find the numerical solutions of the time-dependent nonlinear partial differential equations. So the adaptation of the extended cubic B-spline is done to the collocation method to get numerical solutions of the MRLW equation. We have seen that after the parameter λ has been determined, the cost of the algorithm of the extended cubic B-spline collocation method is the same as the cubic B-spline collocation method. Therefore it is an easy and economical method. The results show that the extended cubic B-spline collocation method with an appropriate λ gives a slightly better result than the cubic B-spline collocation method for solving the MRLW equation.

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