Finite symmetry transformation group and localized structures of the (2+1)-dimensional coupled Burgers equation*

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In this paper, the finite symmetry transformation group of the (2+1)-dimensional coupled Burgers equation is studied by the modified direct method, and with the help of the truncated Painlevé expansion approach, some special localized structures for the (2+1)-dimensional coupled Burgers equation are obtained, in particular, the dromion-like and solitoff-like structures.

Keywords: finite transformation group, modified direct method, truncated Painlevé expansion approach, localized structure

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1. Introduction

Symmetry study[1–5] plays an important role in nonlinear science. For a given partial differential equation (PDE), if we know the symmetry of the equation, we may obtain many other exact solutions, conservation laws,[6] and corresponding localized coherent structures. It is quite difficult and complicated to find simple expressions of the Lie point symmetry groups using the traditional method. In Refs. [7]–[12], a new simple method, namely, the modified direct method has been put forward. By using this modified direct method, a finite transformation group of the nonlinear partial differential equation has been obtained, such as the Konopelchenko–Dubrovsky equation in Ref. [13].

In this paper, we consider the symmetry transformation group of the (2+1)-dimensional Burgers equation[14,15]

$$\begin{cases} u_t = u_{xx} + u_{yy} + 2uu_x + 2vu_y, \\ v_t = v_{xx} + v_{yy} + 2uv_x + 2vv_y, \end{cases} \tag{1}$$

where u and v are functions of x, y, and t. System (1) reduces to the well-known (1+1)-dimensional Burgers equation when taking $y = x$, and it possesses a hereditary structure and may reduce to a hierarchy of Burgers equations under some reductions. A detailed physical description of the (2+1)-dimensional coupled Burgers equations can be found in Ref. [16].

The construction of the localized excitations in (2+1)-dimensional PDEs is still a challenging and rewarding problem, the localized structures of the system (1) are not known completely to the best our knowledge. Therefore, we also try to find the localized structures of system (1) by using the truncated Painlevé expansion approach in this paper.

The organization of this paper is as follows. In Section 2, the finite symmetry transformation group for the system (1) is obtained by using the modified direct method. The general solution of the system (1) is discussed in Section 3. In Section 4, we give some special localized structures for the system (1), in particular, the dromion-like and solitoff-like structures. The summary and discussion are given in the last section.

2. Finite symmetry transformation group of the system (1)

In order to obtain the finite symmetry transformation group of the (2+1)-dimensional coupled burger equation, we let a symmetry group in the form of

$$u = \alpha_1 + \beta_1 U(\xi, \eta, \tau), \tag{2}$$

$$v = \alpha_2 + \beta_2 V(\xi, \eta, \tau), \tag{3}$$

where $\alpha_1, \alpha_2, \xi, \eta, \tau$ are all functions of x, y, t, $\{U(\xi, \eta, \tau), V(\xi, \eta, \tau)\}$ is the solution of the system (1), namely, when restricting $U(\xi, \eta, \tau) = U, V(\xi, \eta, \tau) = V$, $\{U, V\}$ satisfies the following equations

$$U_\tau = 2UU_\xi + U_{\xi\xi} + U_{\eta\eta} + 2VU_\eta, \tag{4}$$

$$V_\tau = 2UV_\xi + V_{\xi\xi} + V_{\eta\eta} + 2VV_\eta. \tag{5}$$

Substituting Eqs. (2)–(5) to system (1), we can get

$$-\beta_1(\tau_x^2 + \tau_y^2)U_{\eta^4} + G(x, y, t, U, V, U_\eta, U_{\eta^2}, U_{\eta^3}, U_\xi, \ldots, V_\xi, \ldots, V_\eta, \ldots, V_\tau, \ldots) = 0. \tag{6}$$

Obviously, β_1 and U_{η^4} should not be zero, so no trivial solution for $\tau_x = 0, \tau_y = 0$ exists, namely,

$$\tau = \tau(t). \tag{7}$$

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Then we have

$$
\begin{aligned}
&(-2\beta_1\beta_2\xi_y V - 2\beta_1(\beta_1\xi_x - \tau_t + \beta_1\eta_x))U\\
&-\beta_1(\xi_{yy} - \xi_t + \xi_{xx} + 2\alpha_2\xi_y + 2\alpha_1\xi_x)U_\xi\\
&-2\beta_1(-\tau_t + \beta_2\eta_y)V\\
&+\beta_1(-\eta_{xx} + \eta_t - \eta_{yy} - 2\alpha_1\eta_x - 2\alpha_2\eta_y)U_\eta\\
&-\beta_1(\eta_y\xi_y + \xi_x\eta_x)U_{\xi\eta} - \beta_1(\eta_x^2 - \tau_t + \eta_y^2)U_{\eta\eta} = 0, \quad (8)\\
&(-2\beta_2^2\xi_y V - 2\beta_2(\beta_1\xi_x - \tau_t + \beta_1\eta_x)U\\
&-\beta_2(\xi_{yy} - \xi_t + \xi_{xx} + 2\alpha_2\xi_y + 2\alpha_1\xi_x)V_\xi\\
&+\beta_2(-\eta_{xx} + \eta_t - \eta_{yy} - 2\alpha_1\eta_x - 2\alpha_2\eta_y)V_\eta\\
&-\beta_2(\eta_x^2 - \tau_t + \eta_y^2)V_{\eta\eta}\\
&-\beta_2(\xi_x^2 + \xi_y^2 - \tau_t)V_{\xi\xi} - 2\beta_2(\eta_y\xi_y + \xi_x\eta_x)V_{\xi\eta}\\
&-2\beta_2(-\tau_t + \beta_2\eta_y)V = 0. \quad (9)
\end{aligned}
$$

By vanishing the coefficients of the polynomials of $\{U,V\}$ and their derivatives, the remaining determining equations on functions ξ, η, τ , α_1, α_2, β_1, and β_2 can be found

$$
\begin{aligned}
&\xi_y = 0,\ \beta_1\xi_x - \tau_t + \beta_1\eta_x = 0,\ \ -\tau_t + \beta_2\eta_y = 0, \quad (10)\\
&\xi_{yy} - \xi_t + \xi_{xx} + 2\alpha_2\xi_y + 2\alpha_1\xi_x = 0, \quad (11)\\
&\eta_y\xi_y + \xi_x\eta_x = 0,\ \eta_x^2 - \tau_t + \eta_y^2 = 0,\\
&\xi_x^2 + \xi_y^2 - \tau_t = 0, \quad (12)\\
&-\eta_{xx} + \eta_t - \eta_{yy} - 2\alpha_1\eta_x - 2\alpha_2\eta_y = 0. \quad (13)
\end{aligned}
$$

It is easy to find the general solution of the determining Eqs. (10)–(13)

$$
\begin{aligned}
&\xi = \tau_t^{1/2}x + f(t),\ \beta_1 = \tau_t^{1/2}, \quad (14)\\
&\eta = \tau_t^{1/2}y + g(t),\ \beta_2 = \tau_t^{1/2}, \quad (15)\\
&\alpha_1 = \frac{1}{4}\tau_t^{-1}\tau_{tt}x + \frac{1}{2}\tau_t^{-1/2}f_t(t), \quad (16)\\
&\alpha_2 = \frac{1}{4}\tau_t^{-1}\tau_{tt}y + \frac{1}{2}\tau_t^{-1/2}g_t(t), \quad (17)
\end{aligned}
$$

where $f(t)$, $g(t)$, τ are arbitrary functions of time t.

In summary, we have the following theorem.

Theorem 1 If $\{U(x,y,t),\ V(x,y,t)\}$ is a solution of the (2+1)-dimensional coupled Burgers equation, then the following expressions

$$
\begin{aligned}
&u = \frac{1}{4}\tau_t^{-1}\tau_{tt}x + \frac{1}{2}\tau_t^{-1/2}f_t(t) + \tau_t^{1/2}U(\xi,\eta,\tau), \quad (18)\\
&v = \frac{1}{4}\tau_t^{-1}\tau_{tt}y + \frac{1}{2}\tau_t^{-1/2}f_t(t) + \tau_t^{1/2}V(\xi,\eta,\tau) \quad (19)
\end{aligned}
$$

hold with Eqs. (14) and (15), where $f(t)$, $g(t), \tau$ are arbitrary functions of t.

If we set

$$
\tau(t) = t + \varepsilon h(t),\ f(t) = \varepsilon p(t),\ g(t) = \varepsilon l(t). \quad (20)
$$

By restricting $h(t) = h$, $p(t) = p$, $l(t) = l$, the general Lie point symmetry of the (2+1)-dimensional coupled Burgers equation have the following form

$$
\begin{aligned}
\sigma &\equiv \sigma_1(h) + \sigma_2(p) + \sigma_3(l)\\
&\equiv \begin{pmatrix} \frac{1}{4}h_{tt}x + \frac{1}{2}h_t xu_x + \frac{1}{2}h_t yu_y + hu_t + \frac{1}{2}h_t u\\ \frac{1}{4}h_{tt}y + \frac{1}{2}h_t xv_x + \frac{1}{2}h_t yv_y + hv_t + \frac{1}{2}h_t v \end{pmatrix}\\
&\quad + \begin{pmatrix} \frac{1}{2}p_t + pu_x\\ \frac{1}{2}p_t + pv_x \end{pmatrix} + \begin{pmatrix} lu_y\\ lv_y \end{pmatrix}. \quad (21)
\end{aligned}
$$

3. General form of the localized structure of the system (1)

According to the truncated Painlevé expansion method,[17,18] we write the solution of the system (1) as the following form

$$
u = \frac{u_0}{\phi} + u_1, \quad (22)
$$

$$
v = \frac{v_0}{\phi} + v_1, \quad (23)
$$

where ϕ is a function of x and t, and $\{u_1, v_1\}$ is an arbitrary seed solution of system (1). Substituting Eqs. (22) and (23) to system (1), and equating the negative powers of ϕ, we can get

$$
u_0 = \phi_x,\ v_0 = \phi_y, \quad (24)
$$

with ϕ satisfying the equation

$$
\phi_t = \phi_{xx} + \phi_{yy} + 2u_1\phi_x + 2v_1\phi_y, \quad (25)
$$

where $u_{1y} = v_{1x}$. If we take $v_1 = \phi$, $u_1 = \partial_y^{-1}\phi_x = w$, then

$$
u = \frac{\phi_x}{\phi} + \partial_y^{-1}\phi_x. \quad (26)
$$

Equation (25) is equal to

$$
\begin{cases} \phi_t = \phi_{xx} + \phi_{yy} + 2w\phi_x + 2\phi\phi_y,\\ \phi_x = w_y. \end{cases} \quad (27)
$$

It is easy to see that the coupled Burger equation is just transformed to the (2+1)-dimensional Burgers equation which degenerated by the (3+1)-dimensional Burgers equation. We may obtain many solution structures of the higher-dimensional coupled Burgers system by Eq. (27).

In the following, we will look for the solutions of system (27). Actually, in Ref. [19], when taking $a = 1$, $b = 1$, $u = 2\phi$, $v = 2w$, the (2+1)-dimensional Burgers equation can be transformed to system (27). So, from Ref. [19], we can get the variable separation solution ($\Phi = \phi_x = w_y$) of the system (27) directly, namely,

$$
\Phi = \frac{\sum_{i=1}^N\sum_{i=1}^N p_i p_{jx}(q_i q_{jx} - q_j q_{iy})}{\sum_{i=1}^N\sum_{i=1}^N p_i q_i p_j q_j}, \quad (28)
$$

where

$$
p_{it} = (\partial_x^2 + 2w\partial_x + A - \alpha_i)p_i, \quad i = 1,2,\ldots,N, \quad (29)
$$

$$q_{it} = q_{iyy} + \alpha_i q_i, \quad i = 1,2,\ldots,N, \tag{30}$$

where $\{\alpha_i, i = 1,2,\ldots,N\}$ are all arbitrary function of t, and $A = A(x,t)$ is an integration function.

So, from Eqs. (29) and (30), many interesting solutions of the system (1) can be obtained. Moreover, because we have the infinite symmetry group of the equation, due to Theorem 1, we can gain more solution structures for the system (1).

4. Special localized structures of the system (1)

In this section, we simply consider some special types of localized structures for the potential field ($U = u_y = v_x$) of the (2+1)-dimensional coupled Burgers equation.

Firstly, we take $N = 4$, $q_1 = a_0$, $q_2 = a_1$, $q_3 = a_2 q$, $q_4 = a_3 q$, $p_1 = p_3 = 1$, $p_2 = p_4 = p$, through some computations, the solution of system (27) reads

$$\phi = -\frac{(a_1 a_2 - a_0 a_3) p_x q_y}{(a_0 + a_1 p + a_2 q + a_3 pq)^2}, \tag{31}$$

where p is an arbitrary function of x, y, t, and q is the solution of the (1+1)-dimensional variable coefficient heat equation.

Substituting Eq. (26) and Eq. (31) into $U = u_y = v_x$, we can get

$$U = \frac{2(a_1 a_2 - a_0 a_3) p_x q_y}{(a_0 + a_1 p + a_2 q + a_3 pq)^2}, \tag{32}$$

where p is an arbitrary function of x, y, t, and q is the solution of the (1+1)-dimensional variable coefficient heat equation.

Then, we take different p and q into Eq. (32), we have the following eight figures.

Taking $p = \mathrm{sech}(k_1 x + k_1^2 t)$, $q = \mathrm{e}^{k_2 y + k_2^2 t} + \mathrm{e}^{-k_3 y + k_3^2 t}$, with parameters $k_1 = 1$, $k_2 = 1$, $k_3 = 1$, $a_0 = 1$, $a_1 = 2$, $a_2 = 1$, $a_3 = 1$, $t = 1$, we have a four-dromions-like solution.

Its evolution is shown in Fig. 1(a). Taking $p = \mathrm{e}^{k_1 x + k_1^2 t}$, $q = \mathrm{e}^{k_2 y + k_2^2 t} + \mathrm{e}^{-k_3 y + k_3^2 t}$, with the same parameters as those in Fig. 1(a), we obtain a fussion phenomenon of the two-dromions-like solution as shown in Fig. 1(b).

By taking have $p = \mathrm{e}^{k_1 x + k_1^2 t}$, $q = \mathrm{e}^{k_2 y + k_2^2 t}$, with the same parameters as those in Fig. 1(a), we obtain a typical one-dromion-like structure as shown in Fig. 1(c).

We choose $p = \mathrm{e}^{\cos(k_1 x)}$, $q = \mathrm{e}^{k_2 y + k_2^2 t} + \mathrm{e}^{-k_3 y + k_3^2 t}$, the parameters are same as those in Fig. 1(a), and a periodic dromions-like solution is obtained as shown in Fig. 1(d).

Taking $p = \sec(k_1 x + k_1^2 t)$, $q = \mathrm{e}^{k_2 y + k_2^2 t}$, with parameters $k_1 = 1, k_2 = 1, k_3 = 1, a_0 = 1, a_1 = 2, a_2 = 1, a_3 = 3, t = 0$, it is another periodic dromions-like solution as shown in Fig. 1(e).

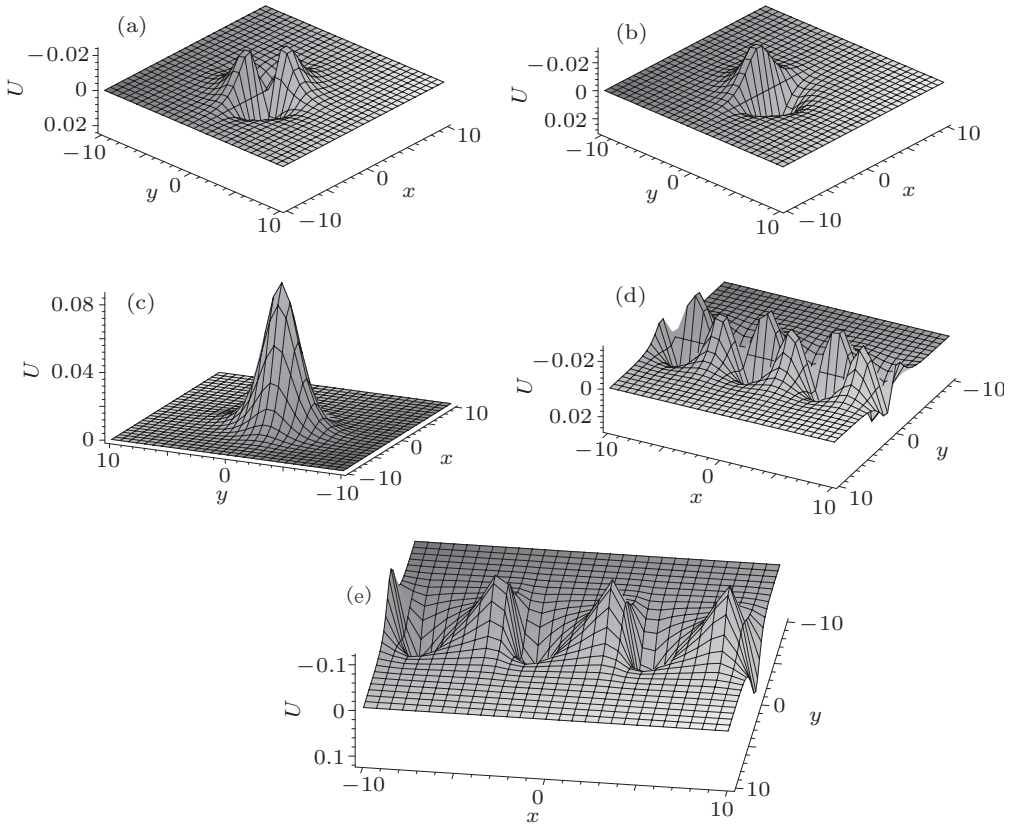


Fig. 1. The evolutions of some dromion-like solutions: (a) a four-dromion-like solution, (b) a two-dromion-like solution with fussion, (c) a one-dromion-like solution, (d) and (e) periodic dromion-like solutions.

By taking $p = e^{k_1 x + k_1^2 t}$, $q = e^{k_2 y + k_2^2 t}$, with parameters $k_1 = 1$, $k_2 = 1$, $k_3 = 1$, $a_0 = 1$, $a_1 = 2$, $a_2 = 1$, $a_3 = 0$, $t = 0$, we obtain a one-solitoff-like structure as shown in Fig. 2(a).

Choosing $p = \mathrm{sech}(k_1 x + k_1^2 t)$, $q = e^{(k_2 y + k_2^2 t)} + e^{(-k_3 y + k_3^2 t)}$, with parameters $k_1 = 1$, $k_2 = 1$, $k_3 = 1$, $a_0 = 1$, $a_2 = 0$, $a_1 = 1$, $a_3 = 3$, $t = 0$, we get a four-solitoff-like structure as shown in Fig. 2(b).

Taking $p = \mathrm{sech}(k_1 x + k_1^2 t)$, $q = e^{(k_2 y + k_2^2 t)}$, with parameters $k_1 = 1$, $k_2 = 1$, $k_3 = 1$, $a_0 = 1$, $a_2 = 0$, $a_1 = 1$, $a_3 = 3$, $t = 0$, a two-solitoff-like structure is obtained as shown in Fig. 2(c).

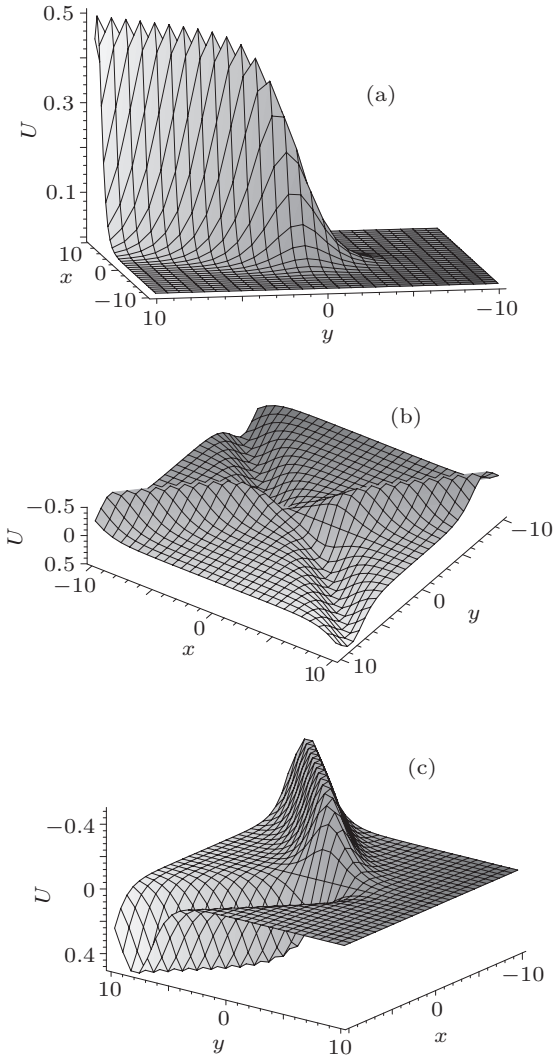


Fig. 2. The evolutions of the solitoff-like solution: (a) a one-soitoff-like solution, (b) a four-solitoff-like solution, and (c) a two-solitoff-like solution.

Figures 2(a)–2(b) exhibit the evolution of the solitoff-like solutions. So far, we have found the dromion-like structure and the solitoff-like structure with the truncated Painlevé expansion approach, and other types of localized structure of the (2+1)-dimensional coupled Burgers equation will be studied in the future.

5. Summary and discussion

We studied the infinite symmetry of the (2+1)-dimensional coupled burgers equation by making use of the modified direct method. The method reported here can be applied in principle to all nonlinear systems including both integrable and non-integrable ones and more details will be reported later. The truncated Painlevé expansion approach is powerful for solving nonlinear PDEs, and the general form of the localized structures of the (2+1)-dimensional coupled burgers equation can be obtained by using this approach. Some types of the localized structures for the equation, especially the dromion-like and solitoff-like structures, are discussed in detail. We believe more localized structures can be obtained through the Painlevé expansion approach.

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