N-soliton solutions of an integrable equation studied by Qiao*

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In this paper, we studied N-soliton solutions of a new integrable equation studied by Qiao [*J. Math. Phys.* **48** 082701 (2007)]. Firstly, we employed the Darboux matrix method to construct a Darboux transformation for the modified Korteweg-de Vries equation. Then we use the Darboux transformation and a transformation, introduced by Sakovich [*J. Math. Phys.* **52** 023509 (2011)], to derive N-soliton solutions of the new integrable equation from the seed solution. In particular, the multiple soliton solutions are explicitly obtained and shown through some figures.

Keywords: soliton solution, Darboux transformation, integrable equation

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1. Introduction

In Ref. [1], a new integrable hierarchy, including nonlinear cubic evolution equations, was proposed with the help of Hamiltonian and recursion operator procedure. Its second positive member is a new completely integrable wave equation

$$m_t = \left(\frac{1}{2m^2}\right)_{xxx} - \left(\frac{1}{2m^2}\right)_x, \tag{1}$$

where m is a scalar potential function and subscripts denote the partial derivatives. This equation has been shown to have bi-Hamiltonian structures and a Lax pair that implies its integrability. Through studying Eq. (1), Qiao developed two new kinds of soliton solutions under the inhomogeneous boundary condition $\lim_{|x|\to\infty} m = B$, where B is a nonzero constant. One is a continuous W/M-shape soliton solution and the other is a one-single-peak soliton.[1,2] The new kinds of peaked soliton solutions cannot be equivalent to the regular peak ones: $c\,\mathrm{e}^{-|x-ct|}$, where c is a constant.[2,3]

In Ref. [4], the bifurcations of traveling wave solutions of Eq. (1) were studied and this equation does possess smooth soliton solutions and periodic solitary wave solutions. Very recently, Sakovich[5] found a transformation, which relates Eq. (1) to the well-known modified Korteweg-de Vries (mKdV) equation, and obtained smooth soliton solutions of Eq. (1) from the known rational and soliton solutions of the mKdV equation.

In the present paper, based on the transformation introduced by Sakovich,[5,6] we seek the N-soliton solutions of Eq. (1) using the Darboux matrix method.[7–24] This paper is organized as follows. In Section 2, a chain of new transformations are introduced to relate Eq. (1) to the mKdV equation. In Section 3, the Lax pair and Darboux transformation (DT) are constructed for the mKdV equation. In Section 4, N-soliton solutions of Eq. (1) are derived from the DT. The paper is concluded in Section 5 with a summary of the results.

2. Transformation of the mKdV equation

Sakovich[5] has found a chain of new transformations, which relate Eq. (1) to the mKdV equation

$$w_t = w_{yyy} - \frac{3}{2}w^2 w_y. \tag{2}$$

All solutions of Eq. (1) can be represented in a parametrical form by all nonzero solutions of the mKdV equation (2) via the following transformations

$$m(x,t) = \frac{-1}{w(y,t)}, \tag{3}$$

$$x = v(y,t): \begin{cases} v_y = w, \\ v_t = w_{yy} - \dfrac{1}{2}w^3, \end{cases} \tag{4}$$

where the independent variable y serves as the parameter.

3. Lax pair and Darboux transformation

In order to construct a DT of the mKdV equation (2), we consider the AKNS-type Lax pair[25] of the mKdV equation (2) in the form of

$$\phi_y = U\phi, \quad \phi_t = V\phi, \quad \phi = (\phi_1, \phi_2)^{\mathrm{T}}, \tag{5}$$

where U and V are the following two matrices

$$U = \begin{pmatrix} \lambda & \frac{1}{2}\mathrm{i}w \\ -\frac{1}{2}\mathrm{i}w & -\lambda \end{pmatrix}, \tag{6}$$

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$$V=\begin{pmatrix} 4\lambda^3-\frac{1}{2}\lambda w^2 & 2\mathrm{i}\lambda^2 w+\mathrm{i}\lambda w_y-\frac{1}{4}\mathrm{i}(w^3-2w_{yy}) \\ -2\mathrm{i}\lambda^2 w+\mathrm{i}\lambda w_y+\frac{1}{4}\mathrm{i}(w^3-2w_{yy}) & -4\lambda^3+\frac{1}{2}\lambda w^2 \end{pmatrix}. \tag{7}$$

Through a direct calculation, the zero curvature equation

$$U_t-V_y+[U,V]=0$$

yields the mKdV equation (2).

In the following, we construct the DT of the mKdV equation (2). Let us introduce a gauge transformation

$$\bar{\phi}=T\phi, \tag{8}$$

which transforms the Lax pair (5) into a new Lax pair of $\bar{\phi}$:

$$\bar{\phi}_y=\bar{U}\phi,\quad \bar{U}=(T_y+TU)T^{-1}, \tag{9}$$

$$\bar{\phi}_t=\bar{V}\phi,\quad \bar{V}=(T_t+TV)T^{-1}. \tag{10}$$

We adopt the known result from Ref. [7] about the elementary Darboux matrix

$$T=\begin{pmatrix} \lambda-\frac{\lambda_1(1-\delta^2)}{1+\delta^2} & -\frac{2\lambda_1\delta}{1+\delta^2} \\ -\frac{2\lambda_1\delta}{1+\delta^2} & \lambda+\frac{\lambda_1(1-\delta^2)}{1+\delta^2} \end{pmatrix},$$
$$\det T=\lambda^2-\lambda_1^2, \tag{11}$$

where

$$\delta=\frac{\phi_2}{\phi_1}, \tag{12}$$

and $\phi=(\phi_1,\phi_2)^{\mathrm{T}}$ satisfies the Lax pair (5) with $\lambda=\lambda_1$. The function δ satisfies the following Riccati equations

$$\begin{aligned} \delta_y &= -\frac{1}{2}\mathrm{i}w-2\lambda_1\delta-\frac{1}{2}\mathrm{i}w\delta^2, \\ \delta_t &= -2\mathrm{i}\lambda^2 w+\mathrm{i}\lambda w_y+\frac{1}{4}\mathrm{i}(w^3-2w_{yy}) \\ &\quad -2\left(4\lambda^3-\frac{1}{2}\lambda w^2\right)\delta \\ &\quad -\left(2\mathrm{i}\lambda^2 w+\mathrm{i}\lambda w_y-\frac{1}{4}\mathrm{i}(w^3-2w_{yy})\right)\delta^2, \end{aligned} \tag{13}$$

which can be obtained directly from Eqs. (12) and (5).

From the above discussions, we have the following propositions.

Proposition 1 The matrix $\bar{U}$ determined by the second expression of Eq. (9) has the same form as U:

$$\bar{U}=\begin{pmatrix} \lambda & \frac{1}{2}\mathrm{i}\bar{w} \\ -\frac{1}{2}\mathrm{i}\bar{w} & -\lambda \end{pmatrix}, \tag{15}$$

where the original potential w in U is transformed into the new one $\bar{w}$ in $\bar{U}$, given by

$$\bar{w}=-w+\frac{4\mathrm{i}\delta_y}{1+\delta^2}=w-\frac{8\mathrm{i}\lambda_1\delta}{1+\delta^2}. \tag{16}$$

Proof Substituting T and U into the right-hand side of the second expression of Eqs. (9) and using Eq. (16), we have the following result

$$\begin{aligned} (T_y+TU)T^{-1} &= \begin{pmatrix} \lambda & \frac{1}{2}\mathrm{i}\left(w-\frac{8\mathrm{i}\lambda_1\delta}{1+\delta^2}\right) \\ -\frac{1}{2}\mathrm{i}\left(w-\frac{8\mathrm{i}\lambda_1\delta}{1+\delta^2}\right) & -\lambda \end{pmatrix} \\ &= \begin{pmatrix} \lambda & \frac{1}{2}\mathrm{i}\bar{w} \\ -\frac{1}{2}\mathrm{i}\bar{w} & -\lambda \end{pmatrix}=\bar{U}, \end{aligned} \tag{17}$$

which completes the proof.

Proposition 2 The matrix $\bar{V}$ determined by the second expression of Eqs. (10) has the same form as V:

$$\bar{V}=\begin{pmatrix} 4\lambda^3-\frac{1}{2}\lambda\bar{w}^2 & 2\mathrm{i}\lambda^2\bar{w}+\mathrm{i}\lambda\bar{w}_y-\frac{1}{4}\mathrm{i}(\bar{w}^3-2\bar{w}_{yy}) \\ -2\mathrm{i}\lambda^2\bar{w}+\mathrm{i}\lambda\bar{w}_y+\frac{1}{4}\mathrm{i}(\bar{w}^3-2\bar{w}_{yy}) & -4\lambda^3+\frac{1}{2}\lambda\bar{w}^2 \end{pmatrix}, \tag{18}$$

where the original potential w is mapped into the new potential $\bar{w}$ according to the transformation (16).

Proof Substituting T and V into the right-hand side of the second expression of Eqs. (10) and using Eqs. (21) and (22), we can get the following result

$$(T_t+TV)T^{-1}=\begin{pmatrix} p_{11}^{(3)}\lambda^3+p_{11}^{(2)}\lambda^2+p_{11}^{(1)}\lambda+p_{11}^{(0)} & p_{12}^{(2)}\lambda^2+p_{12}^{(1)}\lambda+p_{12}^{(0)} \\ p_{21}^{(2)}\lambda^2+p_{21}^{(1)}\lambda+p_{21}^{(0)} & p_{22}^{(3)}\lambda^3+p_{22}^{(2)}\lambda^2+p_{22}^{(1)}\lambda+p_{22}^{(0)} \end{pmatrix}=Q, \tag{19}$$

where

$$
\begin{aligned}
p_{11}^{(0)} &= p_{22}^{(0)} = 0, \quad p_{11}^{(3)} = -p_{22}^{(3)} = 4, \\
p_{22}^{(1)} &= -p_{11}^{(1)} = \frac{1}{2}\left(w - \frac{8\mathrm{i}\lambda_1\delta}{1+\delta^2}\right)^2 = \frac{1}{2}\bar{w}^2, \\
p_{11}^{(2)} &= p_{22}^{(2)} = 0, \quad p_{12}^{(2)} = -p_{21}^{(2)} = 2\mathrm{i}\left(w - \frac{8\mathrm{i}\lambda_1\delta}{1+\delta^2}\right) = 2\mathrm{i}\bar{w}, \\
p_{12}^{(1)} &= p_{21}^{(1)} = \mathrm{i}\left(w_y + \frac{4\lambda_1(-1+\delta^2)(-4\mathrm{i}\lambda_1\delta + w(1+\delta^2))}{(1+\delta^2)^2}\right) \\
&= \mathrm{i}\bar{w}_y, \\
q_{21}^{(0)} &= -q_{12}^{(0)} = \frac{1}{4(1+\delta^2)}(8\lambda_1 w^2\delta - 16\mathrm{i}\lambda_1^2 w(1+\delta^2) \\
&\quad + \mathrm{i}w^3(1+\delta^2) - 2(32\lambda_1^3\delta \\
&\quad - 4\mathrm{i}\lambda_1 w_y + \mathrm{i}w_{yy} + \mathrm{i}\delta^2(4\lambda_1 w_y + w_{yy}))) \\
&= \frac{1}{4}\mathrm{i}(\bar{w}^3 - 2\bar{w}_{yy}). \qquad (20)
\end{aligned}
$$

Thus $Q = \bar{V}$. This completes the proof.

With the aid of Eqs. (16) and (13), we can also calculate the first and second derivatives of $\bar{w}$ with respect to y

$$
\begin{aligned}
\bar{w}_y &= w_y + \frac{4\lambda_1(-1+\delta^2)(-4\mathrm{i}\lambda_1\delta + w(1+\delta^2))}{(1+\delta^2)^2}, \qquad (21) \\
\bar{w}_{yy} &= w_{yy} + \frac{1}{(1+\delta^2)^3}(4\lambda_1(-8\mathrm{i}\lambda_1^2\delta + 48\mathrm{i}\lambda_1^2\delta^3 - 8\mathrm{i}\lambda_1^2\delta^5 \\
&\quad - 2\mathrm{i}w^2\delta(1+\delta^2)^2 + 2\lambda_1 w(1 - 9\delta^2 \\
&\quad - 9\delta^4 + \delta^6) - w_y - \delta^2 w_y + \delta^4 w_y + \delta^6 w_y)). \qquad (22)
\end{aligned}
$$

According to Propositions 1 and 2, the transformation (16) changes the Lax pair (5) into other Lax pairs (9) and (10) in the same form. Therefore, both of the Lax pairs lead to the same mKdV equation (2). The transformation (16) is usually called a DT of the mKdV equation (2). With the help of DT (16), the new solutions of the mKdV equation (2) can be generated from a seed solution.

Remark 1 It is not difficult to see that the transformation $\bar{w} = -w + 4\mathrm{i}\delta_y/(1+\delta^2)$ is equivalent to $\bar{v}_y = -w + 4\mathrm{i}\delta_y/(1+\delta^2)$. Integrating $\bar{v}_y = -w + 4\mathrm{i}\delta_y/(1+\delta^2)$ with respect to y once, we have

$$
\bar{v} = -\int w\,\mathrm{d}y + 4\mathrm{i}\arctan(\delta). \qquad (23)
$$

We also note that if w is a solution of the mKdV equation (2), so is $-w$. Therefore, we have the following theorem.

Theorem 1 If $\bar{w}$ is a solution of the mKdV equation (2), then we have $\bar{m}$:

$$
\bar{m}(x,t) = \frac{-1}{\bar{w}(y,t)}, \qquad (24)
$$

$$
x = \bar{v}(y,t) = -\int w\,\mathrm{d}y + 4\mathrm{i}\arctan(\delta) : \begin{cases} \bar{v}_y = \bar{w}, \\ \bar{v}_t = \bar{w}_{yy} - \frac{1}{2}\bar{w}^3, \end{cases} \qquad (25)
$$

which gives a solution of Eq. (1).

4. Solitons

In this section, we will apply Theorem 1 to construct N-soliton solutions for Eq. (1). As usual, we make a DT starting from a seed solution of the mKdV equation (2). Substituting the seed solution $w = \mathrm{i}\mu$ (μ is a constant) into the Lax pair (5), we choose their basic solutions as follows:

$$
\phi = \begin{pmatrix} \phi_1^{(j)} \\ \phi_2^{(j)} \end{pmatrix} = \begin{pmatrix} \cosh\xi_j \\ \dfrac{2\lambda_j}{\mu}\cosh\xi_j - \dfrac{\sqrt{4\lambda_j^2 - \mu^2}}{\mu}\sinh\xi_j \end{pmatrix},
$$

j is an odd number, (26)

and

$$
\phi = \begin{pmatrix} \phi_1^{(j)} \\ \phi_2^{(j)} \end{pmatrix} = \begin{pmatrix} \sinh\xi_j \\ \dfrac{2\lambda_j}{\mu}\sinh\xi_j - \dfrac{\sqrt{4\lambda_j^2 - \mu^2}}{\mu}\cosh\xi_j \end{pmatrix},
$$

j is an even number, (27)

where

$$
\xi_j = \frac{1}{2}\sqrt{4\lambda_j^2 - \mu^2}\left(y + \frac{1}{2}(8\lambda_j^2 + \mu^2)t\right). \qquad (28)
$$

By applying Theorem 1, we are able to obtain multi-soliton solutions for Eq. (1).

Case 1 ($j = 1$) Noticing equation (12), we have

$$
\delta_1 = \frac{2\lambda_1}{\mu} - \frac{\sqrt{4\lambda_1^2 - \mu^2}}{\mu}\tanh\xi_1, \qquad (29)
$$

where

$$
\xi_1 = \frac{1}{2}\sqrt{4\lambda_1^2 - \mu^2}\left(y + \frac{1}{2}(8\lambda_1^2 + \mu^2)t\right).
$$

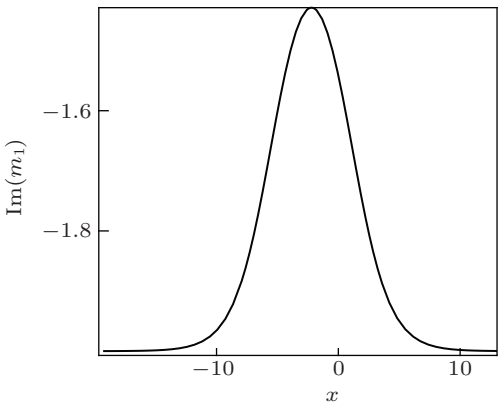


Fig. 1. One-soliton solution (30) with $\mu = 1/2$, $\lambda_1 = -3/10$, and $t = 0$.

With the aid of Eqs. (24), (25), (16), and (29), we obtain a one-soliton solution of Eq. (1)

$$m_1 = \frac{-1}{-w+\dfrac{4\mathrm{i}\delta_{1,y}}{1+\delta_1^2}} = \frac{-\mathrm{i}}{\mu+\dfrac{2(4\lambda_1^2-\mu^2)\operatorname{sech}^2\xi_1}{\mu\left(1+\left(\dfrac{2\lambda_1}{\mu}-\dfrac{\sqrt{4\lambda_1^2-\mu^2}}{\mu}\tanh\xi_1\right)^2\right)}},$$

$$x = -\int w\,\mathrm{d}y + 4\mathrm{i}\arctan(\delta_1) = \mathrm{i}\left(-\mu y + 4\arctan\left(\frac{2\lambda_1}{\mu}-\frac{\sqrt{4\lambda_1^2-\mu^2}}{\mu}\tanh\xi_1\right)\right). \tag{30}$$

The graph of this one-bell-shape-soliton solution (30) of Eq. (1) is given in Fig. 1.

Case 2 $(j=2)$ According to the Lax pairs (9) and (10), their solution is explicitly given as follows:

$$\begin{aligned}\bar{\phi} = T\phi &= \begin{pmatrix} \lambda_2-\dfrac{\lambda_1(1-\delta_1^2)}{1+\delta_1^2} & -\dfrac{2\lambda_1\delta_1}{1+\delta_1^2} \\ -\dfrac{2\lambda_1\delta_1}{1+\delta_1^2} & \lambda_2+\dfrac{\lambda_1(1-\delta_1^2)}{1+\delta_1^2}\end{pmatrix}\begin{pmatrix} \sinh\xi_2 \\ \dfrac{2\lambda_2}{\mu}\sinh\xi_2-\dfrac{\sqrt{4\lambda_2^2-\mu^2}}{\mu}\cosh\xi_2\end{pmatrix} \\ &= \begin{pmatrix} \left(\lambda_2-\dfrac{\lambda_1(1-\delta_1^2)}{1+\delta_1^2}\right)\sinh\xi_2+\left(-\dfrac{2\lambda_1\delta_1}{1+\delta_1^2}\right)\left(\dfrac{2\lambda_2}{\mu}\sinh\xi_2-\dfrac{\sqrt{4\lambda_2^2-\mu^2}}{\mu}\cosh\xi_2\right) \\ \left(-\dfrac{2\lambda_1\delta_1}{1+\delta_1^2}\right)\sinh\xi_2+\left(\lambda_2+\dfrac{\lambda_1(1-\delta_1^2)}{1+\delta_1^2}\right)\left(\dfrac{2\lambda_2}{\mu}\sinh\xi_2-\dfrac{\sqrt{4\lambda_2^2-\mu^2}}{\mu}\cosh\xi_2\right)\end{pmatrix}.\end{aligned} \tag{31}$$

Noticing Eqs. (12) and (31), we have

$$\delta_2 = \frac{\left(-\dfrac{2\lambda_1\delta_1}{1+\delta_1^2}\right)\sinh\xi_2+\left(\lambda_2+\dfrac{\lambda_1(1-\delta_1^2)}{1+\delta_1^2}\right)\left(\dfrac{2\lambda_2}{\mu}\sinh\xi_2-\dfrac{\sqrt{4\lambda_2^2-\mu^2}}{\mu}\cosh\xi_2\right)}{\left(\lambda_2-\dfrac{\lambda_1(1-\delta_1^2)}{1+\delta_1^2}\right)\sinh\xi_2+\left(-\dfrac{2\lambda_1\delta_1}{1+\delta_1^2}\right)\left(\dfrac{2\lambda_2}{\mu}\sinh\xi_2-\dfrac{\sqrt{4\lambda_2^2-\mu^2}}{\mu}\cosh\xi_2\right)}. \tag{32}$$

With the help of Eqs. (24), (25), and (16), taking the one-soliton solution (30) as the new seed solution, we arrive at a two-soliton solution of Eq. (1)

$$m_2 = \frac{-1}{-w_1+\dfrac{4\mathrm{i}\delta_{2,y}}{1+\delta_2^2}} = \frac{-1}{\mathrm{i}\left(\mu-\dfrac{4\delta_{1,y}}{1+\delta_1^2}+\dfrac{4\delta_{2,y}}{1+\delta_2^2}\right)},$$

$$\begin{aligned}x &= -\int w_1\,\mathrm{d}y + 4\mathrm{i}\arctan(\delta_2) \\ &= \mathrm{i}\left(\mu y - 4\arctan(\delta_1)+4\arctan(\delta_2)\right),\end{aligned} \tag{33}$$

where δ_1 and δ_2 are given by Eqs. (29) and (32), respectively. The two soliton graphs are shown in Fig. 2.

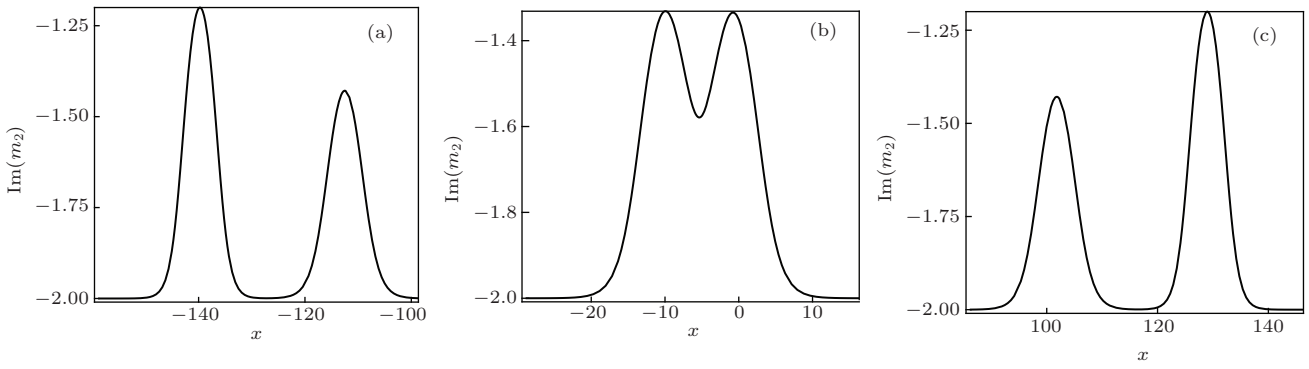


Fig. 2. Two-soliton solution (33) with $\mu=-1/2$, $\lambda_1=-3/10$, $\lambda_2=-1/3$ and (a) at $t=-46$, (b) at $t=0$, (c) at $t=46$.

Case 3 ($j=3$) According to the Lax pairs (9) and (10), their solution can be written as

$$\bar{\phi} = T\phi = \begin{pmatrix} \lambda_3 - \dfrac{\lambda_2(1-\delta_2^2)}{1+\delta_2^2} & -\dfrac{2\lambda_2\delta_2}{1+\delta_2^2} \\ -\dfrac{2\lambda_2\delta_2}{1+\delta_2^2} & \lambda_3 + \dfrac{\lambda_2(1-\delta_2^2)}{1+\delta_2^2} \end{pmatrix} \begin{pmatrix} \cosh\xi_3 \\ \dfrac{2\lambda_3}{\mu}\cosh\xi_3 - \dfrac{\sqrt{4\lambda_3^2-\mu^2}}{\mu}\sinh\xi_3 \end{pmatrix}$$
$$= \begin{pmatrix} \left(\lambda_3 - \dfrac{\lambda_2(1-\delta_2^2)}{1+\delta_2^2}\right)\cosh\xi_3 + \left(-\dfrac{2\lambda_2\delta_2}{1+\delta_2^2}\right)\left(\dfrac{2\lambda_3}{\mu}\cosh\xi_3 - \dfrac{\sqrt{4\lambda_3^2-\mu^2}}{\mu}\sinh\xi_3\right) \\ \left(-\dfrac{2\lambda_2\delta_2}{1+\delta_2^2}\right)\cosh\xi_3 + \left(\lambda_3 + \dfrac{\lambda_2(1-\delta_2^2)}{1+\delta_2^2}\right)\left(\dfrac{2\lambda_3}{\mu}\cosh\xi_3 - \dfrac{\sqrt{4\lambda_3^2-\mu^2}}{\mu}\sinh\xi_3\right) \end{pmatrix}. \tag{34}$$

Noticing Eqs. (12) and (31), we have

$$\delta_3 = \frac{\left(-\dfrac{2\lambda_2\delta_2}{1+\delta_2^2}\right)\cosh\xi_3 + \left(\lambda_3 + \dfrac{\lambda_2(1-\delta_2^2)}{1+\delta_2^2}\right)\left(\dfrac{2\lambda_3}{\mu}\cosh\xi_3 - \dfrac{\sqrt{4\lambda_3^2-\mu^2}}{\mu}\sinh\xi_3\right)}{\left(\lambda_3 - \dfrac{\lambda_2(1-\delta_2^2)}{1+\delta_2^2}\right)\cosh\xi_3 + \left(-\dfrac{2\lambda_2\delta_2}{1+\delta_2^2}\right)\left(\dfrac{2\lambda_3}{\mu}\cosh\xi_3 - \dfrac{\sqrt{4\lambda_3^2-\mu^2}}{\mu}\sinh\xi_3\right)}. \tag{35}$$

With the help of Eqs. (24), (25), and (16), after taking the two-soliton solution (33) as the seed solution, we arrive at a three-soliton solution of Eq. (1)

$$\begin{aligned} m_3 &= \frac{-1}{-w_2 + \dfrac{4\mathrm{i}\delta_{3,y}}{1+\delta_3^2}} \\ &= \frac{-1}{\mathrm{i}\left(-\mu + \dfrac{4\delta_{1,y}}{1+\delta_1^2} - \dfrac{4\delta_{2,y}}{1+\delta_2^2} + \dfrac{4\delta_{3,y}}{1+\delta_3^2}\right)}, \\ x &= -\int w_2\,\mathrm{d}y + 4\mathrm{i}\arctan(\delta_3) \\ &= \mathrm{i}(-\mu y + 4\arctan(\delta_1) - 4\arctan(\delta_2) \\ &\quad + 4\arctan(\delta_3)), \end{aligned} \tag{36}$$

where δ_1, δ_2, and δ_3 are given by Eqs. (29), (32), and (35), respectively. The three-soliton graphs are shown in Fig. 3.

Case N ($j=N$) In general, we can get N-soliton solutions of Eq. (1) in an explicit form:

$$\begin{aligned} m_N &= \frac{-1}{\mathrm{i}\left((-1)^N\mu + 4\sum_{j=1}^{N}\dfrac{(-1)^{N-j}\delta_{j,y}}{1+\delta_j^2}\right)}, \\ x &= \mathrm{i}\left((-1)^N\mu y + 4\sum_{j=1}^{N}(-1)^{N-j}\arctan(\delta_j)\right), \end{aligned} \tag{37}$$

where

$$\delta_1 = \frac{2\lambda_1}{\mu} - \frac{\sqrt{4\lambda_1^2-\mu^2}}{\mu}\tanh\xi_1, \tag{38}$$

$$\delta_j = \frac{-\dfrac{2\lambda_{j-1}\delta_{j-1}}{1+\delta_{j-1}^2}\phi_1^{(j)} + \left(\lambda_j + \lambda_{j-1}\dfrac{1-\delta_{j-1}^2}{1+\delta_{j-1}^2}\right)\phi_2^{(j)}}{\left(\lambda_j - \lambda_{j-1}\dfrac{1-\delta_{j-1}^2}{1+\delta_{j-1}^2}\right)\phi_1^{(j)} - \dfrac{2\lambda_{j-1}\delta_{j-1}}{1+\delta_1^2}\phi_2^{(j)}}, \quad j=2,3,4,\ldots,N, \tag{39}$$

where $\phi_1^{(j)}$ and $\phi_2^{(j)}$ are given by Eqs. (26) and (27).

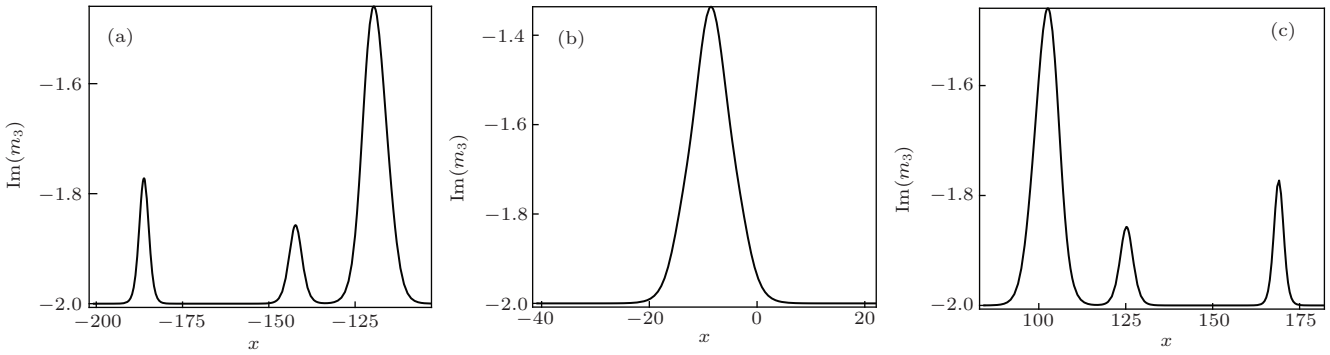


Fig. 3. A three-soliton solution (36) with $\mu=-1/2$, $\lambda_1=-3/10$, $\lambda_2=-1/3$, $\lambda_3=-4/10$ and (a) at $t=-46$, (b) at $t=0$, (c) at $t=46$.

So, with the help of the DT (16), we are able to generate multi-soliton solutions of Eq. (1) through an iteration procedure starting from a seed solution.

5. Conclusions

In this paper, we successfully solve Eq. (1) studied by Qiao[1] through the DT (16) of the mKdV equation (2) and transformations (3) and (4). By means of that DT, we derived multiple smooth soliton solutions of Eq. (1) via the transformation between Eq. (1) and the mKdV equation. The interactions of multiple smooth soliton solutions are graphically figured out (see Figs. 1–3). Similarly to the approach to constructing soliton solutions of Eq. (1), if the different solutions of the Lax pair (5) are chosen, we can find the periodic wave solution and rational solution of Eq. (1).

(i) Choosing the basic solutions of the Lax pair (5)

$$\phi = \begin{pmatrix} \phi_1^{(j)} \\ \phi_2^{(j)} \end{pmatrix} = \begin{pmatrix} \cos\zeta_j \\ \dfrac{2\lambda_j}{\mu}\cos\zeta_j + \dfrac{\sqrt{\mu^2-4\lambda_j^2}}{\mu}\sin\zeta_j \end{pmatrix},$$

j is an odd number,

and

$$\phi = \begin{pmatrix} \phi_1^{(j)} \\ \phi_2^{(j)} \end{pmatrix} = \begin{pmatrix} \sin\zeta_j \\ \dfrac{2\lambda_j}{\mu}\sin\zeta_j - \dfrac{\sqrt{\mu^2-4\lambda_j^2}}{\mu}\cos\zeta_j \end{pmatrix},$$

j is an even number,

where

$$\zeta_j = \frac{1}{2}\sqrt{\mu^2-4\lambda_j^2}\left(y+\frac{1}{2}(8\lambda_j^2+\mu^2)t\right), \quad w = \mathrm{i}\mu,$$

we can get periodic wave solutions of Eq. (1).

(ii) Substituting the solution of the Lax pair (5)

$$\phi = \begin{pmatrix} \phi_1^{(1)} \\ \phi_2^{(1)} \end{pmatrix} = \begin{pmatrix} c_1 + \dfrac{w\mathrm{i}}{2}(c_1+c_2)x - \dfrac{3w^3\mathrm{i}}{4}(c_1+c_2)t \\ c_2 - \dfrac{w\mathrm{i}}{2}(c_1+c_2)x + \dfrac{3w^3\mathrm{i}}{4}(c_1+c_2)t \end{pmatrix},$$

$$\lambda_1 = \frac{\mathrm{i}w}{2},$$

and Eq. (12) into Eq. (16), and using Theorem 1, we can obtain a rational solution, which is reduced to the rational solution in Ref. [5] if $c_1 = c_2 = 1/4$, $w = -a$.

The question of how to construct the cuspon, the interaction of two single peaked solitons, two W-shape-peaks solitons, two M-shape-peaks solitons, one W-shape-peak and one M-shape-peak solitons, and one single peaked and the other M/W solitons by the Darboux transformation method or the Hirota bilinear method or others is an interesting and significant topic, and these problems are still under investigation.

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